On Docile Spaces, Mackey First Countable Spaces, and Sequentially Mackey First Countable Spaces

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Abstract

In this article we discuss the relationship between three types of locally convex spaces: docile spaces, Mackey first countable spaces, and sequentially Mackey first countable spaces. More precisely, we show that docile spaces are sequentially Mackey first countable. We also show the existence of sequentially Mackey first countable spaces that are not Mackey first countable, and we characterize Mackey first countable spaces in terms of normability of certain inductive limits.

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1 Introduction

In this paper, space refers to a Hausdorff locally convex topological vector space $E = (E, \tau)$ over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We assume that $E$ is infinite dimensional unless otherwise indicated. Unspecified notation follows standard locally convex spaces notation such as in [2]. To denote $E$ as a linear subspace of another linear space $F$, we use $E \subseteq F$, and $E \subsetneq F$ if $E$ is a proper subspace of $F$.

The concept of a docile space was introduced in [3]:

**Definition 1.1.** A space $E = (E, \tau)$ is **docile** if every linear subspace of infinite dimension contains a $\tau$-bounded set $B$ such that $\text{span}\{B\}$ is infinite-dimensional.

In one of his seminal papers, G. W. Mackey ([6]) defined the following:

**Definition 1.2.** A space $E = (E, \tau)$ is **first countable**, henceforth referred to as Mackey first countable (MFC in short), if for every sequence of $\tau$-bounded sets $(B_n)$ there exists a sequence of nonzero scalars $(a_n)$, such that $\bigcup_{n=1}^{\infty} a_n B_n$ remains bounded.

A sequential version of MFC space, sequentially Mackey first countable space (Definition 1.3 below), was introduced in [1], mainly to show that a Hausdorff quotient of a MFC space is a sequentially Mackey first countable space. A result that partially answers G. W. Mackey’s conjecture (see [6]) which asks whether a quotient of an MFC space is also an MFC space, was also given in [1].

**Definition 1.3.** A space $E$ is sequentially Mackey first countable (SMFC), if given any sequence $(B_n)$ of bounded sets in $E$, and any sequence $(x_k)$ from $\bigcup_{n=1}^{\infty} B_n$, there exists a sequence $(\alpha_k)$ of scalars with infinitely many nonzero terms such that $\{\alpha_k x_k : k \in \mathbb{N}\}$ is bounded.
Observe that in Definition 1.3 one may consider sequences \((x_k)\) without any reference to a sequence \((B_n)\) of bounded sets. Observe as well that clearly every MFC space is a SMFC space.

The plan of this paper is as follows. First we show that docile spaces are SMFC spaces (Section 2). Then we prove the existence of spaces that are SMFC but not MFC (Section 3). Finally in Section 4 we obtain a nice characterization of Mackey first countability via the normability of certain inductive limits (Theorem 4.4).

2 Docile spaces are sequentially Mackey first countable.

To prove that a docile space \((E, \tau)\) is SMFC let us start with a sequence in \(E\) of nonzero vectors \((x_n)\) and let \(F\) be the linear span of \((x_n)\). If \(F\) is finite dimensional then \(\tau|_F\) is a norm, say \(\rho\), and with \(\alpha_n = 1/\rho(x_n)\) we have that \((\alpha_n x_n)\) is bounded. Assume then, that the vectors \(\{x_n\}\) are linearly independent so that \(F\) is infinite dimensional. By docility we get an infinite dimensional bounded, balanced, and convex subset \(B\), of \(F\). Let \((y_j)\) be a sequence of linearly independent vectors in \(B\) and, for each \(j = 1, 2, \ldots\) write

\[
y_j = \sum_{n=1}^{\infty} \beta_{jn} x_n
\]

where \(\beta_{jn} = 0\) for all \(n\) but finitely many.

Set \(A_j = \{x_n : \beta_{jn} \neq 0\}\) and \(A = \bigcup_{j=1}^{\infty} A_j\). Note that the set \(A\) is infinite dimensional since the set \(\{y_j\}\) is linearly independent.

