Monophonic Domination in Special Graph Structures
and Related Properties

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Abstract

In this paper, we study the monophonic domination number of special graph structures like \( k \)-Dimensional Cube \( Q_k \), triangle-free Graphs, Tree, Middle Graphs and Edge Deleted Graphs. Certain general properties satisfied by the monophonic domination number are discussed. It also, realizes the existence of monophonic domination number of \( G \) in terms of monophonic diameter and girth. Finally, we study how the monophonic domination number is affected when one or more edges are deleted from a complete graph \( K_p \).

Mathematics Subject Classification: 05C69

Keywords: Monophonic Path, Monophonic Number, Domination Number, Monophonic Domination Number, Girth and Monophonic Diameter

1 Introduction

For basic graph theoretic notations and terminology which are not specifically defined here, we recommend the reader refer to Buckley and Harary [1,
5], G. Chatrand and P. Zhang [2]. Throughout this paper, we consider only finite connected simple graphs (with no loops or multiple edges). The concept of dominating set in graphs was introduced by O. Ore [11] and it has a commendable historical background in the game of chess and related game movements. A unique handling of fundamentals of domination in graphs is done by T.W.Haynes et al [9]. For any graph $G = (V, E)$ the vertex set is denoted by $V$ and the edge set is denoted by $E$. A graph $G$ with order $|V| = p$ and size $|E| = q$ is usually called $(p, q)$ graph.

A subset $D$ of $V$ is said to be a dominating set of $G$ if every vertex in $V$ is either an element of $D$ or is adjacent to an element of $D$, or closed neighborhood $N[D] = V$. The minimum of the cardinalities of the dominating sets of $G$ is called domination number of $G$, and is denoted by $\gamma(G)$. A subset $D$ of $V$ is a 2- dominating set of $G$ if every vertex of $V \setminus D$ has at least two neighbors in $D$. The minimum of the cardinalities of 2- dominating sets is called the 2- domination number, and is denoted by $\gamma_2(G)$. For any subset $X$ of $V$, then we call a subset $D$ of $V$ is an $X$- dominating set of $G$ if $X \subseteq N[D]$. The minimum of the cardinalities of $X$- dominating sets of $G$ is called $X$- domination number, and is denoted by $\gamma_X(G)$. For further references on domination parameters in a graph, see [9].

The theory of convexity related to a graph structure can be divided into two types, one is geodetic convexity and other one is monophonic convexity. Geodetic convexity is related to the shortest path in a system of paths, which is defined in a connected graph $G$, but in the monophonic convexity (or an induced path convexity) replaces the shortest path by a chordless path (or an induced path).

A chord of a path $P : [u_1u_2...u_n]$ in a graph $G$ is an edge $u_iu_j$ with $j \geq i + 2$. A $u-v$ path $P$ is said to be a monophonic path (or an induced path), if it is a chordless path. For any two vertices $u$ and $v$ in a graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is the length of the longest $u-v$ monophonic path in $G$. $e_m(v) = \max\{d_m(v, u) : u \in V\}$ is the monophonic eccentricity of a vertex $v$ in $G$. The monophonic radius of any graph $G$ is denoted by $r_m$ or $\text{rad}_m(G)$, and is defined by $r_m = \min\{e_m(v) : v \in V\}$ and the monophonic diameter of $G$ is denoted by $d_m$ or $\text{diam}_m(G)$, and is defined by $d_m = \max\{e_m(v) : v \in V\}$. A monophonic diametral path in a graph $G$ is a monophonic path, the length of which is equal to the diameter of the graph. So the monophonic diameter of the graph $G$ is the length of the longest monophonic diametral path. The monophonic radius and diameter can be easily computed and note that monophonic distance $d_m$ is different from usual distance $d$. The usual distance $d$ is a metric on $V$ but $d_m$ is not a metric on $V$.

The closed monophonic interval $J_G[u, v]$ is the set containing vertices $u, v$ and all vertices lying on some $u-v$ monophonic path, while for a subset $M$ of $V$, the set $J_G[M] = \bigcup_{u,v \in M}(J_G[u,v])$ known as monophonic
Monophonic domination in special graph structures

A subset $M$ of $V$ is said to be a monophonic set of $G$ if $J_G[M] = V$, or every vertex of $G$ is contained in a monophonic path of some pair of vertices of $M$. The minimum of the cardinalities of the monophonic sets is called the monophonic number of $G$. The symbol $m(G)$ denote the monophonic number of $G$. The concepts of monophonic number of a graph appeared in [1] and studied by I.M Palayo et al [ 8 ]. For further references on monophonic number of a graph, see [4,12,13] and for the geodetic sets and related parameters, see[3, 6, 7, 8].

The girth of a connected graph $G$ is the length of its shortest cycle in $G$ and the length of the longest cycle in $G$ is the circumference of $G$. If $G$ has no cycle, then girth and circumference are infinity. The cycle graph $C_3$ is called triangle and a graph $G$ is said to be a Triangle-free graph if it does not contain cycles of length 3. For any graph $G$ containing a cycle satisfies the inequality $c \leq 2d_m + 1$ where $c$ and $d_m$ are the girth and monophonic diameter of $G$ respectively.

