International Journal of Mathematical Analysis
Vol. 11, 2017, no. 17, 825 - 831
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2017.78109

Traveling Wave Solutions for Burgers-Sharma-Tasso-Olver Equation with Variable Coefficients: The Improved tanh-coth Method vs. Exp. Function Method

Cesar A. Gómez S. and Juan C. Hernández

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Kra. 30 No. 45-03, Colombia

Abstract

In this paper, we investigate the Burgers-Sharma-Tasso-Olver equation (B-STO) with variable coefficients from the point of view of its traveling wave solutions using the improved tanh-coth method and the Exp. function method. We show that from the solutions of the B-STO equation obtained applying the first method we can derive solutions to the classical Burgers equation as well as solutions to classical Sharma-Tasso-Olver equation, both with variable coefficients.

Mathematics Subject Classification: 35C05

Keywords: Improved tanh-coth method; Exp. function method; Burgers equation; Sharma-Tasso-Olver equation; combined Burgers-Sharma-Tasso-Olver equation; variable coefficients

1 Introduction

The construction of exact solutions to nonlinear partial differential equations with variable coefficients is an important problem in mathematical physics today. For instance, we can see that in recent years, some equations with variable coefficients have been analyzed due to them are important models associated to physical phenomenons. The following equations are examples the above [1], [2], [3]

$$\begin{cases} u_t(x,t) + k_1 u^2(x,t) u_x(x,t) + u_{xxx}(x,t) = 0, \\ u_t(x,t) + k_1 t^n u^2(x,t) + k_2 t^m u_{xxx}(x,t) = 0, \\ u_t(x,t) + k_1 u^n(x,t) u_x(x,t) + u_{xxx}(x,t) = 0. \end{cases} \tag{1}$$

On the other hand, its very interesting to study generalized models which involucre important equations as particular cases, that is the case of the called Burger-Sharma-Tasso-Olver equation

$$\begin{cases} u_t(x,t) + \delta(t)[u(x,t)^3 + u(x,t)u_x(x,t) + u_{xx}(x,t)]_x \\ +\rho(t)[u(x,t)u_x(x,t) + u_{xx}(x,t)] = 0, \end{cases} \tag{2}$$

which is an integrable nonlinear evolution equation [4] obtained as a combination of the well known equations: the Burgers equation [5]

$$u_t(x,t) - 2\rho u(x,t)u_x(x,t) - \rho u_{xx}(x,t) = 0, \tag{3}$$

and the Sharma-Tasso-Olver equation [5]

$$u_t(x,t) - \delta[u(x,t)^3 + u(x,t)u_x(x,t) + u_{xx}(x,t)]_x = 0. \tag{4}$$

Respect to (2), $u = u(x,t)$ depend on x and t, and the coefficients $\delta(t)$ and $\rho(t)$ are arbitrary functions depending on the time t. In the case $\delta(t) = 1$, $\rho(t) = 0$ we obtain (3) and in the case $\delta(t) = 0$, $\rho(t) = 1$ we obtain (4).

The paper follow the sequence: In section 2., we use the improved tanh-coth method [6] to obtain exact traveling wave to Eq.(2). In Sec. 3., se obtain traveling wave solutions to (2) using the Exp. function method [7]. Finally, we compare the two methods and some conclusions are given.

2 Traveling wave solutions to (Eq.(2)) using the improved tanh-coth method

At first, we consider the transformation

$$\begin{cases} u(x,t) = v(\xi) \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{5}$$

where ξ_0 is an arbitrary constant. Applying (5) to (2) and making one first integration with respect to ξ, we obtain

$$\lambda v(\xi) + \delta(t)[v^3(\xi) + 3v(\xi)v'(\xi) + v''(\xi)] + \rho(t)[v^2(\xi) + v'(\xi)] + C = 0, \quad (6)$$

being C the integration constant. According with the improved tanh-coth method [6], we consider a solution to (6) by means of the expression

$$v(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \quad (7)$$

where M is a positive integer determined by balancing method. In the previous equation $\phi(\xi)$ is solution of the generalized Riccati equation

$$\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2, \quad (8)$$

the $a_i(t)$, $i = 1, 2, \ldots, 2M$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are functions depending only of the variable t to be determined later. The solution of (8) in the case $\beta(t)^2 - 4\alpha(t)\gamma(t) \neq 0$ is given by [8]:

$$\phi(\xi) = \frac{-\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\xi + \xi_0] - \beta(t)}{2\gamma(t)}. \quad (9)$$

