On a Subclass of Univalent Functions Defined by a Generalized Differential Operator

Opoola O. Timothy

Department of Mathematics
University of Ilorin, Ilorin, Nigeria

Abstract

In this paper, we investigate a new subclass of univalent functions defined by a generalized differential operator. An inclusion result and characterization properties of this class of functions are also established.

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1. Introduction and Definitions

Let $A$ denotes the class of functions analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $S$ denote the subclass of functions in $A$ that are univalent and have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

Let $P$ denote the class of functions such that $p(0) = 1$ and $\text{Rep}(z) > 0$ for $z \in U$. The class of functions in $P$ are called Caratheodory functions (see [4]). For a function $f(z) \in A$ and of the form (1) we let

$$D_t f(z) = 1 + \sum_{k=2}^{\infty} [(1 + (k + \beta - \mu - 1)t)a_k z^{k-1}] \quad 0 \leq \mu \leq \beta, \quad t \geq 0. \quad (1.2)$$
Furthermore, we define the differential operator $D^n(\mu, \beta, t)f(z)$ such that
\[ D^0(\mu, \beta, t)f(z) = f(z) \]
\[ D^1(\mu, \beta, t)f(z) = zD_t f(z) = ztf'(z) - z(\mu - \beta)t + (1 + (\beta - \mu - 1)t)f(z) \]
\[ D^2(\mu, \beta, t)f(z) = zD_t(zD_tf(z)) = zD_t(D(\mu, \beta, t)f(z)) \] (1.3)
\[ D^n(\mu, \beta, t)f(z) = zD_t[D^{n-1}(\mu, \beta, t)f(z)] \quad n \in N_0 = N \cup \{0\} \]

If $f(z)$ is given by (1), then from the above definition of $D^n(\mu, \beta, t)f(z)$ we have that
\[ D^n(\mu, \beta, t)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]n a_k z^k \] (1.4)

$0 \leq \mu \leq \beta, \quad t \geq 0$ and $n \in N_0 = N \cup \{0\}$

It should be noted that

(i) when $\mu = \beta$ and $t = 1$, $D^n(\mu, \beta, t)f(z)$ is the Salagean differential operator (see Salagean[5]).

(ii) when $\mu = \beta$, then $D^n(\mu, \beta, t)f(z)$ is the Al-Oboudi differential operator studied in [1].

We say that a function $f(z) \in A$ is in the class $S^n(\mu, \beta, t)$ if
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad Re[D^n(\mu, \beta, t)f(z)] > \lambda \]
for $0 \leq \lambda < 1, \quad 0 \leq \mu \leq \beta, \quad t \geq 0$ and $n \in N_0 = N \cup \{0\}$

For $\mu = \beta$, the class $S^n(\mu, \beta, t)$ is the class of functions considered by Al-Oboudi in [1]. It should further be noted that if $p(z) \in S^n(\mu, \beta, t)$, then $p(0) = 1$ and
\[ Re(p(z)) > \lambda, \quad 0 \leq \lambda < 1. \text{ i.e } p(z) \in P \text{ which is the class of the Caratheodory functions.} \]

**Lemma** We shall need the following lemma to prove our results.

**Lemma 2.1** ([2], p.356 - 358). Let $u = u_1 + iu_2, \quad v = v_1 + iv_2$ and let $\psi(u, v)$ be complex valued function such that

(a) $\psi(u, v)$ is continuous in a domain $\Omega$ of $\mathbb{C}^2$.

(b) $(1, 0) \in \Omega$ and $Re\psi(1, 0) > 0$

(c) $Re\psi(\xi + (1 - \xi)u_2i, \quad v_1) \leq \xi$ when $(\xi + (1 - \xi)u_2i, \quad v_1) \in \Omega$ and $2v_1 \leq -(1 - \xi)(1 + u_2^2)$ for real $\xi, \quad 0 \leq \xi < 1$. If $p \in P$ such that $(p(z), zp'(z)) \in \Omega$ and $Re\psi(p(z), zp'(z)) > \xi$ for $z \in U$, then $Re\ p(z) > \xi$ for $z \in U$.

\[ 2. \text{ Main Results} \]

