On Strong Rainbow Vertex-Coloring of

Generalized Petersen Graphs

\( G(n,2) \) and \( G(n,3) \)

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Abstract

A path in a vertex-colored graph \( G \) is called a rainbow path if no two internal vertices get the same color. A vertex-colored graph \( G \) is strongly rainbow vertex-connected, if for every pair of distinct vertices, there exists at least one shortest rainbow path. The minimum number of colors required to strongly rainbow vertex-color a graph \( G \) is called the strong rainbow vertex-connection number, denoted by \( srvc(G) \). This work presents the exact values of strong rainbow vertex-connection numbers for the generalized Petersen graphs \( G(n,2) \) and \( G(n,3) \).

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1. Introduction

The concept of rainbow connection of a graph was first presented by Chartrand et al. in [4]. The vertex version of the rainbow connection, called the rainbow vertex-connection number \( rvc(G) \), was put forward by M. Krivelevich and R. Yuster in 2009 [7]. Computing the rainbow vertex-connection number of an arbitrary graph is \( NP \) Hard [7]. Let \( G(V,E) \) be a graph. A vertex-coloring of a
graph $G$ is a function from its vertex set to the set of natural numbers. A path in a vertex-colored graph $G$ is called a rainbow path if no two internal vertices get the same color. A vertex-colored graph $G$ is rainbow vertex-connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow vertex-coloring. The rainbow vertex-connection number $rvc(G)$ is the minimum number of colors needed for rainbow vertex coloring of a graph $G$. Similarly, a vertex-colored graph $G$ is strongly rainbow vertex-connected, if for every pair of distinct vertices, there exists a rainbow geodesic (shortest rainbow path). The minimum number of colors required to strongly rainbow vertex color a graph $G$ is called the strong rainbow vertex-connection number, denoted by $srvc(G)$. Let $c(v)$ denote the color of the vertex $v \in V$. The distance between two vertices $u$ and $v$ in $G$, denoted by $d(u,v)$ is the length of a shortest path between them in $G$. The eccentricity $e(v)$ of a vertex $v$ in a connected graph $G$ is $\max\{d(u,v) \mid u \in V\}$. The maximum eccentricity of all vertices in a graph $G$ is called the diameter $diam(G)$ of the graph.

2. An overview of the paper

The Petersen graph which appears throughout the literature of graph theory, is named after Julius Petersen, the Danish Mathematician. It was Watkins [8] who surmised that the class of generalized Petersen graphs have a Tait coloring, in addition to $G(5,2)$. This inference was proven in [3]. From then on this class of graphs is being analyzed extensively because of its applications. The graph $G(n,k)$ with $n$ and $k$ relatively prime was proposed by Coxeter in [5]. The credit for determining all the hamiltonian generalized Petersen graphs goes to Alspach [2]. The generalized Petersen graphs have been the focus of study of several other authors as well. In 2012, Huang et. al studied $L(2,1)$ labeling of generalized Petersen graphs [6]. In 2013, Ahmad et al. [1] investigated the metric dimension of generalized Petersen graphs.

This paper investigates the strong rainbow vertex-connection number of some classes of generalized Petersen graph $G(n,2)$ and $G(n,3)$.

3. Strong Rainbow-Vertex Connection Number of $G(n,2)$ and $G(n,3)$

In this section, we investigate the strong rainbow vertex connection in $G(n,2)$ for odd and even $n$ and $G(n,3)$ for odd $n$.

