New Integrals with Respect to a Fuzzy-Valued Additive Measure

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Abstract

In this paper, the concept of fuzzy-valued integrals of gradual number-valued measurable functions with respect to a new fuzzy-valued additive measure is introduced. Some of its properties are investigated and dominated convergence theorem is obtained.

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1 Introduction

The topic of fuzzy-valued measures and integrals has received much attention because of their usefulness in several applied fields like mathematical economics and optimal control. Significant contributions in this area were made by Stojaković [5], Xue, Ha and Wu [6] and Zhou [10], etc. Recently, Zhou [9] introduced a new fuzzy-valued additive measure by virtue of gradual Hausdorff metric. This paper is a continuity of [9]. In the present paper, we introduce fuzzy-valued integrals of gradual number-valued measurable functions with respect to the fuzzy-valued additive measure in [9] and investigate Some of its properties.
The organization of the paper is as follows. In Section 2, we state some basic results about gradual numbers, fuzzy numbers, gradual number-valued measures, gradual number-valued integrals and fuzzy-valued measures. In Section 3, we introduce the new fuzzy-valued integral. We investigate some of its properties and obtain the Dominated Convergence Theorem.

2 Preliminaries

In this section, we state some basic concepts about gradual numbers, fuzzy numbers, gradual number-valued measures, gradual number-valued integrals and fuzzy-valued measures.

Definition 2.1. [2] A gradual number $\tilde{r}$ is defined by an assignment function

$$A_{\tilde{r}} : (0, 1] \to \mathbb{R}.$$  

Naturally a nonnegative gradual number is defined by its assignment function from $(0, 1]$ to $[0, +\infty)$.

In the sequel, $\tilde{r}(\alpha)$ may be substituted for $A_{\tilde{r}}(\alpha)$. The set of all gradual numbers (resp. nonnegative gradual numbers) is denoted by $\mathbb{R}(I)$ (resp. $\mathbb{R}^+(I)$). A crisp element $b \in \mathbb{R}$ has its own assignment function $\tilde{b} : (0, 1] \to \mathbb{R}$ defined by $\tilde{b}(\alpha) = b$ for each $\alpha \in (0, 1]$. We call such elements in $\mathbb{R}(I)$ constant gradual numbers. In particular, $\tilde{0}$ (resp. $\tilde{1}$) denotes constant gradual number defined by $\tilde{0}(\alpha) = 0$ (resp. $\tilde{1}(\alpha) = 1$) for all $\alpha \in (0, 1]$.

Definition 2.2. [2] Let $\tilde{r}, \tilde{s} \in \mathbb{R}(I)$. The arithmetic operations of $\tilde{r}$ and $\tilde{s}$ are defined as follows:

1. $(\tilde{r} + \tilde{s})(\alpha) = \tilde{r}(\alpha) + \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
2. $(\tilde{r} - \tilde{s})(\alpha) = \tilde{r}(\alpha) - \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
3. $(\tilde{r} \cdot \tilde{s})(\alpha) = \tilde{r}(\alpha) \cdot \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
4. $\left(\frac{\tilde{r}}{\tilde{s}}\right)(\alpha) = \frac{\tilde{r}(\alpha)}{\tilde{s}(\alpha)}, \text{ if } \tilde{s}(\alpha) \neq 0, \forall \alpha \in (0, 1]$.

In the following, we describe some basic results for fuzzy numbers. A fuzzy number is a normal, convex, upper semicontinuous and compactly supported fuzzy set on $\mathbb{R}$. In the sequel, let $\mathcal{F}_c(\mathbb{R})$ denote the family of all fuzzy numbers. According to Fortin and Dubois and Fargier [2], a fuzzy number $\tilde{A}$ can be viewed as a particular gradual interval $\tilde{A} = [\tilde{a}, \tilde{a}^+]$, where $\tilde{a}$ and $\tilde{a}^+$ are defined by

$$\tilde{a}^-(\alpha) = \inf\{x : \tilde{A}(x) \geq \alpha\} \text{ and } \tilde{a}^+(\alpha) = \sup\{x : \tilde{A}(x) \geq \alpha\}$$