To show the existence of nonnegative scalars \((\alpha_n)\), with infinitely many of them nonzero, for which the sequence \((\alpha_n x_n)\) is \(\tau\)-bounded, we will “spread” the sequence \((x_n)\) over a cartesian product as follows. First, it is known that any space \((E, \tau)\) can be embedded as a subspace of a product of Banach spaces (see [4, (7), p.208]). So let \(\{(E_\lambda, \hat{\rho}_\lambda)\}_{\lambda \in \Lambda}\) be a collection of Banach spaces (here \(\hat{\rho}_\lambda\) is meant to be the norm in the corresponding Banach space) and \(\gamma : (E, \tau) \to \prod_\lambda E_\lambda\) a linear continuous map such that \(E\) and \(\gamma(E)\) are topologically homeomorphic. The construction in [4, (7), p.208] shows that if \(\{\rho_\lambda\}_{\lambda \in \Lambda}\) is the family of seminorms that generate \(\tau\) then \(E_\lambda = E/\{x \in E : \rho_\lambda(x) = 0\}\) and \(\gamma(x)(\lambda) = [x]_\lambda\), the equivalence class of \(x\) in \(E_\lambda\), that is \(x \in E\) is identified with the point in the cartesian product whose \(\lambda\)-coordinate is precisely \([x]_\lambda\). For each \(\lambda \in \Lambda\), we denote by \(q_\lambda : \prod_\lambda E_\lambda \to E_\mu\) the canonical projection. We also denote by \(\tau_{prod}\) the product topology.

Note that for each \(\lambda \in \Lambda\), \(q_\lambda(B) := B_\lambda \subset E_\lambda\) is \(\hat{\rho}_\lambda\)-bounded and \(q_\lambda(y_j) = \sum_{n=1}^{\infty} \beta_{jn} q_\lambda(x_n) \in B_\lambda \subset E_\lambda\).
Carlos Bosch et al.

If we can copy nicely the sequence \((x_n)\) into the cartesian product, the desired sequence \((\alpha_n)\) is obtained as follows. When we say copy nicely we mean that for each \(x_n\) there exists \(\lambda_n \in \Lambda\) such that \(\gamma(x_n) \in E_{\alpha_n}\), that is, \(q_\lambda(\gamma(x_n)) = [0]_\lambda\) if \(\lambda \neq \lambda_n\) and \(q_{\lambda_n}(\gamma(x_n)) = [x_n]_{\lambda_n}\).

In this scenario,

\[
q_{\lambda_n}(\gamma(y)) = \beta_{jn} \gamma(x_n) + \sum_{k \neq n} \beta_{jk} \gamma(x_k) \in B_{\lambda_n} \subset E_{\alpha_n}.
\]

Thus

\[
\beta_{jn} \gamma(x_n) = B_{\lambda_n} - \sum_{k \neq n} \beta_{jk} \gamma(x_k) = B'_\lambda \subset E_{\alpha_n}.
\]

It follows that for each \(n \in \mathbb{N}\), \(\beta_{jn} \gamma(x_n) \in \prod_{n \in \mathbb{N}} B'_{\lambda_n}\). Note that the latest set is \(\tau_{\text{prod}}\)-bounded (see, for instance [4, (13) p.155]). Therefore

\[
\bigcup_{n=1}^{\infty} \beta_{jn} \gamma(x_n) \subset \prod_{n \in \mathbb{N}} B'_{\lambda_n} \subset (\prod_{\lambda} E_{\lambda}) \bigcap \gamma(E).
\]

Finally, by considering the union over all indices \(j\) we have:

\[
\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \beta_{jn} \gamma(x_n) \subset \prod_{n \in \mathbb{N}} B'_{\lambda_n} \subset (\prod_{\lambda} E_{\lambda}) \bigcap \gamma(E).
\]

That is, the set \(\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \beta_{jn} \gamma(x_n)\) is \(\tau_{\text{prod}}\)-bounded.

Since \(\gamma\) is a linear homeomorphism we have that \(\{\beta_{jn} x_n\}_{(j,n) \in \mathbb{N} \times \mathbb{N}} \subset E\) is \(\tau\)-bounded, with infinitely many of the coefficients \(\beta_{jn}\) being nonzero. Now, if for each \(j \in \mathbb{N}\) we let \(\alpha_j = \max\{|\beta_{jn}| : n \in \mathbb{N}\}\), the set \(\{\alpha_j x_j : j \in \mathbb{N}\}\) is \(\tau\)-bounded, with infinitely many of the scalars \(\alpha_j\) being nonzero and positive.