Theorem 3.1: Let $G(n,2)$ be the generalized Petersen graph, where $n$ is even and $n \geq 10$. Then $srvc(G(n,2)) = 2 + \frac{n}{2}$.
Proof: Since \( n \) is even, the outer rim is an even cycle of length \( n \) and there are two inner rims each of which is a cycle of length \( \frac{n}{2} \). Our aim is to find the minimum number of colors required for the \( U \) – vertices and \( V \) – vertices. First we shall find the minimum number of colors required for the \( U \) – vertices. We note that in \( G(n,2) \) there exists a unique shortest path \( P = \{v_i,u_i,u_{i+1},v_{i+1} \} \) of length 3, with end vertices \( v_i \) and \( v_{i+1} \). The two internal vertices of \( P \) are \( U \) – vertices. The path \( P \) will be a rainbow path if the two internal vertices are assigned with distinct colors. We also note that any path \( P \) of length greater than 3 with end vertices \( v_i \) and \( v_j \) is not a unique shortest path. Moreover, the path \( P \) consists of \( U \) – vertices and \( V \) – vertices as internal vertices. Since our objective is to minimize the number of colors, we use more number of \( V \) – vertices as they skip in steps of 2. Therefore we conclude that 2 colors are sufficient for the \( U \) – vertices. Therefore \( U \)-vertices are assigned with two colors \( a_1 \) and \( a_2 \) alternately.

Next, to find the minimum number of colors required for the \( V \) - vertices. The two inner rims are cycles of length \( \frac{n}{2} \). Let the two cycles be \( C_{\frac{n}{2}}^1 \) and \( C_{\frac{n}{2}}^2 \). The cycle \( C_{\frac{n}{2}}^1 \) is constituted by the vertices \( v_0,v_2,...,v_{\frac{n}{2}} \) and the cycle \( C_{\frac{n}{2}}^2 \) is constituted by the vertices \( v_1,v_3,...,v_{\frac{n}{2}-1} \). Consider a path \( P \) whose end vertices lie on the outer rim and the length of the path is \( \text{diam}(G(n,2)) = \left\lceil \frac{n}{4} \right\rceil + 2 \). Let \( P \) be a path with end vertices \( u_i \) and \( u_{i+\frac{n}{2}} \) with \( d(u_i,u_{i+\frac{n}{2}}) = \text{diam}(G(n,2)) \) where \( 0 \leq i \leq n-1 \). As there exists more than one shortest path between vertices \( u_i \) and \( u_{i+\frac{n}{2}} \), we choose \( P \) containing at most two \( U \) vertices as internal vertices (Since only two colors are assigned for \( U \) vertices) and the remaining internal vertices as \( V \) vertices. Consider the two cases for which \( \frac{n}{2} \) is odd and even.

Case (i): \( \frac{n}{2} \) is even. If \( P \) is a path with end vertices \( u_i \) and \( u_{i+\frac{n}{2}} \) with \( d(u_i,u_{i+\frac{n}{2}}) = \text{diam}(G(n,2)) \) where \( 0 \leq i \leq n-1 \), then \( P \) is of the form \( P: u_i,v_i,v_{i+2},...,v_{i+\frac{n}{2}},u_{i+\frac{n}{2}} \) where \( 0 \leq i \leq n-1 \). The internal vertices of \( P \) are \( V \) - vertices lying on the cycle (say) \( C_{\frac{n}{2}}^1 \). There are exactly \( \left( \frac{n}{4} \right) + 1 \) internal vertices in \( P \). The path \( P \) will be a rainbow path if all the internal vertices have distinct colors. Since each inner rim is a cycle of length \( \frac{n}{2} \) and exactly \( \left( \frac{n}{4} \right) + 1 \) vertices
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should be distinct in color, assign \( \left( \frac{n}{4} \right) + 1 \) distinct colors to the vertices of \( C_n^1 \) starting from the vertex \( v_0 \). It follows that the remaining uncolored \( \left( \frac{n}{4} \right) - 1 \) vertices in \( C_n^1 \) should also be assigned with new colors for strong rainbow connectivity. That is, the \( V - \) vertices of \( C_n^1 \) in cyclic order namely \( v_0, v_2, ..., v_n = v_0 \) are assigned with \( \frac{n}{2} \) distinct colors totally. We also note that \( c(U) \cap c(V) = \phi \). That is the colors used for the \( U - \) vertices should be distinct from the colors used for the \( V - \) vertices. In addition, the two inner rims are disjoint and there exists no unique shortest path whose internal vertices lie on different inner rims. Therefore we assign the same set of \( \frac{n}{2} \) distinct colors to both the inner rims.

Thus the graph is strongly rainbow vertex connected since any two vertices are connected by at least one shortest rainbow path.