Substituting (7) into (6), and balancing $v(\xi)v'(\xi)$ with $v(\xi)^3$ we obtain

$$2M + 1 = 3M,$$

so that

$$M = 1.$$

So that, (7) reduces to

$$v(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^{-1}. \quad (10)$$

with the substitution of (10) into (6) and using (8) we obtain an algebraic system in the variables $a_0(t)$, $a_1(t)$, $a_2(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, λ. However, by reasons of space, we omit here.

Solving the system with aid of Mathematica and using (10), (9) and (5) we consider only the following solutions to (2):

I:

$$\begin{cases} \gamma(t) = \frac{9\delta(t)^2\beta(t)^2\rho(t)+27\delta(t)^2C-\rho(t)^3}{36\alpha(t)\delta(t)^2\rho(t)}, \ \lambda(t) = \frac{27\delta(t)^2C+2\rho(t)^3}{9\delta(t)\rho(t)}, \\ a_0(t) = -\frac{\rho(t)}{3\delta(t)}, \ a_1(t) = \frac{-9A^2\beta(t)^2\rho(t)-27\delta(t)^2C+\rho(t)^3}{36\alpha(t)\delta(t)^2\rho(t)}, \ a_2(t) = \alpha(t). \\ u(x,t) = -\frac{\rho(t)}{3\delta(t)}+ \\ \frac{1}{2}\left(\sqrt{\frac{\rho(t)^2}{9\delta(t)^2} - \frac{3C}{\rho(t)}}\tanh\left(\frac{1}{2}\sqrt{\frac{\rho(t)^2}{9\delta(t)^2} - \frac{3C}{\rho(t)}}(x + [\frac{27\delta(t)^2C+2\rho(t)^3}{9\delta(t)\rho(t)}]t + \xi_0)\right) + \beta(t)\right) \\ -\frac{9\delta(t)^2\beta^2\rho(t)+27\delta(t)^2C-\rho(t)^3}{18\delta(t)^2\rho(t)\left(\sqrt{\frac{\rho(t)^2}{9\delta(t)^2}-\frac{3C}{\rho(t)}}\tanh\left(\frac{1}{2}\sqrt{\frac{\rho(t)^2}{9\delta(t)^2}-\frac{3C}{\rho(t)}}(x+[\frac{27\delta(t)^2C+2\rho(t)^3}{9\delta(t)\rho(t)}]t+\xi_0)\right)+\beta(t)\right)}. \end{cases}$$

Here, $\alpha(t)$, $\beta(t)$ are arbitrary functions depending on the variable t and C is an arbitrary constant.

II:

$$\begin{cases} a_1(t) = \frac{2a_0^3(t)\delta(t)+a_0^2(t)\rho(t)-C}{4\alpha(t)(2a_0(t)\delta(t)+\rho(t))}, \ a_2(t) = \alpha(t), \ \beta(t) = 0, \\ \gamma(t) = \frac{-2a_0^3(t)\delta(t)-a_0^2(t)\rho(t)+C}{4\alpha(t)(2a_0(t)\delta(t)+\rho(t))}, \ \lambda(t) = \frac{-8a_0^3(t)\delta(t)^2-8a_0^2(t)\delta(t)\rho(t)-2a_0(t)\rho(t)^2+\delta(t)C}{2a_0(t)\delta(t)+\rho(t)}, \\ u(x,t) = a_0(t)+ \\ \frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{2a_0(t)\delta(t)+\rho(t)}}\tanh\left(\frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{2a_0(t)\delta(t)+\rho(t)}}(x + \lambda t + \xi_0)\right) \\ -\frac{\left(-2a_0^3(t)\delta(t)-a_0^2(t)\rho(t)+C\right)\coth\left(\frac{1}{2}\sqrt{a_0^2(t)-\frac{C}{2a_0(t)\delta(t)+\rho(t)}}(x+\lambda t+\xi_0)\right)}{2(2a_0(t)\delta(t)+\rho(t))\sqrt{a_0^2(t)-\frac{C}{2a_0(t)\delta(t)+\rho(t)}}}, \end{cases}$$

being $\lambda = \lambda(t) = \frac{-8a_0^3(t)\delta(t)^2-8a_0^2(t)\delta(t)\rho(t)-2a_0(t)\rho(t)^2+\delta(t)C}{2a_0(t)\delta(t)+\rho(t)}$, $a_0(t)$ arbitrary function depending on variable t and C constant.

