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Parallel Properties of Poles of Positive Functions and those of Discrete Reactance Functions

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Abstract

In the context of stability theory, it has often been noticed in the literature that parallel properties hold for specific parameters of the Routh-Hurwitz stability type and their counterpart parameters in the Schur-Cohn stability type. Such parallelism incited researchers to seek a common framework for these two stability structures. The present work is a contribution in this direction. We establish properties of the poles of positive functions which are Routh-Hurwitz parameters. Similar properties are established on poles of complex discrete reactance functions which are the counterparts of positive functions in the Schur-Cohn stability type.

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1. Introduction

The problem of root distribution of a polynomial with respect to a given curve has been long treated. The most commonly used curves are the imaginary axis and the unit circle due to their importance in a diversity of computer science and engineering applications: control theory, spectral analysis, numerical computations, digital signal processing and systems theory. For some references in this respect see [1 and 3]. The Routh-Hurwitz tests for the imaginary axis settle the stability of a continuous-time system of differential equations by establishing criteria for all eigenvalues of the system to lie in left-half plane. The Schur-Cohn tests for the unit circle settle the stability of a discrete-time system of difference equations by establishing criteria for all eigenvalues to lie inside the unit circle. A variety of methods have been used to derive these root-distribution tests: Sturm chains, index theory, Rouche's theorem, Lyaponov equations, and complex analysis techniques, see for example [7, 10 and 14]. The problem of root distribution of a polynomial in some sub-regions of the complex plane like sectors, ellipses, parabolas has also been investigated [3 and 13].

Many attempts were made to put stability criteria of different natures such as Routh-Hurwitz and Shur-Cohn on common ground, by invoking the intimate relationships that might prevail between these various stability structures [2, 9 and 13]. This is an interesting research topic due to the fact that results on the Shur-Cohn stability type are harder to find and to prove than their counterparts in the Routh-Hurwitz stability case. There are few results on this subject, see for example [4, 8 and 11], but we are still far from constructing a unifying frame for both types of stability. The search for this unified approach has intrigued researchers for a long time. The present work is a contribution in this vein.
Positive functions have played a critical role in deriving stability criteria in the Routh-Hurwitz case, see for example [12 and 15]. In the Schur-Cohn stability analysis, positive functions are mirrored by complex reactance functions which are the discrete-time counterparts of positive functions.

In the present work, we establish two necessary and sufficient conditions for a polynomial to be Hurwitz, one in terms of a complex operation called paraconjugation, and the other in terms of positive functions. It should be noted that positive functions play a key role in the derivation of stability criteria, [6, 12 and 15]. Both Hurwitz polynomials and positive functions have applications in the stability of mechanical and electrical networks, see [1 and 5]. We also establish properties on the location of poles of positive functions with respect to the imaginary axis. It is curiously interesting to discover that parallel properties are satisfied by the poles of a complex reactance function with respect to unit circle.
In section 2, we introduce Hurwitz polynomials, positive functions, Schur polynomials and discrete reactance functions. In section 3, we established two

characterizations of Hurwitz polynomials. In section 4, we offer a geometric description of the poles of a positive function. In section 5, parallel results are established on the poles of a complex discrete reactance function. We end up with some concluding remarks.

2. Basic Definitions and Notations

Definition 1
A non-constant polynomial is a Hurwitz polynomial if all its roots have negative real parts.

Definition 2
The paraconjugate of a rational function $f(s)$ is defined by $f^*(s) = \overline{f(-\bar{s})}$, where $\bar{s}$ denotes the complex conjugate of *s*.
For example, if

$$f(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-2}s^2 + a_{n-1}s + a_n ,$$

then

$$f^*(s) = (-1)^n s^n + (-1)^{n-1}\bar{a}_1 s^{n-1} + \cdots + \bar{a}_{n-2}s^2 - \bar{a}_{n-1}s + \bar{a}_n .$$

Also, if f is written in the factored form

$$f(s) = (s - s_1)(s - s_2)\cdots(s - s_n)$$

then its paraconjugate can be written as

$$f^*(s) = (-1)^n (s + \bar{s}_1)(s + \bar{s}_2)\cdots(s + \bar{s}_n)$$

Definition 3
A rational function *g* is said to be positive if $\operatorname{Re} g(s) > 0$ whenever $\operatorname{Re} s > 0$.

