A Solution of the Two-Dimensional Zakharov-Kuznetsov Equation Using Lattice-Boltzmann and He’s Semi-Inverse Method

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Abstract
In this work we solve the time dependent Zakharov-Kuznetsov equation in two dimensions, using lattice-Boltzmann technique and a $d2q9$ lattice-velocity scheme. Also, using the He’s semi-inverse method we find several families of solutions.

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1 Introduction
The Zakharov-Kuznetsov equation (ZKEq), is a very important nonlinear partial differential equation, which rules weak ion acoustic waves in magnetohydrodynamics, [1]. In the last decades, important analytical methods known as solitary wave solutions applied to the ZKEq solution have been developed in order to find analytical solutions [2]-[3]. On the other hand, lattice-Boltzmann (LB), is a technique originated in the statistical mechanics of the non-equilibrium, which has been applied with great success, to a great variety of problems in engineering and sciences., [4]-[5]. Besides, variational methods has become an elegant tool in order to provide solitary nonlinear wave solutions [6].
2 The lattice Boltzmann model

The lattice Boltzmann equation is given by, [4]-[5]:

\[ f_j(\vec{x} + \vec{v}_j \Delta t, t + \Delta t) - f_j(\vec{x}, t) = \Omega_j(\vec{x}, t) + \omega_j(\vec{x}, t) \] (1)

The term \( \Omega_j(\vec{x}, t) \) represents the B.G.K. approximation, [7]:

\[ \Omega_j(\vec{x}, t) = -\frac{1}{\tau} \left( f_j(\vec{x}, t) - f_j^{eq}(\vec{x}, t) \right) \] (2)

Expanding in a Taylor series, the distribution functions, up to third order, are:

\[ f_j(\vec{x} + \vec{v}_j \epsilon, t + \epsilon) - f_j(\vec{x}, t) = \epsilon \left( \frac{\partial}{\partial t} + \vec{x}_j \cdot \vec{\nabla} \right) f_j \] (3) 
\[ + \frac{\epsilon^2}{2} \left( \frac{\partial}{\partial t} + \vec{x}_j \cdot \vec{\nabla} \right)^2 f_j + \frac{\epsilon^3}{6} \left( \frac{\partial}{\partial t} + \vec{x}_j \cdot \vec{\nabla} \right)^3 f_j + O(\epsilon^4) \]

Doing a perturbative expansion of the derivatives in time in powers of \( \epsilon \), we get:

\[ f_j = f_j^{(0)} + \epsilon f_j^{(1)} + \epsilon^2 f_j^{(2)} + \epsilon^3 f_j^{(3)} \] (4)

And assuming:

\[ f_j^{(0)} = f_j^{eq} \] (5)

Where the temporal scales are defined as:

\[ t_0 = t ; t_1 = \epsilon t ; t_2 = \epsilon t^2 ; t_3 = \epsilon t^3 \] (6)

And the perturbative expansion in parameter \( \epsilon \) of the temporal derivative is:

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon^1 \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \epsilon^3 \frac{\partial}{\partial t_3} \] (7)

The extra terms \( \omega_j \) [5], therefore:

\[ \omega_j = \epsilon^2 S_j \] (8)

Replacing eqs. (2)-(8) in eq. (1), we get at first \( \epsilon \), respectively, the next set of equations:

\[ \frac{\partial f_j^{(0)}}{\partial t_0} + \vec{v}_j \cdot \vec{\nabla} f_j^{(0)} = -\frac{1}{\tau} f_j^{(1)} \] (9)

At second order in \( \epsilon \)

\[ \frac{\partial f_j^{(0)}}{\partial t_1} - \tau(1 - \frac{1}{\tau}) \left( \frac{\partial}{\partial t_0} + \vec{v}_j \cdot \vec{\nabla} \right)^2 f_j = -\frac{1}{\tau} f_j^{(2)} + S_j \] (10)
3 The moments of the distributions

The moments of the distribution are:

\[ \sum_j f_j^{(0)} = \phi = \sum_j f_j^{(eq)} \] (11)

\[ \sum_j \vec{v}_j f_j^{(0)} = 0 \] (12)

\[ \Pi_{m,n}^0 = \sum_j v_{j,m} v_{j,n} f_j^{(0)} = -\lambda \left[ \frac{\partial \phi^m}{\partial x} \ 0 \ \frac{\partial \phi^m}{\partial x} \right] \] (13)

\[ \sum_j f_j^{(k)} = 0, \quad \text{if } k \geq 1 \] (14)

Where \( \phi \) is the microscopic fields and \( \delta_{ij} \) is Kronecker’s delta.