for each $\alpha \in (0, 1]$, respectively. A crisp interval $A = [a, a^+]$ can be regarded as a degenerate fuzzy number bounded by two constant gradual numbers and
a gradual number \( \tilde{r} \) as a degenerate fuzzy number \( \{ \tilde{r} \} \). We call that \( \tilde{A} \) is a nonnegative fuzzy number if \( \tilde{a}^- (\alpha) \geq 0 \) for each \( \alpha \in (0, 1] \). Let \( \mathcal{F}^*_c(\mathbb{R}) \) denote the set of all nonnegative fuzzy numbers.

Note that the boundaries of fuzzy numbers are real numbers, the boundaries of fuzzy numbers are gradual numbers. Thus, in the same way as defining crisp interval, we can define relation, sum and scalar multiplication on the space of fuzzy numbers as follows: Let \( \tilde{A} = [\tilde{a}^-, \tilde{a}^+] \) and \( \tilde{B} = [\tilde{b}^-, \tilde{b}^+] \) be in \( \mathcal{F}^*_c(\mathbb{R}) \) and \( \gamma \in \mathbb{R} \). Define

1. \( \tilde{A} = \tilde{B} \) if and only if \( \tilde{a}^- = \tilde{b}^- \) and \( \tilde{a}^+ = \tilde{b}^+ \);
2. \( \tilde{A} \preceq \tilde{B} \) if and only if \( \tilde{a}^- \preceq \tilde{b}^- \) and \( \tilde{a}^+ \preceq \tilde{b}^+ \);
3. \( \tilde{A} \oplus \tilde{B} = [\tilde{a}^- + \tilde{b}^-, \tilde{a}^+ + \tilde{b}^+] \);
4. \( \gamma \tilde{A} = [\gamma \tilde{a}^-, \gamma \tilde{a}^+] \) if \( \gamma \geq 0 \) and \( \gamma \tilde{A} = [\gamma \tilde{a}^+, \gamma \tilde{a}^-] \) if \( \gamma < 0 \).

**Definition 2.3.** [7] Let \( (X, \mathcal{A}) \) be a measurable space. A mapping \( \tilde{m} : \mathcal{A} \rightarrow \mathcal{F}^*_c(\mathbb{R}) \) is called a gradual number-valued measure if it satisfies the following two conditions:

1. \( \tilde{m}(\emptyset) = \hat{0} \);
2. If \( A_1, A_2, \ldots \) are in \( \mathcal{A} \), with \( A_i \cap A_j = \emptyset \) for \( i \neq j \), then

\[
\tilde{m} \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \tilde{m}(A_i).
\]

The second condition is called countable additivity of the gradual number-valued measure \( \tilde{m} \). We say that \( (X, \mathcal{A}, \tilde{m}) \) is a gradual number-valued measure space.

**Definition 2.4.** [9] Let \( (X, \mathcal{A}) \) be a measurable space. A mapping \( \tilde{M} : \mathcal{A} \rightarrow \mathcal{F}^*_c(\mathbb{R}) \) is called a fuzzy-valued measure if it satisfies the following two conditions:

1. \( \tilde{M}(\emptyset) = \hat{0} \);
2. \( \tilde{M} \left( \bigcup_{n=1}^{\infty} A_n \right) = \bigoplus_{n=1}^{\infty} \tilde{M}(A_n) \) exists for any sequence \( \{A_n\}_{n \in \mathbb{N}} \) of disjoint measurable sets.

The second condition is called countable additivity of the fuzzy-valued measure \( \tilde{M} \). We say that \( (X, \mathcal{A}, \tilde{M}) \) is a fuzzy-valued measure space.