Case (ii): \( \frac{n}{2} \) is odd. Any path \( P \) with end vertices \( u_i \) and \( u_{i + \frac{n}{2}} \) is of the form \( P: u_i, v_i, v_{i+2}, ..., v_{i + \frac{n}{2} - 1}, u_{i + \frac{n}{2} - 1}, u_{i + \frac{n}{2}} \) where \( 0 \leq i \leq n - 1 \) whose length is \( \text{diam}(G(n,2)) \).

The internal vertices of \( P \) are \( \left\lfloor \frac{n}{4} \right\rfloor \) vertices of the inner rim (say the cycle \( C_n^1 \)) in cyclic order and exactly one outer rim vertex. Since the inner rim \( C_n^1 \) is a cycle of length \( \frac{n}{2} \) and \( \left\lfloor \frac{n}{4} \right\rfloor \) vertices in cyclic order should be distinct in color, it follows that the remaining \( \left\lfloor \frac{n}{4} \right\rfloor \) vertices of \( C_n^1 \) should also be distinct in color. Hence totally \( \frac{n}{2} \) distinct colors are assigned for the inner rim \( C_n^1 \). Since both the inner rims are disjoint cycles, it is sufficient to color both the cycles with the same set of \( \frac{n}{2} \) distinct colors as discussed in case (i). Hence the proof.

**Theorem 3.2:** For a generalized Petersen graph \( G(n,2) \), if \( 2 \mid n \), then \( srvc(G(n,2)) = 3 + \left\lceil \frac{n}{4} \right\rceil + r, \ 0 \leq r \leq \left\lceil \frac{n}{4} \right\rceil - 1. \)
**Proof:** \( n \) is odd. It is easy to check that \( \text{srvc}(G(5,2)) = 1 \) and \( \text{srvc}(G(7,2)) = 3 \).

For \( n \geq 9 \), the following cases arise.

Case (i): \( \frac{n-1}{2} \) is even. The outer rim is an odd cycle of length \( n \). From theorem 3.1, every two consecutive vertices in the outer rim should be distinct in color. We color the first \( n-1 \) vertices \( u_0, u_1, \ldots, u_{n-2} \) with the colors \( a_1 \) and \( a_2 \) alternately and the \( n^{th} \) vertex \( u_{n-1} \) is assigned with the color \( a_3 \). Next we color the inner rim vertices. The inner rim is a cycle of odd length \( n \). The vertices of the inner rim in cyclic order are \( v_0, v_2, \ldots, v_{n-1}, v_1, v_3, \ldots, v_{n-2} \). The diameter of \( G(n,2) \) is \( \left\lfloor \frac{n}{4} \right\rfloor + 2 \).