The k-dimensional cube or k-cube or hypercube $Q_k$ is the simple graph with vertex set consists of k-tuples with entries in $\{0,1\}$ and edge set involves the pairs of k-tuples that differ in exactly one position. Thus $Q_k$ is a k-regular bipartite graph with order $|V(Q_k)| = 2^k$, size $|E(Q_k)| = k(2^k - 1)$ and $diam_m(Q_k) = k$. The Middle graph of $G = (V,E)$ is the graph $M(G)$ with vertex set $V \cup E$ and edge set $E'$, where $uv \in E'$ if and only if either $u$ is a vertex of $G$ and $v$ is an edge of $G$ containing $u$ , or $u$ and $v$ are edges in $G$ having a vertex in common.

Definition 1.1. A subset $M$ of $V$ is said to be a monophonic dominating set of a graph $G$ if $M$ is both a monophonic and a dominating set of $G$. The minimum of the cardinalities of the monophonic dominating sets in $G$ is called the monophonic domination number of $G$, and is denoted by $\gamma_m(G)$. A monophonic dominating set of $G$ of order $\gamma_m(G)$ is called $\gamma_m$- set of $G$. Since vertex set is always a monophonic dominating set of any graph $G$, monophonic domination number $\gamma_m(G)$ is well defined.

Example 1.2. For the graph $G$ given in the Figure 1, $M_1 = \{v_1, v_4, v_7\}$ and $M_2 = \{v_1, v_5, v_7\}$ are the 3-element monophonic dominating sets of $G$. Also $M_3 = \{v_2, v_3, v_6, v_8\}$, $M_4 = \{v_2, v_3, v_5, v_7\}$, $M_5 = \{v_2, v_3, v_4, v_7\}$, $M_6 = \{v_1, v_4, v_6, v_8\}$ and $M_7 = \{v_1, v_5, v_6, v_8\}$ are the 4-element monophonic dominating sets of $G$. Since no monophonic dominating set exists with 2- elements, $\gamma_m(G) = 3$. 

Figure 1: Graph $G$ with $\gamma_m(G) = 3$

For unexplained term and symbols see [2, 5]. we cite few results related to monophonic domination, which are to be used in the sequel.

2 Some Basic observations

Observation 2.1. For every monophonic dominating set $M$ in a graph $G$, then

1. $\text{Ext}(G) \subseteq M$, where $\text{Ext}(G)$ is the set of all extreme vertices of $G$.
2. $\text{Pend}(G) \subseteq M$, where $\text{Pend}(G)$ is the set of all pendant vertices of $G$.

Observation 2.2. In a connected graph $G$ with order $p$, if $M = \text{Ext}(G)$ is a monophonic dominating set of $G$, then $M$ is the unique monophonic dominating set of $G$, and $\gamma_m(G) = |M|$.

Observation 2.3. In a connected graph $G$ with order $p$ and at least one pendant vertex, then for every monophonic dominating set $M$ of $G$, $V - M$ is not a monophonic dominating set of $G$.

Observation 2.4. In a connected graph $G$ with a cut vertex $v$, then every monophonic dominating set contains at least one element from each component of $G - \{v\}$.

Next theorems due to J.John et al [10], which are used wherever required

Theorem 2.5 (10). In a connected graph $G$ with order $p \geq 2$, then $\gamma_m(G) = 2 \iff$ there exists a monophonic dominating set $M = \{u, v\}$ of $G$ such that the monophonic distance $d_m(u, v) \leq 3$

Theorem 2.6 (10). Let $G$ be a tree such that $N(v)$ contains a pendant vertex for every internal vertex $v \in V$. Then $\gamma_m(G) = |\text{Pend}(G)|$. 

Theorem 2.7 (10). If \( G \) is a non-complete connected graph of order \( p \) such that it has a minimum cut set, then \( \gamma_m(G) \leq p - k(G) \), where \( k(G) \) is the vertex connectivity of a graph \( G \).

Remark 2.8. The bound in the Theorem 2.7 is sharp and strict.

To prove the sharpness, consider the graph \( G = C_4 \), we have \( \gamma_m(G) = 2 \) and \( k(G) = 2 \). Thus \( \gamma_m(G) = p - k(G) = 2 \). Also, inequality in the Theorem 2.7 is strict, when we consider the graph \( G = P_9 \), it is clear that \( p = 9, k(G) = 2 \) and \( \gamma_m(G) = p - k(G) = 7 \). Thus \( \gamma_m(G) < p - k(G) \).

Corollary 2.9. In a connected non-complete graph \( G \) with order \( p \), if \( G \) has a cut vertex, then \( \gamma_m(G) \leq p - 1 \).

Proof. It follows from the Theorem 2.7.

Next theorem is an extended work of A. Hansberg et al [7].