If we don’t have that for each \(x_n\) there exists \(\lambda_n \in \Lambda\) such that \(\gamma(x_n) \in E_{\alpha_n}\) as before, first note that since each \(x_n \neq 0\), there exists at least one projection \(q_{\lambda_n}\) such that \(0 \neq q_{\lambda_n}(x_n) = [x_n]_{\lambda_n} \in E_{\lambda_n}\). If there are infinitely many terms \(x_n\) with the same coordinate different from zero, then infinitely many terms \(x_n\) can be nicely copied into the cartesian product as before. Thus, we can assume, without lost of generality, that the \(\lambda_n\)'s are all distinct.

Let \(z_n \in \prod_{\lambda} E_{\lambda}\) be defined as follows: \(z_n(\lambda) = [0]_\lambda\) if \(\lambda \neq \lambda_n\) and \(z_n(\lambda_n) = [x_n]_{\lambda_n}\).

By identifying the space \(E_{\lambda}\) in the cartesian product via the inclusion \(i_{\lambda} : E_{\lambda} \to \prod_{\mu} E_{\mu}, \ i_{\lambda}(x_{\lambda} \lambda)(\mu) = [x_{\lambda}]_{\lambda}\) if \(\lambda = \mu\) and \(i_{\lambda}(x_{\lambda} \lambda)(\mu) = [0]_\mu\) if \(\mu \neq \lambda\), we obtain that \(z_n\) is an element of the space \(E_{\lambda_n}\) and, that \(z_n\) is not in \(E_{\lambda}\) if \(\lambda \neq \lambda_n\). Thus, we have \(\bigcup_{n=1}^{\infty} E_{\lambda_n} \subset \prod_{\lambda} E_{\lambda_n}\) and the set \(\{z_n\} \subset \bigcup_{n=1}^{\infty} E_{\lambda_n} < \prod_{\lambda} E_{\lambda}\) is linearly independent.

Let \(G\) be the linear span of \(\{z_n\}\). Note that with our conventions \(G\) is now a subspace of \(\prod_{\lambda} E_{\lambda}\), which in turn is a subspace of \(\prod_{\Lambda} E_{\lambda}\). Observe also that
the spaces \((F, \tau)\) and \((G, \tau_{\text{prod}})\) are linearly isomorphic, via the linear extension of the map \(T(x_n) = z_n\), \(n \in \mathbb{N}\). Note that \(T\) is also continuous.

A few more observations are needed in the scenario of our current discussion to obtain that \(E\) is SMFC. We list them as the following lemmas:

**Lemma 2.1.**
1. \(G \cap E_{\lambda_n} = \{tz_n : t \in \mathbb{R}\} = \{t[x_n]_\lambda : t \in \mathbb{R}\}\) for each \(n \in \mathbb{N}\).
2. \(G \cap E_\lambda = \{0\}\) for \(\lambda \neq \lambda_n\) for all \(n \in \mathbb{N}\).

**Proof.**
1. Since \(z_n \in G \cap E_{\lambda_n}\), \(\{tz_n : t \in \mathbb{R}\} < G \cap E_\lambda\). Let \(z \in G \cap E_{\lambda_n}\),
\[z = \sum_{k=1}^m t_k z_k.\] Then \(z = q_{\lambda_n}(z) = \sum_{k=1}^m t_k q_{\lambda_n}(z_k) = t_n z_n\).

2. Let \(z \in G \cap E_\lambda\), say, \(z = \sum_{k=1}^m t_k z_k\). Then \(z = q_\lambda(z) = \sum_{k=1}^m t_k q_\lambda(z_k) = 0\). \(\square\)

**Lemma 2.2.** The restricted linear map \(T^{-1} : (G \cap E_{\lambda_n}) \to F\) is continuous with respect to the product topology in both domain and codomain.

**Proof.** Recall that \(T^{-1}(z_n) = \gamma(x_n) = x_n\) and that \(\tau_{\text{prod}}|_F = \tau|_F\). Let \(q_\lambda\) be a canonical projection. Then
\[(q_\lambda \circ T^{-1})(z_n) = q_\lambda(x_n) = [x_n]_\lambda \quad \forall z_n \in E_{\lambda_n} \setminus \bigcup_{\lambda \neq \lambda_n} E_\lambda.\]