In this case, if we take $\rho(t) = 0$ for obtain traveling solutions to Sharma-Tasso-Olver equation (Eq. (4)). More exactly, one solution to Eq.(4) is given by

$$\begin{cases} u(x,t) = a_0(t)+ \\ \frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{2a_0(t)\delta(t)}} \tanh\left(\frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{2a_0(t)\delta(t)}}(x+\lambda t+\xi_0)\right) - \\ \frac{\left(C-2a_0^3(t)\delta(t)\right)\coth\left(\frac{1}{2}\sqrt{a_0^2(t)-\frac{C}{2a_0(t)\delta(t)}}(x+\lambda t+\xi_0)\right)}{4a_0(t)\delta(t)\sqrt{a_0^2(t)-\frac{C}{2a_0(t)\delta(t)}}} \end{cases}$$

with $\lambda = \frac{\delta(t)C - 8a_0^3(t)\delta(t)^2}{2a_0(t)\delta(t)}$. If we take $\delta(t) = 0$ we obtain the following solution for Burgers equation (3)

$$\begin{cases} u(x,t) = a_0(t)+ \\ \frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{\rho(t)}} \tanh\left(\frac{1}{2}\sqrt{a_0^2(t) - \frac{C}{\rho(t)}}(x+\lambda t+\xi_0)\right) - \\ \frac{\left(C-a_0^2(t)\rho(t)\right)\coth\left(\frac{1}{2}\sqrt{a_0^2(t)-\frac{C}{\rho(t)}}(x+\lambda t+\xi_0)\right)}{2\rho(t)\sqrt{a_0^2(t)-\frac{C}{\rho(t)}}}, \end{cases}$$

where $\lambda = -2\rho(t)a_0(t)$.

III:

$$\begin{cases} a_1(t) = -2\gamma(t), \;\; a_2(t) = 0, \;\; \beta(t) = \frac{-3a_0(t)\delta(t)-\rho(t)}{3\delta(t)}, \\ \alpha(t) = \frac{-3a_0^2(t)\delta(t)\rho(t)-2a_0(t)\rho(t)^2-9\delta(t)C}{6a_1\delta(t)\rho(t)}, \;\; \lambda(t) = \frac{27\delta(t)^2C+2\rho(t)^3}{9\delta(t)\rho(t)}, \\ u(x,t) = a_0(t)+ \\ \sqrt{\frac{\rho(t)^2}{9\delta(t)^2} - \frac{3C}{\rho(t)}} \tanh\left(\frac{1}{2}\sqrt{\frac{\rho(t)^2}{9\delta(t)^2} - \frac{3C}{\rho(t)}}(x+\lambda t+\xi_0)\right) - \frac{3a_0(t)\delta(t)+\rho(t)}{3\delta(t)}, \end{cases}$$

where $a_0(t)$ is an arbitrary function depending on t, C arbitrary constant and $\lambda(t) = \frac{27\delta(t)^2C+2\rho(t)^3}{9\delta(t)\rho(t)}$.