Theorem 2.10. In a connected graph \( G \) with order \( p \geq 2 \), we have \( \gamma_m(G) = p - 1 \iff \) there exists a vertex \( v \in V \) such that \( v \) is adjacent to every other vertex of \( G \) and \( G - \{v\} \) is the union of at least two complete graphs.

Proof. Assume that \( \gamma_m(G) = p - 1 \), then there exists a monophonic dominating set \( M \) such that \( |M| = |V| - 1 \). Clearly \( V - M = \{v\} \) for any \( v \in V \). Let \( u \) and \( w \) be non adjacent vertices of \( v \). Also, suppose that component \( H \) of \( G - \{v\} \) contains \( u \) and \( w \). Let \([u = x_1, x_2, \ldots, x_t = w]\) be a chordless path in \( H \). Then, clearly \( t \geq 3 \), and there exists a monophonic dominating set \( M_1 = V - \{v, x_2\} \) such that \( |M_1| = p - 2 \), which is a contradiction to the assumption. Note that \( N(v) \cap V(H) \) induces a complete graph. If we assume \( G - \{v\} \) is connected (only one component), then we get an extreme vertex say \( v \), again we arrive at a contradiction. Hence \( G - \{v\} \) is the disjoint union of graphs \( H_1, H_2, \ldots, H_p \) for \( p \geq 2 \). Finally show that \( v \) is not adjacent to some vertex in \( H_i \) say in \( H_1 \). Let \( w, u \in V(H_1) \). Then there exists a \( u - v \) path \([u, w, v]\) such that \( uv \) is not an edge in \( G \). Connectivity of \( G \) shows that component \( H_2 \) contains a neighbor \( y \) of \( v \) such that \( d_m(u, y) = 3 \). If follows that there is monophonic dominating set \( M_2 = V - \{v, w\} \) such that \( |M_2| = p - 2 \), which is a contradiction. Hence \( G - \{v\} \) is the union of at least two complete graphs and \( v \) is adjacent to every other vertex of \( G \).

Conversely, there is a vertex \( v \in V \) such that \( d(v) = p - 1 \) and \( G - \{v\} \) is the union of (at least 2) complete graphs, then by definition 1, \( \gamma_m(G) = p - 1 \).

In the next section, we initiate to characterize the relation between the monophonic diameter and monophonic domination number of a graph \( G \).
3 Monophonic Domination in Special Graphs

Let $G$ be a $(p,q)$ connected graph with $|Ext(G)| = m$. Then by definition 1.1, $2 \leq \max\{\gamma(G), m(G)\} \leq \gamma_m(G) \leq p$ and by Theorem 1.1, $\gamma_m(G) \geq m$. This upper bound is sharp, when we consider the star graph $G = K_{1,p-1}$. It is clear that $\gamma_m(G) = p - 1$ and $m = p - 1$, that is $\gamma_m(G) = m$. Also since $\gamma(K_p) = Ext(K_p)$, $\gamma_m(K_p) = p$. For any $G$ with specific $\gamma_m(G) = p$, then we can identify that $m(G) = p$ and graph must be $G = K_p$. When we consider monophonic diameter $d_m \leq 3$ of $G$, no cut vertex of $G$ belongs to any minimum monophonic dominating set of $G$ and, if $\gamma(G) = 2$, then $\gamma_m(G) = m(G)$. But converse need not be true in the case of wheel graph $W_5$. Also if $d_m > 3$ and $\gamma(G) = 2$, then cut vertex of $G$ belongs to every monophonic dominating set of $G$ and $\gamma_m(G) > m(G)$. If $d_m = 1, 2$ or $3$, then every monophonic set is also a monophonic dominating set of $G$. But in the case of caterpillar converse need not be true. For any non complete graph of order $p \geq 4$ with $d_m \geq 3$, then monophonic domination number of $G$ is at most $p - 2$. If $M(G)$ is the middle graph of $G$ with order $p$. Then $\gamma_m(M(G)) = p$ since $V(G) = Ext(M(G))$.

**Theorem 3.1.** In a complete bipartite graph $G = K_{p,q}$

\[
\gamma_m(G) = \begin{cases} 
  p & \text{when } q = 1 \text{ and } p > 1 \\
  q & \text{when } p = 1 \text{ and } q > 1 \\
  2 & \text{when } p = q = 1 \text{ or } p = 2 \text{ or } q = 2 \\
  3 & \text{when } p = 3 \text{ and } q \geq 3 \text{ or } q = 3 \text{ and } p \geq 3 \\
  4 & \text{when } p, q > 3
\end{cases}
\]

**Proof.** Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$ be the bipartite sets of the complete bipartite graph $G = K_{p,q}$. If we consider the case $p = 1$ and $q > 1$ or $(q = 1$ and $p > 1)$, then $B$ or $A$ is the unique minimum monophonic dominating set of $G$. Therefore, $\gamma_m(G) = q$ or $\gamma_m(G) = p$. Next, consider the case $p = q = 1$, then $G = K_2$, a complete graph on 2 vertices. Clearly, $\gamma_m(G) = 2$. Also, if $p = 2$ or $q = 2$, then the set $\{a_1, a_2\}$ or $\{b_1, b_2\}$ is a minimum monophonic dominating set of $G$ and $\gamma_m(G) = 2$. Similarly, when $p = 3$ and $q \geq 3$ or $q = 3$ and $p \geq 3$, the set $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ is a minimum monophonic dominating set of $G$, and so $\gamma_m(G) = 3$.