Let \(U = D_{\varepsilon\lambda_n}(0) \times (\prod_{\lambda \neq \lambda_n} E_\lambda)\) be a subbasic neighborhood of zero in the product so that \(U \cap F\) is a neighborhood of zero in \(F\). We claim that \(T^{-1}(D_{\varepsilon\lambda_n}(0)) \subset (U \cap F)\). Indeed, \(z = tz_n \in D_{\varepsilon\lambda_n}(0)\), that is,
\[
\hat{\rho}_{\lambda_n}(tz_n) = |t| \hat{\rho}_{\lambda_n}(z_n) = |t| \hat{\rho}_{\lambda_n}([x_n]_{\lambda_n}) < \varepsilon,
\]
and
\[
\hat{\rho}_{\lambda_n}(q_{\lambda_n}(T^{-1}tz_n)) = \hat{\rho}_{\lambda_n}(t[x_n]_{\lambda_n}) < \varepsilon.
\]
This implies that \(T^{-1}(z) \in U \cap F\).

Let \(U = D_{\varepsilon\mu}(0) \times (\prod_{\lambda \neq \lambda_n} E_\lambda)\), with \(\mu \neq \lambda_n\), a subbasic neighborhood of zero in the product space so that \(U \cap F\) is a neighborhood of zero in \(F\). We have \(T^{-1}(G \cap E_{\lambda_n}) \subset (U \cap F)\). Indeed, let \(z = tz_n \in G \cap E_{\lambda_n}\). Then
\[
q_{\mu}(T^{-1}(z)) = q_{\mu}(tz_n) = q_{\mu}([t x_n]_{\lambda_n}) = 0 \in U \cap F.
\]
\(\square\)

**Lemma 2.3.** The linear function \(T^{-1} : G = \bigoplus_{n=1}^\infty (G \cap E_{\lambda_n}) \to F\) is continuous with respect to the product topology in both, the domain and the codomain.
Proof. Let $U = D_{E_1}(0) \times \cdots \times D_{E_m}(0) \times (\prod_{\lambda \neq \lambda_i} E_\lambda)$ be a basic neighborhood of zero in the product so that $U \cap F$ is a neighborhood of zero in $F$.

Let $U_k = D_{E_k}(0) \times (\prod_{\lambda \neq \lambda_k} E_\lambda)$ and $U = \bigcap_{j=1}^m U_k$. By Lemma 2.2, $T^{-1}(D_{E_k}(0)) \subset U_k \cap F$ for each $k = 1, \ldots, m$. Thus, $T^{-1}(\bigcap_{j=1}^m U_k) \cap G \subset U \cap F$ where $(\bigcap_{j=1}^m U_k) \cap G)$ is a neighborhood of zero in $G$.

Corollary 2.4. For each $\tau$-bounded set, $B \subset F$, its image under the topological isomorphism $T$, $T(B) = C \subset G$ is a $\tau_{\text{prod}}$-bounded set and vice versa $B = T^{-1}(C)$.

Finally, observe that we got the sequence $(z_n)$ nicely copied in the product space as in our initial discussion so that there exists a sequence of scalars $(\alpha_{n_k})$ with infinitely many of them positive and such that the set $C = \{\alpha_{n_k} z_{n_k}\} \subset G$ is $\tau_{\text{prod}}$-bounded. Thus the set $\{\alpha_{n_k} x_{n_k}\} \subset F$ is $\tau$-bounded which gives that the space $(E, \tau)$ is SMFC. That SMFC spaces are docile was proved in [5]. Thus we have,

Theorem 2.5. A space $(E, \tau)$ is docile if and only if it is SMFC.

Note. The implication that docile spaces are SMFC was already suggested in [1], but the proof there contained a gap that now is filled with the contents of this section.

3 Sequentially Mackey first countable spaces that are not Mackey first countable.

The existence of SMFC spaces that are not MFC is the content of Corollary 3.2 below, which in turn is a consequence of the following main result:

Theorem 3.1. The weak dual of a non-normable metrizable space of algebraic dimension greater or equal to $\aleph_0$ is not MFC.

In [4, ch.5, p.393] it is proved that every non-normable metrizable space is MFC and that the strong dual of an infinite dimensional metrizable space, $(E', \beta(E', E))$, is not Mackey first countable. In fact, it is easy to see that any regular inductive limit (an inductive limit is regular if every bounded set is both, contained in, and bounded in some step subspace) cannot be MFC.