3 Traveling wave solutions to (2) using the Exp. function method

In this section, we use the Exp. function method [7] to solve (2). First, we use the wave transformation given by (5), to obtain (6). The Exp. function method use the expression

$$v(\xi)=\frac{\sum_{n=-c}^{d}a_n\exp(nk\xi)}{\sum_{n=-p}^{q}b_n\exp(nk\xi)}, \tag{11}$$

where c, d, p and q are positive integers which can be freely chosen. The a_n, b_n and k are unknown constants to be determinate later. Several cases can be considered, however, with the aim to illustrate the method and compare with the improved tanh-cont method for sake of simplicity we consider only the case $p=c=1$ and $d=q=1$. In this order the ideas, (11) converts to

$$v(\xi)=\frac{a_0(t)+a_1(t)\exp(k\xi)+a_{-1}(t)\exp(-k\xi)}{b_0(t)+b_1(t)\exp(k\xi)+b_{-1}(t)\exp(-k\xi)}. \tag{12}$$

Substituting (12) into (6) and after simplifications, we obtain an algebraic system, which by reasons of space we omit here. Solving the system with aid of the Mathematica, we obtain a lot of solutions, however, we consider only the following non trivial solution

$$\begin{cases}a_0(t)=0,\ b_0(t)=0,\ C=0,\ \ a_{-1}(t)=0,\\ b_1(t)=-\frac{3\delta(t)a_1(t)}{2\rho(t)},\ \ \lambda=\frac{2\rho(t)^2}{9\delta(t)},\ \ k=-\frac{\rho(t)}{6\delta(t)}.\end{cases} \tag{13}$$

According with this result, the solution to (2) is given by

$$u(\xi)=\frac{a_1(t)\exp(-\frac{\rho(t)}{6\delta(t)}\xi)}{-\frac{3\delta(t)a_1(t)}{2\rho(t)}\exp(-\frac{\rho(t)}{6\delta(t)}\xi)+b_{-1}(t)\exp(\frac{\rho(t)}{6\delta(t)}\xi)}, \tag{14}$$

where $\xi=x+\frac{2\rho(t)^2}{9\delta(t)}t+\xi_0$ and $a_1(t)$, $b_{-1}(t)$arbitrary functions depending on variable t (or as in the standard models can be constants).

4 Conclusions

Using the computational improved tanh-coth method and the Exp. function method, we have obtained exact solutions for the Burgers-Sharma-Tasso-Olver equation (B-STO) with variable coefficients. According with the second case (case II) we can to derive solutions for the particular cases given by (3) and (4). This is a remarkable fact due to the mathematical structure of (2) obtain great relevance. On the other hand, the Exp. function method give us a new solution (respect to obtained using the improved tanh-coth method), however, from that solutions, we cannot derive solutions to the particular cases (4 and 2).

References

[1] R.M. Miura, The Korteweg-de Vries equations and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Phys.*, **9** (1968), 1202-1204. https://doi.org/10.1063/1.1664700

[2] N. Nirmala, M.J. Vedan and B.V. Baby, Auto-Bäcklund transformation, Lax Pairs Painlevé property or a variable coefficient Korteweg-de Vries equation. I, *J. Math. Phys.*, **27** (986), 2640-2646. https://doi.org/10.1063/1.527282

[3] Zhengrong Liu, Chenxi Yang, The applications of bifurvation method to a higher-order KdV equation, *J. Math. Anal. Appl.*, **275** (2002), 1-12. https://doi.org/10.1016/s0022-247x(02)00210-x

[4] P.J. Olver, Evolution equations possessing infinite many symmetries, *J. Mathem. Phys.*, **18** (1977), 1212-1215. https://doi.org/10.1063/1.523393

[5] Song Wang, Xiao-Yan Tang, Sen-Yue Lou, Soliton fission and fussion: Burgers equation and Sharma-Tasso-Olver equation, *Chaos Solitons & Fractals,* **21** (2004), 231-239. https://doi.org/10.1016/j.chaos.2003.10.014

[6] Cesar A. Gomez S., A. H. Salas, The Cole-Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6), *Appl. Math, and Comp.*, **204** (2008), 957-962. https://doi.org/10.1016/j.amc.2008.08.006

[7] A. H. Salas S., Cesar A. Gomez S., Exact solutions for a third-order KdV equation with variable coefficients and forcing term, *Mathematical Problems in Engineering*, **2009** (2009), 1-13. https://doi.org/10.1155/2009/737928

[8] C. A. Gomez S. and A. Salas, Special symmetries to standard Riccati equations and applications, *Appl. Math. and Comp.*, **216** (2010), no. 10, 3089-3096. https://doi.org/10.1016/j.amc.2010.04.039

Received: August 18, 2017; Published: September 6, 2017