Finally, consider the case $p, q > 3$. Clearly, $\{a_1, a_2, b_1, b_2\}$ is a monophonic set of $G$ and so $\gamma_m(G) \leq 4$. This shows that there is no monophonic dominating set of $G$ with two elements. Next we have to prove that there exists no monophonic dominating set $M$ of $G$ such that $|M| = 3$. Let $A_1$ be the subset of $A$ or $B$. Then $A_1$ is not dominating the vertices of $A - A_1$ or $A_1$ is not dominating the vertices of $B - A_1$. Suppose $A_1 \cap A = \{a_i, a_j\}$ and $A_1 \cap B = \{b_k\}$, then every $a \in A - A_1$ is not covered by any monophonic
path joining pair of vertices of $A_1$. Therefore, $A_1$ is not a monophonic set of $G$ and so $A_1$ is not a monophonic dominating set of $G$. Also, if $A_1 \cap A = \{a_i\}$ and $A_1 \cap B = \{b_j, b_k\}$, then $A_1$ is not a monophonic dominating set of $G$. Therefore, no three element subset of $A$ is a monophonic dominating set of $K_{p,q}$ and $\gamma_m(G) \geq 4$. Hence we conclude that $\gamma_m(G) = 4$ if $p, q > 3$. 

More generally, for $p, q \geq 2$ we reached at an immediate conclusion.

**Corollary 3.2.** For any two integers $p, q \geq 2$, we have $\gamma_m(Kp, q) = \min\{p, q, 4\}$.

**Proof.** It follows from the Theorem 3.1.

**Theorem 3.3.** For any integer $k \geq 3$, then $\gamma_m(Q_k) = 2^{k-2}$.

**Proof.** For each $a_i \in \{0 \text{ or } 1\}$ where $i = 1, 2, 3, \ldots k$. each vertex of $Q_k$ is labeled by $(a_1, a_2, \ldots, a_k)$. Two vertices in $Q_k$ are adjacent if and only if their binary representations differ in exactly one place. We know that $V(Q_k) = 2^k$ and $V(Q_k)$ can be partitioned into two subsets $A$ and $B$ such that $|A| = |B| = 2^{k-1}$, $A \cup B = V$ and $A \cap B = \varnothing$. Next, we construct two vertex subsets $V(A)$ and $V(B)$ where, $V(A) = \{a_1, a_2, \ldots, a_{2^{k-1} - 1}\}$ and $V(B) = \{b_1, b_2, \ldots, b_{2^{k-1} - 1}\}$ such that for all $i \in \{1, 2, \ldots, (2^{k-1} - 1)\}$, edges $a_1b_{i+1}, b_{i+1}a_{i+1}, a_{2^{k-1}-1}$ and $b_{2^{k-1}-1}a_1$ are in $Q_{k-1}$ and for all $i \in \{1, 2, \ldots, 2^{k-1}\}$, edge $a_ib_i$ is in $Q_k$ where the binary representations of $a_i$ and $b_i$ differ only in the last place. Now let $M = \{a_1, a_3, a_5, a_7, \ldots, a_{(2^{k-1}-3)}, b_{(2^{k-1}-1)}\}$, so that $|M| = \frac{2^{k-1} - 1}{2} = 2^{k-2}$. We show that $M$ is a minimum monophonic dominating set of $Q_k$. It is straightforward to verify that $d_m(a_i, b_{i+2}) = \min\{p, q\}$, $d_m(a_i, b_{i+4}) = 3, 1 \leq i \leq (2^{k-1} - 2)$. Let $a_i \in M$, then it is clear that for all $i \in \{1, 2, \ldots, (2^{k-1} - 3)\}$, vertices $b_i, a_{i+1}, b_{i+1}$ and $a_{i+2}$ are lying on the closed monophonic interval $J[a_i, b_{i+2}] \subseteq J[M]$ and $b_i, b_{i+1} \in N[a_i] \subseteq N[M]$ and $a_{i+1}, a_{i+2} \in N[b_{i+2}] \subseteq N[M]$. Also, vertices $a_{2^{k-1}+1}$ and $b_{2^{k-1}+1}$ are lying on $J[b_{2^{k-1}+1}, a_1] \subseteq J[M]$, $a_{2^{k-1}+1} \in N[b_{2^{k-1}+1}] \subseteq N[M]$ and $b_{2^{k-1}+1} \in N[a_1] \subseteq N[M]$. Therefore $M$ is a monophonic dominating set of $Q_k$. Thus, $\gamma_m(Q_k) \leq |M| = 2^{k-2}$.