Definition 4

A non-constant polynomial is a Schur polynomial if all its roots lie inside the unit disc.

Definition 5
A rational function *K*(*z*) having complex coefficients is called a complex discrete reactance function if $\operatorname{Re}[k(z)] > 0$ whenever $|z| > 1$.

3. Two Characterizations of Hurwitz Polynomials

In this section, we established two characterizations of Hurwitz polynomials, one in terms of paraconjugation, and the other in terms of positive functions.
The following two lemmas are needed.

Lemma 1

Given a non-constant polynomial f and its paraconjugate f^*, then the roots of f^* are mirror reflection of the roots of f with respect to the imaginary axis. In particular, any imaginary root of f or f^* is common to both.

Proof
From the factored forms of f and f^*, it is obvious that:
s_j is a root of f if and only if $-\bar{s}_j$ is a root of f^*.
But if $s_j = x + iy$ then $-\bar{s}_j = -x + iy$.

Therefore, s_j and $-\bar{s}_j$ are symmetric with respect to the imaginary axis.
Now assume that $s = iy$ is a pure imaginary root of f, then since $-\bar{s} = iy$, it follows that s is also a root of f^*, and that proves the second part of the lemma.

Lemma 2
Suppose $\operatorname{Re} s_j < 0$, then $|s - s_j| > |s + \bar{s}_j|$ whenever $\operatorname{Re} s > 0$.

Proof
Assuming $\operatorname{Re} s_j < 0$, and $\operatorname{Re} s > 0$, then $\operatorname{Re} s_j \cdot \operatorname{Re} s < 0$.
It therefore follows that $(s_j + \bar{s}_j)(s + \bar{s}) < 0$. By expanding, we get
$-s\bar{s}_j - s_j\bar{s} > ss_j + \bar{s}\,\bar{s}_j$.
Add to both sides of this inequality the expression $s\,\bar{s} + s_j\bar{s}_j$ to get
$s\,\bar{s} - s\,\bar{s}_j - s_j\bar{s} + s_j\bar{s}_j > s\,\bar{s} + ss_j + \bar{s}\,\bar{s}_j + s_j\bar{s}_j$.

The last inequality can be written in the form:
$(s - s_j)(\bar{s} - \bar{s}_j) > (s + \bar{s}_j)(\bar{s} + s_j)$, or equivalently
$|s - s_j|^2 > |s + \bar{s}_j|^2$ which implies $|s - s_j| > |s + \bar{s}_j|$.

Theorem 1.

Assume f is a non-constant polynomial, f and f^* have no common roots.

let $g(s) = \frac{f^*(s)}{f(s)}$. Then, f is a Hurwitz polynomial if and only if g maps the right-half plane into the unit circle.

Proof.

Suppose f is a Hurwitz polynomial. Write f and its paraconjugate f^* in their factored forms,

$f(s) = (s - s_1)(s - s_2)\cdots(s - s_n)$ and $f^*(s) = (-1)^n(s + \bar{s}_1)(s + \bar{s}_2)\cdots(s + \bar{s}_n)$

Since $\operatorname{Re} s_j < 0$, for all $1 \le j \le n$, then by Lemma 2

$\left|s - s_j\right| > \left|s + \bar{s}_j\right|$ for all $1 \le j \le n$, whenever $\operatorname{Re} s > 0$.

We therefore obtain

$$\left|f(s)\right| > \left|f^*(s)\right| \text{ whenever } \operatorname{Re} s > 0.$$

Equivalently, $\left|g(s)\right| < 1$.

We proved that, $\left|g(s)\right| < 1$ whenever $\operatorname{Re} s > 0$.

In other words, g maps the right-half plane into the unit circle.

Conversely suppose that, $\left|g(s)\right| < 1$ whenever $\operatorname{Re} s > 0$, then $\left|f(s)\right| > \left|f^*(s)\right|$ for $\operatorname{Re} s > 0$.

It follows that f cannot have any roots for Re $s > 0$, and therefore, the only possible roots of f when Re $s \ge 0$ are pure imaginary.

But by Lemma 1, any pure imaginary root of f is also a root of f^*, and that contradicts the assumption that f and f^* have no common roots.