4 The construction of the Zakharov-Kuznetsov Equation

Summing on \( j \) in eq. (9), and using eq. (14), supposing an irrotational fluid, we obtain:

\[ \frac{\partial}{\partial t_0} \sum_j f_j^0 + \vec{\nabla} \cdot \sum_j \vec{v}_j f_j^0 = -\frac{1}{\tau} \sum_j f_j^1 \rightarrow \frac{\partial \phi}{\partial t_0} = 0 \] (15)

Again, doing some algebra in eq. (10), and using eqs. (11-13), we have:

\[ \epsilon \frac{\partial \phi}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \nabla_m \nabla_n \Pi_{m,n}^0 = -\frac{\epsilon}{\tau} \sum_j f_j^{(2)} + \epsilon \sum_j S_j \] (16)

Using eq. (14), then, in eq. (16)

\[ \epsilon \frac{\partial \phi}{\partial t_1} + \lambda \epsilon \tau (1 - \frac{1}{\tau}) \left( \frac{\partial^3 \phi^m}{\partial x^3} + \frac{\partial^3 \phi^m}{\partial y^2 \partial x} \right) = \epsilon \sum_j S_j \] (17)

Summing eqs. (15) and (17), and using eq. (7), we obtain:

\[ \frac{\partial \phi}{\partial t} + \lambda \epsilon \tau (1 - \frac{1}{2\tau}) \left( \frac{\partial^3 \phi^m}{\partial x^3} + \frac{\partial^3 \phi^m}{\partial y^2 \partial x} \right) = \epsilon \sum_j S_j \] (18)

If we define:
\[ \epsilon \sum_j S_j = \epsilon (b_1 + 1) S; S = -\frac{\partial \phi^n}{\partial x} - \frac{b}{a} \left( \frac{\partial^3 \phi^m}{\partial x^2 \partial y} - \frac{\partial^3 \phi^m}{\partial y^2 \partial x} \right), \]  

\[ b = \epsilon \lambda (\tau - \frac{1}{2}), a = \epsilon (b_1 + 1) \]

Then, the Zakharov-Kuznetsov equation, [1], is:

\[ \frac{\partial \phi}{\partial t} + a \frac{\partial \phi^n}{\partial x} + b \frac{\partial^3 \phi^m}{\partial x^3} + b \frac{\partial^3 \phi^m}{\partial x^2 \partial y} = 0 \]  

(20)

5 The equilibrium function in the \(d2q9\) velocity scheme.

We use the \(d2q9\) scheme shown in fig. (1), [4]-[5]. For the directions \(v_{i,j}\) and weights \(w_i\) on each cell it is assumed the next values:

\[ w_i = \begin{cases} 
\frac{4}{9}, & \text{if } i = 0; \\
\frac{1}{9}, & \text{if } i = 1, \ldots, 4; \\
\frac{1}{36}, & \text{if } i = 5, \ldots, 8 
\end{cases} \]  

(21)

Both, directions \(v_{i,j}\) and weights \(w_i\), follow the next tensorial relations:

\[ \sum_i w_i v_{i,m} = 0, \sum_i w_i v_{i,m} v_{i,n} = \frac{1}{3} \delta_{m,n}, \sum_i w_i v_{i,m} v_{i,n} v_{i,p} = 0 \]  

(22)

The equilibrium distribution functions \(f_i^{(eq)}\), is define as:

\[ f_i^{(eq)} = f_i^{(0)} = \begin{cases} 
\frac{w_i [A \vec{u} \cdot \vec{u} + B]}{w_0 D}, & \text{if } i > 0 \\
 otherwise & \text{if } i = 0 
\end{cases} \]  

(23)

Using the tensorial relations eqs. (22)

\[ B = -3 \lambda \frac{\partial \phi^m}{\partial x}; \ A = 0; \ D = \frac{9 \phi}{4} + \lambda \frac{15 \partial \phi^m}{4 \partial x} \]  

(24)
Figure 2: The spatiotemporal Lattice-Boltzmann for $\phi(x,t)$ using a $d2q9$ lattice velocity, for one initial profile given by eq. (31).