We show that $M$ is a minimum monophonic dominating set of $Q_k$. Suppose, to the contrary, that there exists a subset $M_1$ of $V$ such that $M_1$ is a monophonic dominating set of $Q_k$ with $|M_1| < 2^{k-2}$. Now we consider a subset $M_1$ of partitioned sets $A$ or $B$. If $M_1 \subseteq A$, then there exists a vertex $a_i \in A$ such that $a_i \notin M_1$. Since no two vertices of $A$ are adjacent, $a_i \notin N[M_1]$. If $M_1 \subseteq B$, then there exists a vertex $b_j \in B$ such that $b_j \notin M_1$. Since no two vertices of $B$ are adjacent, $b_j \notin N[M_1]$. On the other hand suppose that $M_1 \subseteq A \cup B$. Since $|M_1| < 2^{k-2}$, some elements of partitioned sets $A$ and $B$ do not belong to $M_1$. Since $Q_k$ is not a complete bipartite graph, there exists at least one vertex (say $a_i$) such that $a_i \notin M_1$ and $a_i \notin N[M_1]$.

In all aspects, $M_1$ is not a monophonic dominating set of $Q_k$, which is a contradiction. Hence, $\gamma_m(Q_k) = 2^{k-2}$.
Theorem 3.4. In a triangle-free graph $G$ with minimum degree $\delta(G) \geq 2$, $\gamma_m(G) \leq 2|P|$ where $P$ is the maximal matching of $G$.

Proof. Since $P$ is the matching of $G$, each vertex of $G$ is incident with utmost one edge in $P$. The maximality of $P$ gives that no other edges of $G$ can be added to $P$. Let $M = \{v \in V : v$ is incident with an edge of $P\}$. Then clearly $V - M$ is independent. This shows that $M$ is a dominating set since $\delta(G) \geq 2$. Since $G$ is triangle-free, the path $[x, v, y]$ is an $x - y$ monophonic path. Therefore, $M$ is a monophonic dominating set and $N[M] = V$. Hence $M$ is a monophonic dominating set with $\gamma_m(G) \leq 2|P|$.

Remark 3.5. The upper bound in the Theorem 3.4 is sharp.

For $q > p \geq 2$, we consider $G_1 = K_{p,q}$ with partite sets $A_1 = \{u_1, u_2, \ldots, u_q\}$ and $A_2 = \{s_1, s_2, \ldots, s_p\}$. Similarly, for $r > p \geq 2$, $G_2 = K_{p,r}$ with partite sets $B_1 = \{v_1, v_2, \ldots, v_r\}$ and $B_2 = \{t_1, t_2, \ldots, t_p\}$. Define a new graph $G$ by $G = G_1 \cup G_2$ such that $G_1$ and $G_2$ are disjoint with the edge set $P' = \{s_1t_1, s_2t_2, \ldots, s_pt_p\}$. Clearly $G$ is a triangle-free graph and $M = \{s_1, s_2, \ldots, s_p\} \cup \{t_1, t_2, \ldots, t_p\}$ is a minimum monophonic dominating set of $G$ with the maximal matching $P'$. Thus $\gamma_m(G) = 2|P'|$. Hence the upper bound is sharp.

Ingredients of Theorem 3.4 leads to another upper bound in terms of $\gamma_2(G)$ and $\alpha(G)$ of a triangle-free graph $G$.

Theorem 3.6. In a triangle-free graph $G$ with $\gamma_2(G)$, then $\gamma_m(G) \leq \gamma_2(G)$.

Proof. Let $M$ be a $\gamma_2$-set of $G$, clearly, $v \in |N(v) \cap M|$ for all $v \in V - M$ and also, $M$ is a dominating set of $G$. Since $G$ is triangle-free, $v \in J[N(v) \cap M]$ for all $v \in V$. Therefore, $M$ is a monophonic set of $G$. Hence $M$ is a monophonic dominating set of $G$ with $\gamma_m(G) \leq \gamma_2(G)$.

Corollary 3.7. In a triangle-free graph $G$ with order $p$ and $\delta(G) \geq 2$, then $\gamma_m(G) \leq p - i(G)$, where $i(G)$ is the independence number of $G$.

Proof. It follows from Theorem 3.6.

Next, we study how the monophonic domination number is affected when some edges are deleted from the complete graph.

Theorem 3.8. For a complete graph $K_p$ and $e \in E(K_p)$, then $\gamma_m(K'_p) = 2$ where $K'_p = K_p - \{e\}$ is the edge deleted graph.

Proof. Consider an edge $e = uv \in E(K_p)$ and edge deleted graph $K'_p = K_p - \{e\}$. Let $M = \{u, v\}$ be a subset of $K'_p$. Then for every vertex $w \in V(K'_p) - M$, there exists a $u - v$ monophonic path of length 2 containing $w$. Since $d(u) = d(v) = p - 2$ in the edge deleted graph $K'_p$, $M$ is a monophonic set of $K'_p$. Also, $N[M] = V(K'_p)$. Therefore, $M$ is a monophonic dominating set and it follows that $\gamma_m(K'_p) \leq 2$. Hence $\gamma_m(K'_p) = 2$. 