In this paper we will make use of a bounding cardinal, $b$, defined as $b = \min\{|F| : F \subset \omega^\omega$ such that for each $f \in \omega^\omega$ there exists $g \in F$ for which $g$ is not less than $f\}$, where $\omega$ represents the ordinal of the natural numbers, $\aleph_0$, and $|F|$ denotes the cardinality of $F$. It is known, from the ZFC setting, that $\aleph_1 \leq b \leq \aleph$ where $\aleph = \text{card}(\mathbb{R})$.

From [3, Theorem 7.4] we have that if the space $E$ is of algebraic dimension less than $b$, then its weak dual, $(E', \sigma(E', E))$ is docile. Thus, we have,
**Corollary 3.2.** If $E = (E, \tau)$ is metrizable and of dimension $\aleph_0$, then $(E', \sigma(E', E))$ is SMFC but not MFC.

We recall that for an absolutely convex subset $C$ of a space $E$, the linear subspace generated by $C$ is $E_C = \text{span}\{C\} = \bigcup_{n=1}^{\infty} nC$. Some simple observations of this construction are: If $a > 0$ then $E_{aC} = E_C$, and if $A \subset C$, then $E_A \leq E_C$.

The proof of Theorem 3.1 requires two preliminary lemmas:

**Lemma 3.3.** Suppose that $C_1 \subset C_2 \subset \cdots$ is a sequence of absolutely convex subsets of a space $E$, and that $a_1, a_2, \ldots$ is a sequence of positive numbers. If $C = \bigcup_{n=1}^{\infty} a_n C_n$ then $E_C = \bigcup_{n=1}^{\infty} E_{C_n}$.

**Proof.** Since $a_n C_n \subset E_{C_n}$, we have $C \subset \bigcup_{n=1}^{\infty} E_{C_n}$. Hence, $E_C \subset \bigcup_{n=1}^{\infty} E_{C_n}$. Now, let $x \in \bigcup_{n=1}^{\infty} E_{C_n}$. Then $x \in E_{C_n}$, for some $n \in \mathbb{N}$. It follows that for some $r > 0$, $x \in r C_n$, which implies that there exists $y \in C_n$ with $x = ry$. From this,

$$
x = ry = \frac{r}{a_n} (a_n y) \in \frac{r}{a_n} a_n C_n \subset \frac{r}{a_n} C \subset E_C.
$$

Recall that if $C$ is bounded, then we can define a norm on $E_C$ via the Minkowski sublinear functional of $C$:

$$
\rho_C(x) = \inf\{r > 0 : x \in r \cdot C\}.
$$

**Lemma 3.4.** Under the same hypotheses of the lemma 3.3, suppose additionally that each $C_n$ is bounded in $E$. Consider the increasing sequence of linear subspaces $E_1 \subset E_2 \subset \cdots$, where $E_n = (E_{C_n}, \rho_{C_n})$, $n \in \mathbb{N}$. Suppose that $F = \bigcup_{n=1}^{\infty} E_n$ and that $C = \bigcup_{n=1}^{\infty} a_n C_n$ is bounded. Then $F = E_C$ and the topology induced by the norm $\rho_C$ is weaker than the inductive limit topology formed by the sequence $(E_{C_n}, \rho_{C_n})$.

**Proof.** The equality $F = E_C$ was proved in the previous lemma. Now, observe that $a_n C_n \subset a_n C_{n+1} \subset a_n C$, and that from

$$
E_n = \bigcup_{m=1}^{\infty} mC_n.
$$

we have that for each $x \in E_n = \bigcup_{r>0} rC_n$, $x = rz$ for some $r > 0$ and $z \in C_n$. From this, we have $\rho_{C_n}(x) \leq r$, and

$$
x = \frac{r}{a_n} (a_n z) \in \frac{r}{a_n} a_n C_n \subset \frac{r}{a_n} C.
$$

This last line implies that $\rho_C(x) \leq \frac{r}{a_n}$, that is, $a_n \rho_C(x) \leq r$. By taking the infimum, we conclude that $a_n \rho_C(x) \leq \rho_{C_n}(x)$ implies that $\rho_C(x) \leq \frac{1}{a_n} \rho_{C_n}(x)$,
and we deduce that the topology induced by $\rho_C$ on the subspace $E_n$ is weaker than the normed topology of $\rho_{C_n}$, that is, $\rho_{C_n}$ is weaker than $\tau_{\text{ind}}$, which in turn gives us that $\rho_C$ is weaker than $\tau_{\text{ind}}$. \hfill $\Box$