Our main results in this paper are the following

**Theorem 2.1.** Let $0 \leq \mu \leq \beta, \quad 0 \leq \alpha < 1, \quad t \geq 0$ and $n \in N = N \cup \{0\}$. Then
\[ S^{n+1}(\mu, \beta, t) \subset S^n(\mu, \beta, t) \]
**Proof:** Suppose \( f(z) \in S^{n+1}(\mu, \beta, t) \). Then by the definition of the class \( S^{n+1}(\mu, \beta, t) \) we have that
\[
Re[D^{n+1}(\mu, \beta, t)f(z)]' > \lambda , \quad 0 \leq \lambda < 1 \tag{2.1}
\]
from (1.3) we obtain that
\[
D^{n+1}(\mu, \beta, t)f(z) = zt[D^n(\mu, \beta, t)f(z)]' + (1 + (\beta - \mu - 1)t)[D^n(\mu, \beta, t)f(z)]
\]
Therefore,
\[
[D^{n+1}(\mu, \beta, t)f(z)]' = t[D^n(\mu, \beta, t)f(z)]' + zt[D^n(\mu, \beta, t)f(z)]' + (1 + (\beta - \mu - 1)t)[D^n(\mu, \beta, t)f(z)]' - (\beta - \mu)t \tag{2.2}
\]
Let
\[
p(z) = [D^n(\mu, \beta, t)f(z)]'
\]
Then, from (2.1) and (2.2) we obtain that
\[
Re[(1 + (\beta - \mu)t)p(z) + tzp'(z) - (\beta - \mu)t] > \lambda \tag{2.3}
\]
for \( 0 \leq \lambda < 1 \)
Define
\[
\psi(u, v) = (1 + (\beta - \mu)t)u + tv - (\beta - \mu)t , \quad 0 \leq \mu \leq \beta , \quad t \geq 0 \tag{2.4}
\]
Then, from (2.3)
\[
Re\psi(p(z), zp'(z)) > \lambda , \quad 0 \leq \lambda < 1 \tag{2.5}
\]
Also,
\[
\psi(1, 0) = 1 > 0
\]
\[
\psi(\lambda + (1 - \lambda)u_2i, \ v_1) = (1 + (\beta - \mu)t)(\lambda + (1 - \lambda)u_2i + tv_1 - (\beta - \mu)t
\]
and
\[
Re\psi(\lambda + (1 - \lambda)u_2i, \ v_1) = \lambda + (\beta - \mu)t\lambda + tv_1 - (\beta - \mu)t
\leq \lambda + (\beta - \mu)t\lambda - \frac{(1 - \lambda)(1 + u_2^2)}{2} - (\beta - \mu)t
\]
for \( 2v_1 \leq -(1 - \lambda)(1 + u_2^2) \), \( 0 \leq \lambda < 1 \)
i.e
\[
Re \psi(\lambda + (1 - \lambda)u_2i, v_1) \leq \lambda - \frac{(1 - \lambda)(1 + u_2^2)}{2} + (\beta - \mu)t\lambda - (\beta - \mu)t \leq \lambda
\]
for $0 \leq \lambda < 1$, \quad \nu_1 \leq \frac{(1-\lambda)(1+u_2^2)}{2} \quad 0 \leq \mu \leq \beta$ and \quad $t \geq 0$

Hence, we see that $\psi(u, v)$ as define in (2.4) satisfies all the conditions of lemma (2.1). Therefore by lemma 2.1

$$\text{Re}\; p(z) > \lambda \quad , \quad 0 \leq \lambda < 1 \quad (2.6)$$

which gives that

$$\text{Re}\; [D^n(\mu, \beta, t)f(z)]' > \lambda \quad , \quad 0 \leq \lambda < 1 \quad \text{i.e.}$$

$$f(z) \in S^n(\mu, \beta, t).$$

Therefore, $f(z) \in S^{n+1}(\mu, \beta, t) \implies f(z) \in S^n(\mu, \beta, t)$

which implies that

$$S^{n+1}(\mu, \beta, t) \subset S^n(\mu, \beta, t)$$

**Theorem 2.2.** The class $S^n(\mu, \beta, t) \subset S$ for $\leq \mu \leq \beta$, \quad $t \geq 0$ and $n \in N_0 = N \cup \{0\}$

Proof. Let $f(z) \in S^n(\mu, \beta, t)$

then by the definition of the class $S^n(\mu, \beta, t)$, $f(z)$ has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad z \in U$$

By applying theorem 2.1

$$S^n(\mu, \beta, t) \subset S^{n-1}(\mu, \beta, t) \subset \ldots \subset S^0(\mu, \beta, t)$$

This gives that

$$f(z) \in S^0(\mu, \beta, t)$$

which implies that

$$\text{Re}\; [D^0(\mu, \beta, t)f(z)]' > \lambda \quad , \quad 0 \leq \lambda < 1$$

which from (1.3) implies that

$$\text{Re}\; f'(z) > \lambda$$

It is known that if $f(z) \in A$ and $\text{Re}\; f'(z) > \lambda$, \quad $0 \leq \lambda < 1$ and $z \in U$ then $f(z)$ is univalent in $U$(see[4]).