Theorem 3.9. In a complete graph $K_p$ $(p \geq 4)$ and edges $\{e_1, e_2\}$ is a subset $E(K_p)$, then

$$\gamma_m(K''_p) = \begin{cases} 2 & \text{edges } e_1 \text{ and } e_2 \text{ are non-adjacent} \\ 3 & \text{edges } e_1 \text{ and } e_2 \text{ are adjacent} \end{cases}$$

where $K''_p = K_p - \{e_1, e_2\}$ is the edge deleted graph.

Proof. Consider the edge deleted graph $K''_p = K_p - \{e_1, e_2\}$ where the edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ are in $E(K_p)$. We consider two cases

Case(i): Edges $e_1$ and $e_2$ are non-adjacent. Let $M = \{u_1, v_1\}$ be a subset of $V(K''_p)$. Clearly, $d(u_1) = d(v_1) = p - 2$. By Theorem 3.8, it is seen that $M$ is a minimum monophonic dominating set of $G$. Therefore, $\gamma_m(K''_p) = 2$.

Case(ii): Edges $e_1$ and $e_2$ are adjacent. In this case, $e_1$ and $e_2$ have a common vertex, say, $v_i = v_2$. Let $M = \{u_1, v_1, v_2\}$ be a subset of $V(K''_p)$. Then, every vertex in $V(K''_p) - M$ lies in a $u_1 - v_1$ monophonic path of length 2. That is, every vertex in $V(K''_p) - M$ is both monophonic and dominated by the vertices of $M$ and so, $M$ is a monophonic dominating set of $K''_p$. Therefore, $2 \leq \gamma_m(K''_p) \leq 3$.

Finally, we have to show that $\gamma_m(K''_p) = 3$. Let $M = \{x, y\}$ be a monophonic dominating set of $K''_p$. Since $p \geq 4$, vertices $x$ and $y$ are not adjacent in $K''_p$. Then, clearly $M$ is either $\{u_1, u_2\}$ or $\{v_1, v_2\}$. In all cases, there is a vertex in $\{u_1, v_1, v_2\} - M$, which is not monophonic dominated by $M$. Therefore, no two element vertex set of $K''_p$ is a monophonic dominating set of $K''_p$. Hence $\gamma_m(K''_p) = 3$.

Theorem 3.10. In any non trivial tree $T$ with order $p \geq 2$, then there exists a subset $X = V - N[L(T)]$ of $V$ such that the variation $\gamma_m(T) - \gamma_X(T) = l_T$, where $l_T = |L(T)|$ and $\gamma_X(T)$ is the X dominating number of $T$.

Proof. Let $L(T) = \{u \in V : uv \in E(T) \text{ with } d(u) = 1 \text{ and } d(v) > 1\}$ be the set of all leaves of $T$ and let $M$ be a monophonic dominating set of $T$. Since $L(T) \subseteq M$, $L(T)$ only dominates the vertices of $N[L(T)]$. Also note that $V - N[L(T)] \subseteq M - L(T)$. Thus set $M - L(T)$ is a minimum $X-$ dominating set of $T$ where $X = V - N[L(T)]$. This implies that X- domination number $\gamma_X(T) = |M| - |L(T)| = \gamma_m(T) - l_T$. Hence we have $\gamma_m(T) - \gamma_X(T) = l_T$.

The monophonic domination number of some standard class of graph can be found, and are given below.

1. For the cycle graph $C_p$, $\gamma_m(C_p) = \lceil \frac{p}{3} \rceil$ for all $p \geq 5$

2. For the complement of cycle graph $C_p$, $\gamma_m(C_p^c) = 3$ for $p \geq 6$. 
3. For the path graph $P_p$, $\gamma_m(P_p) = \lceil \frac{p+2}{3} \rceil$ for all $p$.

4. For the complement of the path $P_p$, $\gamma_m(P_p) = 3$ for $p \geq 5$.

5. For any non-trivial tree $T$, the $\gamma_m(T) = t$ where $t = |\text{Pend}(G)|$.

6. For the wheel graph $W_p$, $\gamma_m(W_p) = \lceil \frac{p-1}{2} \rceil$ for all $p \geq 5$.

7. For the $p-fan$ graph $F_p$, $\gamma_m(F_p) = \lceil \frac{p}{2} \rceil$ for all $p \geq 4$.

8. For the Petersen graph $G$, $\gamma_m(G) = 4$.

9. For the star graph $K_{1,p-1}$, $\gamma_m(K_{1,p-1}) = p - 1$.

10. Let $G_T$ be the standard Tietze’s graph on 12 vertices with $d_m = r_m = c = 3$. Then, we have $\gamma_m(G_T) = 4$.

4 Realization Problems

In this section, we give realization theorems related to the monophonic domination number. First establish the existence of a monophonic dominating set in a connected graph $G$ of order $p$ with $s \leq \gamma_m(G) \leq p - t$, where $|\text{Supp}(G)| = s$ and $|\text{Pend}(G)| = t$.