The proof of Theorem 3.1 goes as follows:

**Proof of Theorem 3.1.** Let $V_1 \supset V_2 \supset \cdots$ be a decreasing countable base of zero neighborhoods of the space $E$. We suppose that these neighborhoods are all absolutely convex and $\tau$-closed. For each $n \in \mathbb{N}$, let $B_n = V_n^\circ$, the polar of the neighborhood $V_n$. By the Alaoglu-Bourbaki theorem, each $B_n$ is $\sigma(E',E)$-compact, hence, by the special case of Banach-Mackey theorem (see, [4, 5(1), p.262]), each $B_n$ is both bounded and complete with respect to the strong topology $\beta(E',E)$.

In [4, (8), p.394] it is shown that when $E$ is a non-normable metrizable space, $(B_n)$ forms a fundamental sequence of bounded sets in $\beta(E',E)$ such that for each $n \in \mathbb{N}$, $B_n$ does not absorb $B_{n+1}$, and the linear subspace generated by each $B_n$, $(E'_n)$ is a proper subspace of the subspace generated by $B_{n+1}$.

Note that $\bigcap_{n=1}^{\infty} V_n = \{0\}$ in $E$ implies that $E' = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} E'_n$.

From the previous statements we conclude that the sequence $(B_n)$ exhibits $(E', \beta(E',E))$ as a space that is neither MFC nor SMFC. Also, $(E', \sigma(E',E))$ is not MFC. Indeed, suppose there exists a sequence $(a_n)$ of positive numbers such that $\bigcup_{n=1}^{\infty} a_n B_n$ is $\sigma(E',E)$-bounded. Consider

$$B = \text{absconv} \left( \bigcup_{n=1}^{\infty} a_n B_n \right)^{\sigma(E',E)},$$

the absolutely convex $\sigma(E',E)$-hull of $\bigcup_{n=1}^{\infty} a_n B_n$. We will show that the assumption that $B$ is $\sigma(E',E)$-bounded leads to the conclusion that $E'$ is finite dimensional.

By Lemma 3.3, $E' = \bigcup_{n=1}^{\infty} E'_n = E'_B$, which means that $B$ is absorbing in $E'$. Lemma 3.4 implies that the topology induced by the norm $\rho_B$ in $E'$ is weaker than $\tau_{\text{ind}}$. Moreover, because $B$ is a $\rho_B$-zero neighborhood, there exists a $\tau_{\text{ind}}$-zero neighborhood $U$ such that $U \subset B$. Thus, $B$ is a $\tau_{\text{ind}}$-bornivore, in particular, $B$ is absorbing in $E'$.

On the other hand, the $\sigma(E',E)$-completeness of each $B_n$ implies that each $(E'_n, \rho_{B_n})$ is a Banach space, giving us that the inductive limit of these spaces is barrelled (see [2, 2, p.214]). Thus, as $B \subset E'$ is an absolutely convex, absorbing, $\sigma(E',E)$-closed (hence, $\tau_{\text{ind}}$-closed) set, i.e., a barrel, $B$ is a zero neighborhood with respect to $\tau_{\text{ind}}$. We conclude that $\beta(E',E) = \tau_{\text{ind}}$ in $E'$. Meanwhile, $(E'' = (E'[\beta'])'$ is metrizable (see [4, (5), p.301]) and each $\sigma(E'',E)$-bounded subset is $\beta'$-equicontinuous (see [2, 3.6.2, p.212]). Because $U$ is absorbing in $E'$, its polar in $(E', \tau_{\text{ind}})' = E''$ is $\sigma(E'',E)$-compact, which
On docile spaces and Mackey first countable spaces

385

means \( U^\circ \) is \( \beta(E'',E) \)-bounded. From \( U \subset B \), \( B^\circ \subset U^\circ \subset E'' \), implying that \( B^\circ \) is \( \beta(E'',E) \)-bounded, and \( \sigma(E'',E) \)-bounded as well. Because \( B \) is \( \sigma(E',E) \)-bounded, \( B^\circ \subset E'' \) is absorbing, that is, \( E''_{B^\circ} = E'' \). Furthermore, \( B^\circ \subset E'' \) is \( \beta' \)-equicontinuous and thus \( C = E \cap B^\circ \subset E \) is \( \beta' \)-equicontinuous, hence, \( \tau \)-precompact. The equicontinuity of \( C \) gives us that \( C \) is a \( \tau \)-zero neighborhood, and therefore, \( E \) is finite dimensional. This contradiction completes the proof.