Then, the equilibrium distribution function is:

$$f_i^{(eq)} = \begin{cases} 
-w_i 3 \lambda \frac{\partial \phi^m}{\partial x} \rightarrow i > 0 \\
w_0 \left( \frac{9 \phi}{4} + \lambda \frac{15}{4} \frac{\partial \phi^m}{\partial x} \right) \rightarrow i = 0 
\end{cases}$$

(25)

6 He’s semi-inverse method, Solitary wave solution

Using the next coordinate transformation:

$$u = x + y - kt$$

(26)

$$\frac{\partial}{\partial t} = -k \frac{d}{du}; \quad \frac{\partial}{\partial x} = \frac{d}{du}; \quad \frac{\partial}{\partial y} = \frac{d}{du}; \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{du^2}; \quad \frac{\partial^2}{\partial x^3} = \frac{d^3}{du^3}$$

(27)

Then, eq. (20)

$$-k \frac{d \phi}{du} + a \frac{d \phi^m}{du} + b \frac{d^2 \phi^m}{du^2} + b \frac{d^3 \phi^m}{du^3} = 0$$

(28)

$$k_1 - k \phi + a \phi^n + 2b \frac{d^2 \phi^m}{du^2} = 0$$

(29)

According to He’s semi inverse method [6], we postulate one functional that satisfy eq. (29). So:
\[ J(\phi) = \int \left( b \left( \frac{d\phi^m}{du} \right)^2 + \frac{k}{2} \phi^2 - \frac{a}{n+1} \phi^{n+1} - k_1 \phi \right) du \]  

(30)

We have the field, \( \phi \), to be determined, and we select:

\[ \phi = p \sin(\nu u^2) \exp(-\nu u^2) \]  

(31)

Using \( m = 2 \) in eq. (29)

Defining \( A_1 = \left( \sqrt{\pi/2}(-2+2^{3/4} \cos \pi/8) \right)/8n = 3, \ A_2 = \left( \frac{1}{640} + \frac{i}{640} \right)((30-30i) - 8\sqrt{5} - 10i + 5\sqrt{-1} - i - 8\sqrt{5} - 10i) \sqrt{\pi} \), and \( A_3 = (\sqrt{\pi} \sin(\pi/8))/(2(2)^{1/4}) \).

Then, the entire action is:

\[ J(q, p) = -\frac{k p^2}{2 q^{1/2}} A_1 - \frac{a p^4}{4 q^{1/2}} A_2 - \frac{k_1 p}{q^{1/2}} A_3 \]  

(32)

\( J \) must be stationary:

\[ \frac{\partial J}{\partial p} = 0 \rightarrow a A_2 p^3 + k A_1 p + k_1 A_3 = 0 \]  

(33)

defining \( r_1 = -9a^2 A_2^2 A_3 k_1 + \sqrt{3} \sqrt{4a^3 A_1^2 A_2^3 k_1^3 + 27a^4 A_2^4 A_3^2 k_1^2} \)

\[ p_1 = -\left( \frac{3^2}{2} \right)^{1/3} A_1 k (r_1)^{1/3} + \frac{(r_1)^{1/3}}{2^{1/3} 3^{2/3} a A_2} \]  

(34)

\[ p_2 = \left( 1 + i \sqrt{3} \right) A_1 k \frac{2^{2/3} 3^{1/3} (r_1)^{1/3}}{2^{1/3} 3^{2/3} a A_2} - \left( 1 - i \sqrt{3} \right) \frac{(r_1)^{1/3}}{2^{1/3} 3^{2/3} a A_2} \]  

(35)

\[ p_3 = \left( 1 - i \sqrt{3} \right) A_1 k \frac{2^{2/3} 3^{1/3} (r_1)^{1/3}}{2^{1/3} 3^{2/3} a A_2} - \left( 1 + i \sqrt{3} \right) \frac{(r_1)^{1/3}}{2^{1/3} 3^{2/3} a A_2} \]  

(36)

\[ \frac{\partial J}{\partial q} = 0 \rightarrow a A_2 p^3 + 2 A_1 k p + 4k_1 A_3 = 0 \]  

(37)

defining \( r_2 = -9a^2 A_2^2 A_3 k_1 + \sqrt{3} \sqrt{2a^3 A_1^2 A_2^3 k_1^3 + 27a^4 A_2^4 A_3^2 k_1^2} \)

\[ p_4 = -\frac{2^{2/3} A_1 k}{3^{1/3} (r_2)^{1/3}} + \frac{(r_2)^{1/3}}{3^{2/3} a A_2} \]  