Hence, $f(z) \in S^n(\mu, \beta, t)$ implies that $f(z)$ is univalent in $U$.

Therefore,

$$S^n(\mu, \beta, t) \subset S.$$
Theorem 2.3. Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A \)
If
\[
\sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^{n} |a_k| z^k < 1 - \lambda
\]
then \( f(z) \in S^n(\mu, \beta, t) \).

Proof: It suffices to show that
\[
|[D^n(\mu, \beta, t)f(z)]' - 1| < 1 - \lambda, \quad 0 \leq \lambda < 1
\]
We have that
\[
|[D^n(\mu, \beta, t)f(z)]' - 1| = \left| \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^{n} |a_k| z^k \right| < 1 - \lambda
\]
By the condition of the theorem
Thus, we obtain
\[
|[D^n(\mu, \beta, t)f(z)]' - 1| < 1 - \lambda
\]
and the proof is complete.

Theorem 2.4. A function \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \) is in \( S^n(\mu, \beta, t) \) if and only if
\[
\sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^{n} a_k z^{k-1} < 1 - \lambda
\]
for \( 0 \leq \lambda < 1 \)

Proof: Let \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in S^n(\mu, \beta, t), \quad a_k \geq 0 \).
Then we have that
\[
Re[D^n(\mu, \beta, t)f(z)]' > \lambda \tag{2.7}
\]
which implies that
\[
|[D^n(\mu, \beta, t)f(z)]' - 1| < 1 - \lambda \tag{2.8}
\]
\[
|(D^n(\mu, \beta, t)f(z))' - 1| = \left| \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^{n} a_k z^{k-1} \right|
\]
Thus,
\[
Re \left( \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^{n} a_k z^{k-1} \right) < 1 - \lambda \tag{2.9}
\]
Taking values of $z$ on real axis and letting $z \rightarrow 1$ through real values we have from (2.9) that

$$\sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^n a_k < 1 - \lambda$$

Conversely,

$$\left| \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^n a_k z^{k-1} \right|$$

$$\leq \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^n |a_k|$$

$$= \sum_{k=2}^{\infty} k[1 + (k + \beta - \mu - 1)t]^n a_k$$

Hence, by the condition of the theorem we have that

$$\left| [D^n(\mu, \beta, t)f(z)]' - 1 \right| < 1 - \lambda$$

which gives that

$$\text{Re}[D^n(\mu, \beta, t)f(z)]' > \lambda$$

Therefore, $f(z) \in S^n(\mu, \beta, t)$.

**NEIGHBORHOODS FOR THE CLASS $S^n(\mu, \beta, t)$**

For a function $f(z) \in S^n(\mu, \beta, t)$ and $\delta \geq 0$, the $\delta$-neighborhood of $f(z)$ is define as

$$N_\delta(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^n(\mu, \beta, t) : \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta \right\} \quad (2.10)$$

In the particular case of the identity function $e(z) = z$, we have that

$$N_\delta(e) = \left\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^n(\mu, \beta, t) : \sum_{k=2}^{\infty} k|b_k| \leq \delta \right\} \quad (2.11)$$

The concept of neighborhoods of analytic functions was first introduced by Goodman [3].

**Theorem 2.5** If

$$\delta = \frac{1 - \lambda}{[1 + (1 + \beta - \mu)t]^n} \quad (2.12)$$
then \( S^n(\mu, \beta, t) \subset N_\delta(e) \).

**Proof:** Let \( f(z) \in S^n(\mu, \beta, t) \), then from theorem (2.3) we have that
\[
\sum_{k=2}^{\infty} k [1 + (k + \beta - \mu - 1)t] \ |a_k| < 1 - \lambda
\]
which implies that
\[
[1 + (1 + \beta - \mu)t] \sum_{k=2}^{\infty} k |a_k| < 1 - \lambda
\]
i.e
\[
\sum_{k=2}^{\infty} k |a_k| < \frac{1 - \lambda}{[1 + (1 + \beta - \mu)t]^n}
\]
which by (2.11) gives that \( f(z) \in N_\delta(e) \)
Therefore, \( S^n(\mu, \beta, t) \subset N_\delta(e) \).

### 3. Conclusions

In this work, we define a differential operator which generalizes the well known Salagean and Al-Oboudi differential operators. Using the generalized differential operator we investigate the properties of a new subclass of analytic functions and showed that this new subclass of analytic functions is a class of univalent functions.

The new generalized differential operator can be used to obtain new subclasses of univalent functions and their properties.

**Competing Interests.** No competing interests.

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### References


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