Consider a path \( P \) whose end vertices are \( U \) – vertices with length \( \text{diam}(G(n,2)) \). Since our aim is to minimize the number of colors, we pass through the inner rim vertices as it skips in steps of 2. Let \( P \) be a path \( P: u_i, v_i, v_{i+2}, \ldots, v_{i+ \frac{n-1}{2}}, u_{i+ \frac{n-1}{2}}, v_{i+ \frac{n-1}{2} + 1}, \ldots, v_{i+ \frac{n-1}{2} + \frac{n}{4}}, u_{i+ \frac{n-1}{2} + \frac{n}{4}} \), \( 0 \leq i \leq n-1 \) of length \( \left\lfloor \frac{n}{4} \right\rfloor + 2 \). The path \( P \) contains \( \left\lfloor \frac{n}{4} \right\rfloor \) inner rim vertices in cyclic order. For strong rainbow vertex coloring, all the internal vertices of \( P \) should be distinct in color. We color the inner rim vertices in cyclic order with the colors \( b_1, b_2, \ldots, b_{\left\lfloor \frac{n}{4} \right\rfloor} \) starting from the vertex \( v_0 \), \( \left\lfloor \frac{n}{m} \right\rfloor \) times, where \( m = \left\lfloor \frac{n}{4} \right\rfloor \). If \( \left\lfloor \frac{n}{4} \right\rfloor \) divides \( n \), then all the inner rim vertices will be assigned with colors \( b_1, b_2, \ldots, b_{\left\lfloor \frac{n}{4} \right\rfloor} \) cyclically in the clockwise direction. If not, there are at most \( \left\lfloor \frac{n}{4} \right\rfloor - 1 \) remaining vertices left uncolored at the end of the cycle. Assign \( \left\lfloor \frac{n}{4} \right\rfloor - 1 \) new colors to the remaining vertices such that it preserves rainbow connectivity. Consequently, if \( n \equiv 0 \pmod{\left\lfloor \frac{n}{4} \right\rfloor} \), then the inner rim vertices are assigned with \( \left\lfloor \frac{n}{4} \right\rfloor \) distinct colors \( \left\lfloor \frac{n}{\left\lfloor \frac{n}{4} \right\rfloor} \right\rfloor \) times cyclically. In this case, \( \text{srvc}(G(n,2)) = 3 + \left\lfloor \frac{n}{4} \right\rfloor \). Similarly, \( n \equiv 1 \pmod{\left\lfloor \frac{n}{4} \right\rfloor} \) implies that the inner rim vertices are assigned with \( \left\lfloor \frac{n}{4} \right\rfloor \) distinct colors \( \left\lfloor \frac{n}{\left\lfloor \frac{n}{4} \right\rfloor} \right\rfloor \) times cyclically starting from the vertex \( v_0 \) and only one vertex \( v_{n-2} \) is left uncolored.
Assigning a new color to that vertex, we see that \( srvc(G(n, 2)) = 3 + \left\lceil \frac{n}{4} \right\rceil + 1 \). In general, \( n = r \mod \left\lceil \frac{n}{4} \right\rceil \) implies that \( srvc(G(n, 2)) = 3 + \left\lceil \frac{n}{4} \right\rceil + r \) where \( r \) is an integer, \( 0 \leq r \leq \left\lceil \frac{n}{4} \right\rceil - 1 \). Thus \( srvc(G(n, 2)) = 3 + \left\lceil \frac{n}{4} \right\rceil + r, \) \( 0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1 \).

Case (ii): \( \frac{n-1}{2} \) is odd. The \( U \)-vertices can be assigned with the colors as discussed in case (i). Now we proceed to color the \( V \)-vertices. The diameter of \( G(n, 2) \) is \( \left\lceil \frac{n}{4} \right\rceil + 2 \). Any path \( P \) with length \( \text{diam}(G(n, 2)) \) and having \( U \)-vertices as end vertices is of the form \( P : u_i, v_i, v_{i+2}, \ldots, v_{i+\frac{n-3}{2}}, u_i, v_{i+\frac{n-3}{2}}, \ldots, v_{i+\frac{n-1}{2}}, 0 \leq i \leq n - 1 \). The internal vertices of \( P \) consists of \( \left\lfloor \frac{n}{4} \right\rfloor \) inner rim vertices and exactly one outer rim vertex. For strong rainbow vertex coloring, all the internal vertices of \( P \) should be distinct in color. We color the inner rim vertices in cyclic order starting from the vertex \( v_0 \) as given in case (i). Using the argument discussed in case (i), it follows that \( srvc(G(n, 2)) = 3 + \left\lceil \frac{n}{4} \right\rceil + r \) where \( 0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1 \).

In the next theorem, we characterize the strong rainbow-vertex connection number of \( G(n, 3) \) if \( 3 \| n \) and \( n \) is odd.