(38)
\[ p_5 = \frac{(1 + i\sqrt{3}) A_1 k}{6^{1/3} (r_2)^{1/3}} - \frac{(1 - i\sqrt{3}) (r_2)^{1/3}}{6^{2/3} a A_2} \]  

\[ p_6 = \frac{(1 - i\sqrt{3}) A_1 k}{6^{1/3} (r_2)^{1/3}} - \frac{(1 + i\sqrt{3}) (r_2)^{1/3}}{6^{2/3} a A_2} \]  

We found six families of solutions for eq. (31). Also, we select \( \phi \) as:

\[ \phi = psech(q u) \]  

Also, using \( m = 2 \) in eq. (29). Then, the entire action is:

\[ J(q, p) = \frac{8bp^4 q}{15} + \frac{kp^2}{2q} - \frac{ap^4}{6q} - \frac{k_1 p \pi}{2q} \]  

Also, \( J \) must be stationary:

\[ \frac{\partial J}{\partial p} = 0 \rightarrow \frac{(32bp^3 q^2)}{15} + \frac{kp}{q} - \frac{2ap^3}{3q} - \frac{k_1 \pi}{2q} = 0 \]  

\[ \frac{(32bp^3 q^2)}{15} + kp - \frac{2ap^3}{3} - \frac{k_1 \pi}{2} = 0 \]  

\[ \frac{\partial J}{\partial q} = 0 \rightarrow \frac{(8bp^4)}{15} - \frac{kp^2}{2q^2} + \frac{ap^4}{6q^2} + \frac{k_1 p \pi}{2q^2} = 0 \]  

\[ \frac{(8bp^4)}{15} - \frac{kp}{2} + \frac{ap^3}{6} + \frac{k_1 \pi}{2} = 0 \]

defining \( l_1 = -3a^2 k_1 \pi + \sqrt{3}\sqrt{-4a^3 k^3 + 3a^4 k_1^2 \pi^2} \)

\[ p_1 = -\left(\frac{3}{2}\right)^{2/3} k \left(\frac{3}{2}\right)^{1/3} (l_1)^{1/3} - \frac{\left(\frac{3}{2}\right)^{1/3} (l_1)^{1/3}}{2a}, \]  

\[ q_1 = -(-60k^3 + 25a k_1^2 \pi^2 + \frac{602^{2/3} 3^{1/3} a k^4}{(l_1)^{2/3}} + \frac{202^{1/3} 3^{2/3} a k_1^2 \pi}{(l_1)^{1/3}}) + 102^{2/3} 3^{1/3} k k_1 \pi \]  