**Theorem 2.5.** [9] Let \( (X, \mathcal{A}) \) be a measurable space and \( \tilde{M} : \mathcal{A} \rightarrow \mathcal{F}^*_c(\mathbb{R}) \) a mapping. \( \tilde{M} \) is a fuzzy-valued measure if and only if \( \tilde{M}^-, \tilde{M}^+ : \mathcal{A} \rightarrow \mathbb{R}^+(I) \) defined by

\[
\tilde{M}^-(A) = \left( \tilde{M}(A) \right)^- \quad \text{and} \quad \tilde{M}^+(A) = \left( \tilde{M}(A) \right)^+
\]

are two gradual number-valued measures.

**Definition 2.6.** [10] Let \( (X, \mathcal{A}, \tilde{M}) \) be a gradual number-valued measure space, \( f : X \rightarrow \mathbb{R}(I) \) a gradual number-valued measurable function and \( A \in \mathcal{A} \)
The gradual number-valued integral \( \tilde{\int}_{A} f \, d\tilde{m} : (0,1] \to \mathbb{R}(I) \) of \( f \) with respect to \( \tilde{m} \) on \( A \) is defined as follows:

\[
\left( \tilde{\int}_{A} f \, d\tilde{m} \right)(\alpha) = \int_{A} f_{\alpha} \, d\tilde{m}_{\alpha}, \ \forall \alpha \in (0,1].
\]

For more details on gradual numbers, fuzzy numbers, gradual number-valued measures, gradual number-valued integrals and fuzzy-valued measures, we refer the reader to [1–4,7–10].

### 3 Main Results

Firstly, we introduce the new fuzzy-valued integral as follows:

**Definition 3.1.** Let \((X, \mathcal{A}, \tilde{M})\) be a fuzzy-valued measure space, \( f : X \to \mathbb{R}(I) \) a gradual number-valued measurable function and \( A \in \mathcal{A} \). We call that \( f \) is integrable with respect to \( \tilde{M} \) on \( A \) if there exists \( \tilde{\int}_{A} f \, d\tilde{M} \in \mathcal{F}_{c}(\mathbb{R}) \) such that

\[
\tilde{\int}_{A} f \, d\tilde{M} = \left[ \tilde{\int}_{A} f \, d\tilde{M}^{-}, \tilde{\int}_{A} f \, d\tilde{M}^{+} \right].
\]

**Theorem 3.2.** Let \( f : X \to \mathbb{R}(I) \) be a gradual number-valued measurable function on the fuzzy-valued measure space \((X, \mathcal{A}, \tilde{M})\) and \( A \in \mathcal{A} \). Then

\[
\tilde{\int}_{A} f \, d\tilde{M} = \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M},
\]

where \((\tilde{\chi}_{A} f)(x) = f(x)\) if \( x \in A \) and \((\tilde{\chi}_{A} f)(x) = 0\) if \( x \notin A \).

**Proof.** From Theorem 3.2 [10], we have,

\[
\tilde{\int}_{A} f \, d\tilde{M}^{-} = \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M}^{-}, \tilde{\int}_{A} f \, d\tilde{M}^{+} = \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M}^{+}.
\]

It follows that

\[
\tilde{\int}_{A} f \, d\tilde{M} = \left[ \tilde{\int}_{A} f \, d\tilde{M}^{-}, \tilde{\int}_{A} f \, d\tilde{M}^{+} \right] = \left[ \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M}^{-}, \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M}^{+} \right] = \tilde{\int}_{X} \tilde{\chi}_{A} f \, d\tilde{M}.
\]

This completes the proof.
Theorem 3.3. Let $f, g$ and $\{f_n\}_{n \in \mathbb{N}}$ be gradual number-valued measurable functions on the fuzzy-valued measure space $(X, \mathcal{A}, \tilde{M})$.