**Theorem 4.1.** Let $G$ be a $(p, q)$ graph with $|\text{Supp}(G)| = s$ and $|\text{Pend}(G)| = t$. Then there exists a monophonic dominating set $M$ of $G$ such that $t \leq \gamma_m(G) \leq p - s$, where $p = |V|$.

**Proof.** Consider the followings two subsets of the vertex set $V$ $\text{Pend}(G) = \{v_i \in V : v_i$ is a pendant vertex of $G$ for $1 \leq i \leq t \}$ with cardinality $|\text{Pend}(G)| = t$ and $\text{Supp}(G) = \{v_j \in V : v_j$ is a support vertex of $G$ for $1 \leq j \leq s \}$ with cardinality $|\text{Supp}(G)| = s$. Clearly, $t \geq s$. By Observation 2.1, $\text{Pend}(G)$ is a subset of every monophonic dominating set of $G$ and so $t \leq \gamma_m(G)$. Further, for $1 \leq j \leq s$, every vertex $v_j$ of $\text{Supp}(G)$ lies in a monophonic path joining two vertices of $\text{Pend}(G)$. Also, each vertex $v_j$ of $\text{Supp}(G)$ is dominated by the vertices $v_i$ of $\text{Pend}(G)$, where $1 \leq i \leq t$. This shows that $V - \text{Supp}(G)$ is a monophonic dominating set of $G$. Therefore, $\gamma_m(G) \leq |V - |\text{Supp}(G)|| = p - s$. Hence $t \leq \gamma_m(G) \leq p - s$. \hfill \Box

**Remark 4.2.** The lower and upper bounds in the Theorem 4.1 are sharp.

The upper bound is sharp when we consider the graph $G$ of order $p = 12$ given in the Figure 2. Here, $\{v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is a minimum monophonic dominating set $G$ and $|\text{Supp}(G)| = s = 4$. Therefore, $\gamma_m(G) = 8 = 12 - 4$. Hence $\gamma_m(G) = p - s$, sharpness occurred.
Monophonic domination in special graph structures

Figure 2: A Graph $G$ with $|\text{Supp}(G)| = 4$ and $\gamma_m(G) = 8$

The lower bound is sharp when we consider a comb graph. A comb graph $G$ is a caterpillar in which $|\text{Supp}(G)| = |\text{Pend}(G)|$. Therefore $\gamma_m(G) = |\text{Supp}(G)| = s$. Hence lower bound is also sharp.

**Theorem 4.3.** Let $G$ be a connected graph of order $p$ with monophonic diameter $d_m$. Then there exists a monophonic dominating set $M$ such that $\gamma_m(G) \leq p - \left\lfloor \frac{2d_m}{3} \right\rfloor$

**Proof.** For any two integers $r,t$ we define monophonic diameter $d_m$ by $d_m = 3t+r$ where $r \in \{0, 1, 2\}$. First we select two vertices $u_0$ and $u_{d_m}$ in $G$ such that $d(u_0, u_{d_m}) = d_m$. Let $P : [u_0, u_1 \ldots u_{d_m}]$ be $u_0 - u_{d_m}$ monophonic path, and let $W = \{u_0, u_3, \ldots, u_{3t}, u_{3t+r}\}$. Then we can easily verify that $D = V - W_1$ is a monophonic set of $G$ where $W_1 = (V(P) - W) = \{u_1, u_2, u_4 \ldots \}$. Also $N[D] = V$. It means that $D$ is a monophonic dominating set of $G$. If

$$|W| = \begin{cases} t + 1 & \text{when } r = 0 \\ t + 2 & \text{when } r = 1, 2 \end{cases}$$

then we can find that $|D| = |V| - |W_1| = p - \left\lfloor \frac{6t+2r}{3} \right\rfloor = p - \left\lfloor \frac{2d_m}{3} \right\rfloor$. It follows that $\gamma_m(G) \leq p - \left\lfloor \frac{2d_m}{3} \right\rfloor$.

**Remark 4.4.** The upper bound in Theorem 4.3 is sharp.

To prove the sharpness, consider a path graph $G = P_p$ of order $p$, $\gamma_m(G) = \left\lceil \frac{p+2}{3} \right\rceil = p - \left\lfloor \frac{2(p-1)}{3} \right\rfloor = p - \left\lfloor \frac{2d_m}{3} \right\rfloor$. Thus equality occurred.

**Theorem 4.5.** Let $G$ be a connected graph of order $p$ with girth $c \geq 6$. Then there exists a monophonic dominating set $M$ such that $\gamma_m(G) \leq p - \left\lceil \frac{2c}{3} \right\rceil$.