Note that Corollary 3.2 follows from Theorem 3.1 above, and from the proof that such a space is docile (see [3, Thm. 7.4]) since \( \aleph_0 < b \).

4 A characterization of Mackey first countable spaces.

Let \( B_1 \subset B_2 \subset \cdots \) be an increasing sequence of absolutely convex closed bounded sets in a space \( E \). Set \( (F = \bigcup_{n=1}^\infty E_{B_n}, \tau_{\text{ind}}) \) as the inductive limit of the normed linear subspaces generated by this sequence. By [2, 7.3.4, p.222], the space \( (F, \tau_{\text{ind}}) \) is bornological. In the main result of this section we will prove that \( E \) is MFC if and only if \( (F, \tau_{\text{ind}}) \) is normed.

Proposition 4.1. Suppose \((B_n)\) is any sequence of closed, convex subsets of \( E \), such that \( B_1 \) is balanced. Define, for each \( n \in \mathbb{N} \), \( B_n^\# = \bigcup_{k=1}^n B_k \). If \((a_n)\) is a sequence of scalars such that \( 0 < a_n < 1 \) for each \( n \in \mathbb{N} \), then \( \bigcup_{n=1}^\infty a_n B_n \) is bounded if and only if \( \bigcup_{n=1}^\infty a_n B_n^\# \) is bounded.

Proof. The necessity is clear since \( B_n \subset B_n^\# \). For the other implication, without loss of generality, we may suppose that \( a_1 > a_2 > \cdots \). Let \( A \subset E \) be \( \tau \)-bounded such that \( \bigcup_{n=1}^\infty a_n B_n \subset A \). Observe that \( a_1 B_1 = a_1 B_1^\# \) and that

\[
a_2 B_2^\# = a_2 (B_1 \cup B_2) = a_2 (B_1 \cup a_2 B_2) \subset a_1 B_1 \cup a_2 B_2 \subset A.
\]

Thus,

\[
a_n B_n^\# = a_n \left( \bigcup_{k=1}^n B_k \right) = \bigcup_{k=1}^n a_n B_k \subset \bigcup_{k=1}^n a_k B_k \subset A,
\]

which shows that \( \bigcup_{k=1}^\infty a_k B_k^\# \subset A \).

Corollary 4.2. Suppose \((B_n)\) is any sequence of closed, convex subsets of \( E \), with \( B_1 \) balanced. Define \( B_n^\# \) by:

\[
B_n^\# = \overline{\text{absconv}} \left\{ \bigcup_{k=1}^n B_k \right\}^\tau.
\]

If \((a_n)\) is such that \( 0 < a_n < 1 \) for each \( n \in \mathbb{N} \) then \( \bigcup_{n=1}^\infty a_n B_n \) is bounded if and only if \( \bigcup_{n=1}^\infty a_n B_n^\# \) is bounded.
Proof. If $\bigcup_{n=1}^{\infty} a_nB_n^\# \equiv B_n^{\#}$ is bounded then we obtain as before that $B_n \subset B_n^{\#}$. On the other hand, suppose that $\bigcup_{n=1}^{\infty} a_nB_n$ is bounded. Consider $A \subset E$, an absolutely convex closed, bounded set, such that $\bigcup_{n=1}^{\infty} a_nB_n \subset A$. By proposition 4.1 we have

$$\bigcup_{n=1}^{\infty} a_n(B_1 \cup \cdots \cup B_n) \subset A.$$ 

Observe that $\text{absconv}\{a_n(B_1 \cup \cdots \cup B_n)\}^{\tau} = a_nB_n^{\#}$. It follows that $a_nB_n^{\#} \subset A$ for each $n \in \mathbb{N}$. Hence,

$$\bigcup_{n=1}^{\infty} a_nB_n^{\#} \subset A.$$

\[\square\]

Corollary 4.3.

$$\sum_{n \in \mathbb{N}} E_{B_n} = \bigcup_{n=1}^{\infty} E_{B_n}^{\#} \leq E.$$ 