**Theorem 3.3:** If \( 3 \| n \) and \( n \) is odd in \( G(n, 3) \), then \( srvc(G(n, 3)) = 3 + \frac{n}{3} \).

Proof: If \( 3 \| n \) in \( G(n, 3) \), then there are three inner rims. Each inner rim is a cycle of length \( \frac{n}{3} \). The outer rim is a cycle of length \( n \). We shall first find the minimum number of colors required for the outer rim vertices. We note that in \( G(n, 3) \) there exists a unique shortest path \( P = \{ v_i, u_i, v_{i+1}, u_{i+1} \mid 0 \leq i \leq n - 1 \} \) of length 3, with end vertices \( v_i \) and \( v_{i+1} \). The two internal vertices of \( P \) are \( U \)-vertices. The path \( P \) will be a rainbow path if the two internal vertices are assigned with distinct colors. We also note that any path \( P \) of length greater than 3 with end vertices \( v_i \) and \( v_j \) is not a unique shortest path. Moreover, the path \( P \) consists of \( U \)-vertices and \( V \)-vertices as internal vertices. Since our objective is to minimize the number of colors, we use more number of \( V \)-vertices as they skip in steps of 3. Therefore \( U \)-vertices are assigned with two colors \( a_i \) and \( a_2 \) alternately. Since \( n \) is odd, we
color the first \( n-1 \) vertices \( u_0, u_1, \ldots, u_{n-2} \) with the colors \( a_1 \) and \( a_2 \) alternately and the \( n^{th} \) vertex \( u_{n-1} \) is assigned with the color \( a_3 \). Hence the number of colors required to color the \( U \) - vertices are 3.

Now we proceed to find the minimum number of colors required for the inner rims in order to make \( G(n,3) \) a strongly rainbow vertex connected graph. The three inner rims are cycles of length \( \frac{n}{3} \). Consider two farthest \( U \) – vertices with distance \( \text{diam}(G(n,3)) \) apart. Let \( u_i \) and \( u_{i+\frac{n-1}{2}} \) be two vertices with \( d(u_i, u_{i+\frac{n-1}{2}}) = \text{diam}(G(n,3)) \) where \( 0 \leq i \leq n-1 \). The shortest path connecting the vertices \( u_i \) and \( u_{i+\frac{n-1}{2}} \) is not unique. We choose the path \( P \) connecting the vertices \( u_i \) and \( u_{i+\frac{n-1}{2}} \) in such a way that it cannot have more than two outer rim vertices as internal vertices. Then the path \( P \) is of the form \( P: u_i, v_i, v_{i+3}, \ldots, v_{i+n-3}, u_{i+\frac{n-1}{2}}, u_{i+\frac{n-3}{2}}, u_{i+\frac{n-1}{2}} \).

The internal vertices of \( P \) are \( \left\lceil \frac{n}{6} \right\rceil \) inner rim vertices in cyclic order and exactly one outer rim vertex. This implies that \( \left\lfloor \frac{n}{6} \right\rfloor \) inner rim vertices in cyclic order should acquire distinct colors. Suppose the internal vertices of \( P \) which passes through the inner rim lie on the cycle, say \( C_{\frac{n}{3}}^1 \). Out of \( \frac{n}{3} \) inner rim vertices, only \( \left\lfloor \frac{n}{6} \right\rfloor \) vertices are assigned with colors. That is, \( \left\lfloor \frac{n}{6} \right\rfloor \) remaining vertices of \( C_{\frac{n}{3}}^1 \) are left uncolored. Since \( \left\lfloor \frac{n}{6} \right\rfloor \) vertices in cyclic order should be distinct in color to preserve rainbow connectivity, they should also be assigned with new colors. That is totally \( \frac{n}{3} \) distinct colors are assigned for the vertices of \( C_{\frac{n}{3}}^1 \). Also we note that the three inner rims \( C_{\frac{n}{3}}^1, C_{\frac{n}{3}}^2 \) and \( C_{\frac{n}{3}}^3 \) are disjoint cycles and there exists no unique shortest path whose internal vertices lie on different cycles. Therefore the same set of \( \frac{n}{3} \) distinct colors are assigned for the inner rims \( C_{\frac{n}{3}}^1, C_{\frac{n}{3}}^2 \) and \( C_{\frac{n}{3}}^3 \). We also note that \( c(U) \cap c(V) = \phi \). It is easy to see that the above coloring scheme gives a strong rainbow vertex coloring, with \( 3 + \frac{n}{3} \) colors.
Concluding remarks: In this paper, the exact values of strong rainbow vertex-connection numbers of $G(n,2)$ and $G(n,3)$ have been computed. It would be interesting to find the strong rainbow vertex connection number of $G(n,k)$ for various values of $n$ and $k$.

References


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