1. If $f$ is non-negative, then $\tilde{\int}_X f \, \text{d}\tilde{M}$ is non-negative;
2. $\int_X (f + g) \, \text{d}\tilde{M} = \tilde{\int}_X f \, \text{d}\tilde{M} \bigoplus \tilde{\int}_X g \, \text{d}\tilde{M}$;
3. if $f \preceq g$, then $\tilde{\int}_X f \, \text{d}\tilde{M} \preceq \tilde{\int}_X g \, \text{d}\tilde{M}$;
4. if $f = \tilde{0}$, then $\tilde{\int}_X f \, \text{d}\tilde{M} = \tilde{0}$;
5. if $\tilde{M}(A) = \tilde{0}$, then $\tilde{\int}_A f \, \text{d}\tilde{M} = \tilde{0}$;
6. if $A, B \in \mathcal{A}$, $A \subseteq B$ and $f$ is non-negative, then

$$\tilde{\int}_A f \, \text{d}\tilde{M} \preceq \tilde{\int}_B f \, \text{d}\tilde{M};$$

7. if $A, B \in \mathcal{A}$, $A \cup B = C$ and $A \cap B = \emptyset$, then

$$\tilde{\int}_C f \, \text{d}\tilde{M} = \tilde{\int}_A f \, \text{d}\tilde{M} \bigoplus \tilde{\int}_B f \, \text{d}\tilde{M}.$$

Proof. (1) Since $f$ is non-negative, It follows from Theorem 3.3 [10] that

$$\tilde{\int}_A f \, \text{d}\tilde{M} \succeq 0.$$

This implies that $\tilde{\int}_X f \, \text{d}\tilde{M}$ is nonnegative.

(2) It follows from Theorem 3.3 [10] that

$$\tilde{\int}_X (f + g) \, \text{d}\tilde{M} = [\tilde{\int}_X f + \tilde{\int}_X g \, \text{d}\tilde{M}] \bigoplus [\tilde{\int}_X f + \tilde{\int}_X g \, \text{d}\tilde{M}]$$

(3) If $f \preceq g$, then $\tilde{\int}_X f \, \text{d}\tilde{M} \preceq \tilde{\int}_X g \, \text{d}\tilde{M}$. It follows that

$$\tilde{\int}_X f \, \text{d}\tilde{M} \preceq \tilde{\int}_X g \, \text{d}\tilde{M}.$$
(4) If $f = \tilde{0}$, then $\int_X f \, d\tilde{\mathcal{M}}^- = 0$ and $\int_X f \, d\tilde{\mathcal{M}}^+ = 0$. Hence $\int_X f \, d\tilde{\mathcal{M}} = 0$.

(5) If $\tilde{\mathcal{M}}(A) = \tilde{0}$, then $\tilde{\mathcal{M}}^-(A) = \tilde{0}$ and $\tilde{\mathcal{M}}^+(A) = \tilde{0}$. It follows from Theorem 3.3 [10] that

\[
\int_A f \, d\tilde{\mathcal{M}}^- = 0, \quad \int_A f \, d\tilde{\mathcal{M}}^+ = 0.
\]

This implies that $\int_A f \, d\tilde{\mathcal{M}} = 0$.

(6) If $A, B \in \mathcal{A}$, $A \subseteq B$ and $f$ is non-negative, then $\int_A f \, d\tilde{\mathcal{M}}^- \leq \int_B f \, d\tilde{\mathcal{M}}^-$ and $\int_A f \, d\tilde{\mathcal{M}}^+ \leq \int_B f \, d\tilde{\mathcal{M}}^+$. It follows that

\[
\int_A f \, d\tilde{\mathcal{M}} \leq \int_B f \, d\tilde{\mathcal{M}}.
\]

(7) If $A, B \in \mathcal{A}$, $A \cup B = C$ and $A \cap B = \emptyset$, then

\[
\int_C f \, d\tilde{\mathcal{M}}^- = \int_A f \, d\tilde{\mathcal{M}}^- + \int_B f \, d\tilde{\mathcal{M}}^- \\
\int_C f \, d\tilde{\mathcal{M}}^+ = \int_A f \, d\tilde{\mathcal{M}}^+ + \int_B f \, d\tilde{\mathcal{M}}^+.
\]

It follows that

\[
\int_C f \, d\tilde{\mathcal{M}} = \left[ \int_A f \, d\tilde{\mathcal{M}}^-, \int_B f \, d\tilde{\mathcal{M}}^-, \int_A f \, d\tilde{\mathcal{M}}^+, \int_B f \, d\tilde{\mathcal{M}}^+ \right] \\
= \int_A f \, d\tilde{\mathcal{M}} \oplus \int_B f \, d\tilde{\mathcal{M}}.
\]