The only option is for f to have only roots with negative real parts, and therefore f is a Hurwitz polynomial.

Theorem 2.

Let f be a non-constant polynomial, f and f^* have no common roots. Define,

$$h(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)}.$$

Then f is a Hurwitz polynomial if and only if h is a positive function.

Proof.

$$h = \frac{f - f^*}{f + f^*} \Leftrightarrow h = \frac{1-g}{1+g} \text{ where } g = \frac{f^*}{f}.$$

Obviously $h = \dfrac{1-g}{1+g} \Leftrightarrow g = \dfrac{1-h}{1+h}$.

By direct computations we can prove that:

$$h + \bar{h} = \frac{2(1-g\bar{g})}{|g+1|^2} \text{ and } 1 - g\bar{g} = \frac{2(h+\bar{h})}{|h+1|^2}.$$

Now suppose that f is a Hurwitz polynomial, then by Theorem 1, this is equivalent to

$|g(s)| < 1$ whenever $\operatorname{Re} s > 0$.

$$\text{But } |g| < 1 \Leftrightarrow g\bar{g} < 1 \Leftrightarrow 1 - g\bar{g} > 0 \Leftrightarrow h + \bar{h} = \frac{2(1-g\bar{g})}{|g+1|^2} > 0 \Leftrightarrow \operatorname{Re} h > 0.$$

Therefore, f is a Hurwitz polynomial if and only if Re $h(s) > 0$ whenever Re $s > 0$. In other words, f is a Hurwitz polynomial if and only if h is a positive function.

4. A Routh-Hurwitz Property of Positive Functions

The following theorem offers an interesting property of the poles of a positive function.

Theorem 3.

A positive function can have no poles in the right half-plane, and any poles on the imaginary axis are simple.

Proof.

Let h be a positive function and suppose a is a zero of $h(s)$ of multiplicity k. Then $h(s)$ can be written in the form

$$h(s) = (s-a)^k H(s),$$

where $H(a) \neq 0$ and k is an integer.

It is obvious that $\lim_{s \to a} \frac{H(s)}{H(a)} = 1$. Therefore, we can write

$$H(s) = \rho e^{i(\varepsilon + \theta_0)}$$

where $\theta_0 = \text{Arg} H(a)$, Arg denoting the principal value of the argument and such that $\rho \to |H(a)|$ as $\varepsilon \to 0$.

Now if $s - a = re^{i\theta}$, then

$$h(s) = r^k e^{ik\theta} \rho e^{i(\varepsilon + \theta_0)} = r^k \rho e^{i(\varepsilon + k\theta + \theta_0)}.$$

Here it is understood that $\varepsilon \to 0$ and $\rho \to |H(a)|$ as $r \to 0$.

From Definition 3, it is obvious that a positive function cannot have any zeros with positive real parts.

If Re $a > 0$ and $k \geq 2$, then the last equation above implies that for small values of ε,

Re $h(s)$ changes sign as θ crosses the interval $(-\pi/2, \pi/2)$.

It therefore follows that the imaginary zeros of a positive function are simple. In other words, $k = 1$.

In the same way, we show that $\theta_0 = 0$, so that $H(a) > 0$ for imaginary zero, a.

Since for any nonzero complex number $s = x + iy$ we have,

$\frac{1}{s} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$, it follows that Re s and Re $(1/s)$ have the same sign.

Therefore, $1/h$ is positive if h is positive. It follows that the denominator of h cannot have any zeros in the right half-plane and that any imaginary poles of h must be simple.

5. A Parallel Schur-Cohn Property

The next theorem establishes a property on complex reactance functions which mirrors the property established in Theorem 3.

Theorem 4.

A complex reactance function cannot have any poles outside the unit circle, and any imaginary poles must be simple.

Proof.

Let k be a complex reactance function and suppose a is a pole of $k(z)$ of multiplicity k. Then for z sufficiently close to a, the function $k(z)$ can be approximated by $r / (z - a)^k$, or

$$k(z) \approx \frac{r}{(z-a)^k} .$$

Let's write

$$z - a = \rho e^{i\theta} \text{ and } r = |r| e^{i\theta_0} ,$$

we therefore obtain,

$$k(z) = r \rho^{-k} e^{-ik\theta} = |r| \rho^{-k} e^{i(\theta_0 - k\theta)} ,$$

whose real part is

$$\mathrm{Re}[k(z)] = |r| \rho^{-k} \cos(\theta_0 - k\theta).$$

It is obvious that for any value of k, Re[$k(z)$] changes sign, which implies that a cannot be outside the unit circle.