**Proof.** For any two integers $r,t$ we define the girth $c$ or $c(G)$ by $c = 3t+r$, where $r \in \{0, 1, 2\}$. Consider an induced cycle $C : [u_1, u_2 \ldots u_c, u_1]$ of length $c$
Further, we consider the sets

\[ W = \begin{cases} 
\{u_1, u_4, \ldots, u_{3t-2}\} & \text{when } r = 0 \\
\{u_1, u_4, \ldots, u_{3t-2}, u_{3t+1}\} & \text{when } r = 1, 2 
\end{cases} \]

and \( D = V - (V(C) - W) \). Clearly seen that \( D \) is a monophonic set and \( N[D] = V \). It means that \( D \) is a monophonic dominating set of \( G \). If

\[ |W| = \begin{cases} 
t & \text{when } r = 0 \\
t + 1 & \text{when } r = 1, 2 
\end{cases} \]

then we can find that \( |D| = |V| - |V(C) - W| = p - \left\lfloor \frac{6t+2r}{3} \right\rfloor = p - \left\lfloor \frac{2t}{3} \right\rfloor \). It follows that \( \gamma_m(G) \leq p - \left\lfloor \frac{2t}{3} \right\rfloor \).

**Remark 4.6.** The upper bound in the Theorem 4.5 is sharp

To prove the sharpness, consider \( G = C_p \) a cycle graph on \( p \) vertices, where \( p \geq 6 \). We know that \( \gamma_m(C_p) = \left\lceil \frac{p}{3} \right\rceil = p - \left\lfloor \frac{2p}{3} \right\rfloor = p - \left\lfloor \frac{2t}{3} \right\rfloor \).

**Theorem 4.7.** Let \( G \) be a connected graph of order \( p \) with monophonic diameter \( d_m \). Then there exists a monophonic dominating set \( M \) such that \( \gamma_m(G) \leq p - d_m + \left\lceil \frac{d_m}{3} \right\rceil \).

**Proof.** Consider the path \( P : [u = v_0, v_1, \ldots, v_d = v] \) of length \( d_m \) in the graph \( G \). If \( M = \{v_1, v_2, \ldots, v_{d_m-1}\} \) is a subset of \( V \), then \( V - M \) is a monophonic set of \( G \). Also, it is a dominating set of induced subgraph \( <V - (M - M_1)> \) where, \( M_1 = \{v_1, v_{d_m-1}\} \). Let \( P' = [v_2, v_3, \ldots, v_{d_m-2}] \) with \( |V(P')| = d_{m-3} \). Let \( D' \) be a minimum dominating set of \( P' \). Then \( |D'| = \left\lceil \frac{(d_m-3)}{3} \right\rceil \). Let \( D = (V - M) \cup D' \). Clearly, \( D \) is a monophonic dominating set of \( G \). Therefore,

\[
\gamma_m(G) \leq |D| = |V| - |M| + |D'|
\leq p - d_m + 1 + \left\lceil \frac{(d_m-3)}{3} \right\rceil
\leq p - d_m + 1 + \left\lceil \frac{d_m}{3} \right\rceil - 1
\leq p - d_m + \left\lceil \frac{d_m}{3} \right\rceil
\]

**Remark 4.8.** Equality hold in the Theorem 4.7.

Consider the graph \( G \) of order \( p = 14 \) is given in Figure 3. Here \( d_m = 8 \) and so \( p - d_m + \left\lceil \frac{d_m}{3} \right\rceil = 9 \). Also \( M = \{v_0, v_3, v_5, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\} \) is a minimum monophonic dominating set of \( G \). Hence \( \gamma_m(G) = 9 = p - d_m + \left\lceil \frac{d_m}{3} \right\rceil \).
Corollary 4.9. In a \((p,q)\) connected graph \(G\) with \(d_m\). If \(\delta \geq 3\), then \(\gamma_m(G) \leq p - d_m + 1\).

Proof. Given that \(\text{diam}_m(G) = d_m\), clearly there exists a monophonic path \(P : [u = v_0, v_1, \ldots, v_{d_m} = v]\) of length \(d_m\) in the graph \(G\). If \(M = \{v_1, v_2, \ldots, v_{d_m-1}\}\), then \(V - M\) is a monophonic set of \(G\). Clearly \(V - M \subseteq V\) and \(|V - M| = p - (d_m - 1)\). Since \(\delta \geq 3\), each vertex of \(M\) is adjacent to at least one vertex of \(V - M\). Also, every vertex of \(G\) is contained in the \(u - v\) monophonic path. Therefore, \(V - M\) is a monophonic set of \(G\) and \(M\) is the minimal dominating set of \(G\).

Next we have to show that \(V - M\) is also a dominating set. Assume that \(V - M\) is not dominating set of \(G\), that is for some vertex \(v \in M\) there is no edge from \(v\) to any vertex in \(V - M\). But \(M - \{v\}\) would be a dominating set, contradicting the minimality of \(M\). Therefore \(V - M\) is a dominating set of \(G\). Hence \(V - M\) is a monophonic dominating set of \(G\) and clearly

\[
\gamma_m(G) \leq |V - M| \leq p - (d_m - 1) = p - d_m + 1
\]

\[\Box\]

Acknowledgments. The authors are grateful and thankful to the referees and for their priceless comments and insight

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https://doi.org/10.1016/0895-7177(93)90259-2


Received: October 11, 2017; Published: December 11, 2017