Proof. For each $n \in \mathbb{N}$ we have $E_{B_n} \leq E_{B_n}^{\#}$, which gives

$$\sum_{n \in \mathbb{N}} E_{B_n} \leq \bigcup_{n=1}^{\infty} E_{B_n}^{\#}.$$ 

Let $x \in \bigcup_{n=1}^{\infty} E_{B_n}^{\#}$. Then $x \in E_{B_n}^{\#}$ for some $n \in \mathbb{N}$. By [4, (1), p. 173], $x = \sum_{k=1}^{m} \alpha_k x_k$ with $\alpha_k \geq 0$, $\sum_{k=1}^{m} \alpha_k = 1$ and $x_k \in \bigcup_{k=1}^{m} E_{B_k}$ for each $k = 1, 2, \ldots, m$. We obtain $x \in \sum_{n \in \mathbb{N}} E_{B_n}$. 

Theorem 4.4. The space $(E, \tau)$ is MFC if and only if for each increasing sequence $(B_n)$ of bounded, absolutely convex closed sets in $E$, the inductive limit, $\text{ind}_{n \in \mathbb{N}}(E_{B_n}^{\#}, \rho_{B_n}^{\#})$, is normable.

Proof. Let $F = \bigcup_{n=1}^{\infty} E_{B_n}^{\#} \subset E$, and let $\tau_{\text{ind}}$ denote the corresponding inductive limit topology on $F$. Assume that for each increasing collection $(B_n)$ of bounded, absolutely convex closed sets in $E$, $\text{ind}_{n \in \mathbb{N}}(E_{B_n}^{\#}, \rho_{B_n}^{\#})$, is normable. Let $(B_n)$ be a sequence of bounded subsets of $E$. Using $\rho$ to denote the normed topology on $F$, we have $\tau_{\text{ind}} = \rho$. If $D \subset F$ is the $\rho$-closed unit ball, the for all $n \in \mathbb{N}$ there exists $a_n \subset 0$ such that $a_nB_n^{\#} \subset D \cap E_{B_n}^{\#} \subset D$. From this, $\bigcup_{n=1}^{\infty} a_nB_n^{\#} \subset D$. Moreover, $B_n \subset B_n^{\#}$ implies that $\bigcup_{n=1}^{\infty} a_nB_n \subset D$. Meanwhile, $D$ is $\rho$-bounded, with $\tau \leq \tau_{\text{ind}} = \rho$, and this tells us that $D$ is $\tau$-bounded as well.
Conversely, if \((E, \tau)\) is MFC, then we will prove that there exists a norm \(\rho\) on \(F\) such that \(\tau_{\text{ind}} = \rho\). To this end, let \(B_1 \subset B_2 \subset \cdots\) be an increasing sequence of closed, absolutely convex bounded subsets of \(E\). Then there exists a sequence \((a_n)\) in \((0, 1]\), with \(a_1 > a_2 > \cdots > 0\) such that \(\bigcup_{n=1}^{\infty} a_n B_n\) is bounded in \(E\). Let

\[
B = \text{absconv} \left\{ \bigcup_{n=1}^{\infty} a_n B_n \right\}
\]

We claim that \(\rho_B = \tau_{\text{ind}}\). Indeed, we have that \(B\) is absolutely convex, closed and bounded. By Lemma 3.4, \(F = E_B\). On the other hand, by the definition of the inductive limit topology, \(\rho_B \leq \tau_{\text{ind}}\). Hence, \((F, \rho_B)\) and \((F, \tau_{\text{ind}})\) are both bornological. We conclude via [2, 3.7.1 (a), p.220], that \(\rho_B = \tau_{\text{ind}}\).

Note that the previous result applies to any countable collection of absolutely convex closed and bounded subsets. Indeed, consider the family \(\varphi\) of finite subsets of \(\mathbb{N}\). The family \(\varphi\) is ordered by inclusion. Moreover, given \(S \in \varphi\), \(C_S = \sum_{k \in S} E_{B_k}\) satisfies: If \(S \subset S'\), then \(C_S \leq C_{S'}\). Therefore, we can speak of the inductive limits of such linear subspaces of the form \(\{C_S\}_{S \in \varphi}\): \(G = \text{ind} C_S\), with the respective topology \(t_{\text{ind}}\). By Corollary 4.3, \(F = G\), and because \(C_S\) is a linear subspace of some \(E_{B_n}\), we have: \(\tau_{\text{ind}} \leq t_{\text{ind}}\).

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