\[ + 102^{2/3} 3^{1/3} k k_1 \pi \left(\frac{l_1}{a}\right)^{1/2})/(4\sqrt{3}\sqrt{b k_1 \pi}) \]
\[ p_2 = -\left(\frac{3}{2}\right)^{2/3} k \left(l_1\right)^{1/3} - \left(\frac{3}{2}\right)^{1/3} \left(l_1\right)^{1/3} \frac{k}{2a}, \]  
\[ q_2 = ((-60k^3 + 25ak^2\xi^2 + \frac{602^{2/3}3^{1/3}ak^4}{(l_1)^{2/3}} + \frac{202^{1/3}3^{2/3}ak^2k_1\pi}{(l_1)^{1/3}} + 102^{2/3}3^{1/3}kk_1\pi (l_1)^{1/3} + \frac{102^{1/3}3^{2/3}k^2 (l_1)^{2/3}}{a})(4\sqrt{3}\sqrt{b}k_1\pi) \]  
\[ p_3 = \frac{\left(\frac{3}{2}\right)^{2/3} (1 + i\sqrt{3}) k}{2 (l_1)^{1/3}} + \frac{\left(\frac{3}{2}\right)^{1/3} (1 - i\sqrt{3}) (l_1)^{1/3}}{4a}, \]  
\[ q_3 = -((-60k^3 + 25ak^2\xi^2 - \frac{302^{2/3}3^{1/3}ak^4}{(l_1)^{2/3}} + \frac{30i2^{2/3}3^{5/6}ak^4}{(l_1)^{2/3}} - \frac{30i2^{1/3}3^{1/6}ak^2k_1\pi}{(l_1)^{1/3}} - \frac{102^{1/3}3^{2/3}ak^2k_1\pi}{(l_1)^{1/3}} - 52^{2/3}3^{1/3}kk_1\pi (l_1)^{1/3} - \frac{15i2^{1/3}3^{1/6}k^2 (l_1)^{2/3}}{a} - \frac{52^{1/3}3^{2/3}k^2 (l_1)^{2/3}}{a})(4\sqrt{3}\sqrt{b}k_1\pi) \]  
\[ p_4 = \frac{\left(\frac{3}{2}\right)^{2/3} (1 + i\sqrt{3}) k}{2 (l_1)^{1/3}} + \frac{\left(\frac{3}{2}\right)^{1/3} (1 - i\sqrt{3}) (l_1)^{1/3}}{4a}, \]  
\[ q_4 = ((-60k^3 + 25ak^2\xi^2 - \frac{302^{2/3}3^{1/3}ak^4}{(l_1)^{2/3}} + \frac{30i2^{2/3}3^{5/6}ak^4}{(l_1)^{2/3}} - \frac{30i2^{1/3}3^{1/6}ak^2k_1\pi}{(l_1)^{1/3}} - \frac{102^{1/3}3^{2/3}ak^2k_1\pi}{(l_1)^{1/3}} - 52^{2/3}3^{1/3}kk_1\pi (l_1)^{1/3} - \frac{15i2^{1/3}3^{1/6}k^2 (l_1)^{2/3}}{a} - \frac{52^{1/3}3^{2/3}k^2 (l_1)^{2/3}}{a})(4\sqrt{3}\sqrt{b}k_1\pi) \]  
\[ p_5 = \frac{\left(\frac{3}{2}\right)^{2/3} (1 - i\sqrt{3}) k}{2 (l_1)^{1/3}} + \frac{\left(\frac{3}{2}\right)^{1/3} (1 + i\sqrt{3}) (l_1)^{1/3}}{4a}, \]
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\[ q_5 = -(-60k^3 + 25ak_1^2\pi^2 - \frac{302^{2/3}3^{1/3}ak^4}{(l_1)^{2/3}} - \frac{30i2^{2/3}3^{5/6}ak^4}{(l_1)^{2/3}}) \]
\[ + \frac{30i2^{1/3}3^{1/6}ak^2k_1\pi}{(l_1)^{1/3}} - \frac{102^{1/3}3^{2/3}ak^2k_1\pi}{(l_1)^{1/3}} - 52^{2/3}3^{1/3}kk_1\pi (l_1)^{1/3} \]
\[ - 5i2^{2/3}3^{5/6}kk_1\pi (l_1)^{1/3} + \frac{15i2^{1/3}3^{1/6}k^2(l_1)^{2/3}}{a} \]
\[ - \frac{5 2^{1/3}3^{2/3}k^2(l_1)^{2/3}}{a} (l_1)^{1/3}(4\sqrt{3}\sqrt{b}k_1\pi) \]

\[ p_6 = \left(\frac{3}{2}\right)^{2/3} \left(1 - i\sqrt{3}\right) k \]
\[ + \left(\frac{3}{2}\right)^{1/3} \left(1 + i\sqrt{3}\right) (l_1)^{1/3} \]
\[ \frac{4a}{2(l_1)^{1/3}} \]

\[ q_6 = ((-60k^3 + 25ak_1^2\pi^2 - \frac{302^{2/3}3^{1/3}ak^4}{(l_1)^{2/3}} - \frac{30i2^{2/3}3^{5/6}ak^4}{(l_1)^{2/3}}) \]
\[ + \frac{30i2^{1/3}3^{1/6}ak^2k_1\pi}{(l_1)^{1/3}} - \frac{102^{1/3}3^{2/3}ak^2k_1\pi}{(l_1)^{1/3}} - 52^{2/3}3^{1/3}kk_1\pi (l_1)^{1/3} \]
\[ - 5i2^{2/3}3^{5/6}kk_1\pi (l_1)^{1/3} + \frac{15i2^{1/3}3^{1/6}k^2(l_1)^{2/3}}{a} \]
\[ - \frac{5 2^{1/3}3^{2/3}k^2(l_1)^{2/3}}{a} (l_1)^{1/3}(4\sqrt{3}\sqrt{b}k_1\pi) \]

We found six families of solutions for eq. (35).

7 Conclusions

Therefore, this paper presents a solution to the Zakharov-Kuznetsov equation in two dimensions using lattice-Boltzmann and He’s semi inverse methods. Figure (2), shows the LB result given by an initial profile, eq. (29). Also, we find twelve families of solutions \((i, j = 1, ..., 6)\), using He’s semi inverse method. Furthermore, the extension to 3 dimensions is straightforward.

\[ \phi(\xi)_i = p_i \sin(qu^2) \exp(-qu^2); \quad \phi(\xi)_j = p_j \text{sech}(q_j u) \]

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