Now assume that a lies on the unit circle, i.e. $|a| = 1$, then Re[$k(z)$] changes sign only once.

When ρ is chosen sufficiently small, the unit circle is approximated by its tangent at a. Therefore by choosing r in an appropriate way, we can see that Re[$k(z)$] can be made positive in the exterior of the unit circle if $k = 1$.

Conclusion

The unification of various stability criteria has been advocated by several eminent researchers in the field. Specifically, several attempts have been made to give common interpretations to the algorithms for testing the Routh-Hurwitz and the Schur-Cohn types of stability. The current work is a further thrust in this direction and it aims at gainig new insights into the nature of the different stability types.

References

[1] S. Barnett, *Some Modern Applications of Mathematics*, Ellis Horwood, London, 1995.

[2] B. Datta, On the Routh-Hurwitz-Fujiwara and the Schur-Cohn-Fujiwara theorems for the root-separation problem, *Linear Algebra and its Applications*, **22** (1978), 235-245. https://doi.org/10.1016/0024-3795(78)90074-5

[3] S. Gutman, Root clustering of a complex matrix in an algebraic region, *IEEE Trans. Automat. Cont.*, **24** (1979), 647-650. https://doi.org/10.1109/tac.1979.1102109

[4] G. Heinig, U. Jungnickel, On the Routh-Hurwitz and Schur-Cohn Problems for Matrix Polynomials and Generalized Bezoutians, *Math. Nachr*., **116** (1984), 185-196. https://doi.org/10.1002/mana.19841160114

[5] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991. https://doi.org/10.1017/cbo9780511840371

[6] R.M. Hovstad, A short proof of a continued fraction test for the stability of polynomials, *Proceedings of the Amer. Math. Soc*., **105** (1989), no. 1, 76-79. https://doi.org/10.1090/s0002-9939-1989-0973839-7

[7] A. Hurwitz, *On The Conditions Under Which an Equation Has Only Roots with Negative Real Parts,* Selected Papers on Mathematical Trends in Control Theory, Edited by R. Bellman and R. Kalaba, Dover, New York, 1964.

[8] M. G. Krein and M. A. Naimark, The Method of Symmetric and Hermitian Forms in the Separation of Roots of Algebraic Equations, *Linear and Multilinear Algebra*, **10** (1981), 265-308. https://doi.org/10.1080/03081088108817420

[9] M. Mansour, B.D.O. Anderson, New Relations Between the Schur-Cohn-Fugiwara and Nour Eldin Stability Criteria, *IEEE Trans. Automat. Cont*., **36** (1991), 1117-1120. https://doi.org/10.1109/9.83553

[10] J. Mawhin, Some remarks on the Ω-stability for families of polynomials, *Archivum Mathematicum*, **33** (1997), no. 1-2, 139-145.

[11] Z. Zahreddine, Explicit relationships between Routh-Hurwitz and Schur-Cohn types of stability, *Bulletin of the Irish Mathematical Society*, **29** (1992), 49-54.

[12] Z. Zahreddine, An extension of the Routh array for the asymptotic stability of a system of differential equations with complex coefficients, *Applicable Analysis*, **49** (1993), 61-72. https://doi.org/10.1080/00036819308840165

[13] Z. Zahreddine, On the Γ-stability of systems of differential equations in the Routh-Hurwitz and Schur-Cohn cases, *Bulletin of the Belgian Mathematical Society*, **3** (1996), 363-368.

[14] Z. Zahreddine, On some properties of Hurwitz polynomials with application to stability theory, *Soochow Journal of Mathematics*, **25** (1999), no. 1, 19-28.

[15] Z. Zahreddine, Matrix Measure and Application to Stability of matrices and interval dynamical systems, *International Journal of Mathematics and Mathematical Sciences*, **2003** (2003), no. 2, 75- 85.
https://doi.org/10.1155/s0161171203202295

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