This completes the proof. \qed

**Theorem 3.4.** Let $(X, \mathcal{A}, \tilde{\mathcal{M}})$ be a fuzzy-valued measure space. Then

1. $\int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(A), \forall A \in \mathcal{A}$;
2. $\int_A \gamma \, d\tilde{\mathcal{M}} = \gamma \tilde{\mathcal{M}}(A), \forall A \in \mathcal{A}$ and $\gamma > 0$.

**Proof.** (1) For Theorem 3.4 [10], we have $\int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}}^- = \tilde{\mathcal{M}}^-(A)$ and $\int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}}^+ = \tilde{\mathcal{M}}^+(A)$. It follows that

\[
\int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}} = \left[ \int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}}^-, \int_X \tilde{\chi}_A \, d\tilde{\mathcal{M}}^+ \right] \\
= \left[ \tilde{\mathcal{M}}^-(A), \tilde{\mathcal{M}}^+(A) \right] \\
= \tilde{\mathcal{M}}(A).
\]
(2) For any \( A \in \mathcal{A} \) and \( \gamma > 0 \), it follows from Theorem 3.4 [10] that
\[
\tilde{\int}_A \gamma \, d\tilde{\mathcal{M}}^- = \gamma \cdot \tilde{\mathcal{M}}^-(A) \quad \text{and} \quad \tilde{\int}_A \gamma \, d\tilde{\mathcal{M}}^+ = \gamma \cdot \tilde{\mathcal{M}}^+(A).
\]
Then we have
\[
\tilde{\int}_A \gamma \, d\tilde{\mathcal{M}} = \left[ \gamma \cdot \tilde{\mathcal{M}}^-(A), \gamma \cdot \tilde{\mathcal{M}}^+(A) \right] = \gamma \cdot \tilde{\mathcal{M}}(A).
\]
This completes the proof. \( \square \)

In the following, we generalize the Dominated Convergence Theorem to the new integral.

**Theorem 3.5.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of gradual number-valued measurable functions on the fuzzy-valued measure space \((X, \mathcal{A}, \tilde{\mathcal{M}})\), such that \( \lim_{n \to \infty} f_n = f \). If there exists a non-negative integrable gradual number-valued function \( g \) such that \( |f_n| \preceq g \) for any \( n \in \mathbb{N} \) and \( x \in X \), then
\[
\lim_{n \to \infty} \tilde{\int}_X f_n \, d\tilde{\mathcal{M}} = \tilde{\int}_X f \, d\tilde{\mathcal{M}}.
\]

**Proof.** By Theorem 3.7 [10], if there exists a non-negative integrable gradual number-valued function \( g \) such that \( |f_n| \preceq g \) for any \( n \in \mathbb{N} \) and \( x \in X \), then we have
\[
\lim_{n \to \infty} \tilde{\int}_X f_n \, d\tilde{\mathcal{M}}^- = \tilde{\int}_X f \, d\tilde{\mathcal{M}}^-,
\]
\[
\lim_{n \to \infty} \tilde{\int}_X f_n \, d\tilde{\mathcal{M}}^+ = \tilde{\int}_X f \, d\tilde{\mathcal{M}}^+,
\]
i.e., \( \tilde{\int}_X f_n \, d\tilde{\mathcal{M}}^- \) converges to \( \tilde{\int}_X f \, d\tilde{\mathcal{M}}^- \) and \( \tilde{\int}_X f_n \, d\tilde{\mathcal{M}}^+ \) converges to \( \tilde{\int}_X f \, d\tilde{\mathcal{M}}^+ \) simultaneously. It follows from Theorem 3.6 [8] that
\[
\lim_{n \to \infty} \tilde{\int}_X f_n \, d\tilde{\mathcal{M}} = \tilde{\int}_X f \, d\tilde{\mathcal{M}}.
\]
This completes the proof. \( \square \)

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