Quasi Fuzzy Neighborhood Groups and Their Relationships to Fuzzy Neighborhood Groups

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Abstract

Introducing a notion of quasi fuzzy neighborhood group, we provide conditions under which it gives rise to a fuzzy neighborhood group. In doing so, we present the notion of weakly locally symmetric in a fuzzy quasi-uniform space, and show that this notion is a good extension of its classical counterpart. We introduce a concept of quasi bi-fuzzy neighborhood group, and give connection with fuzzy quasi-uniform structures. Furthermore, we provide natural example of quasi bi-fuzzy neighborhood groups induced by left(right) invariant fuzzy quasi pseudo-metric spaces.

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1. Introduction

Following the appearance of the notion of fuzzy neighborhood systems - a localizable structure [16] within the frame work of fuzzy topological spaces [14] attributed to R. Lowen, quite a good amount of research work on the compatibility of fuzzy neighborhood systems with algebraic structures were undertaken. Among others, A. K. Katsaras [12], N. N. Morsi [22] and T. M.
G. Ahsanullah used this notion of fuzzy neighborhood system from different perspective. Katsaras introduced a notion of linear fuzzy neighborhood space and its fuzzy uniformizability, while Ahsanullah introduced a notion of fuzzy neighborhood group [1] and studied its fuzzy uniformizability. N. N. Morsi [22] studied the notion of fuzzy neighborhood system in much wider context. Furthermore, A. S. Mashhour and Morsi [21] investigated fuzzy neighborhood metric spaces and following this in [3], we discussed invariant probabilistic metrizability of fuzzy neighborhood groups. Classically, quasi-topological groups originally known as paratopological groups in the sense of Bourbaki [6] were studied by many authors (cf. [19, 20, 23]) over the years with an aim to obtain conditions under which a quasi-topological group is a topological group. With the same idea in mind, our aim in this paper is to weaken the notion of fuzzy neighborhood group by introducing a notion herein called quasi fuzzy neighborhood group, and see under what conditions it yields a fuzzy neighborhood group. In so doing, we introduce a concept of weakly locally symmetric in fuzzy quasi-uniform space, and showed that this a good extension in the sense of Lowen’s good extension criterion [15]. Moreover, we look into the relationship between quasi bi-fuzzy neighborhood groups and quasi bi-topological groups - the notions widely studied by many researchers over the the years following the introduction of bi-topological spaces by J. C. Kelly [13](see also [8]). We present here an example of quasi bi-fuzzy neighborhood group that arises naturally from left(or right)invariant fuzzy quasi pseudo-metric spaces.

2. Preliminaries

Let $X$ be a set, $I_0 = ]0, 1]$, $I_1 = [0, 1[$, and $I = [0, 1]$. A fuzzy set is an element of $I^X(= \{ \mu: X \rightarrow I \})$. We follow the definition of fuzzy topology due to R. Lowen [14]. We recall some well-known notions from [16, 17] for the convenience of the reader. If $A \subseteq X$, then its characteristic function is given by $1_A$, and in case $A = \{x\}$, we simply write $1_{\{x\}}$ or even simply by just $x$. We refer the reader to [16] and [17] for the notions of prefilter and prefilterbasis including some other related notions.

If $B$ is a prefilterbase in $I^X$, then the so-called saturation operation, denoted by $\tilde{\cdot}$ is given as $\tilde{B} = \{ \mu: X \rightarrow I; \forall \epsilon \in I_0 \exists \mu_\epsilon \in B \ni \mu_\epsilon - \epsilon \leq \mu \}$. 

**Definition 2.1.** [16] A family $\Sigma = (\Sigma(x))_{x \in X}$ of prefilters in a set $X$ is called a fuzzy neighborhood system on $X$ if the following conditions are fulfilled:

(N1) for all $x \in X$ and for all $\mu \in \Sigma(x)$, $\mu(x) = 1$.

(N2) for all $x \in X$: $\Sigma(x) = \Sigma(x)$.

(N3) for all $x \in X$, for all $\mu \in \Sigma(x)$ and for all $\epsilon \in I_0$, there exists a family $(\mu_\epsilon^z)_{z \in Z}$ such that for all $y, z \in X$, $\mu_\epsilon^z \in \Sigma(z)$ and $\mu_\epsilon^z(z) \wedge \mu_\epsilon^y(y) \leq \mu(y) + \epsilon$.

The pair $(X, \Sigma)$ is called a fuzzy neighborhood space and the elements of $\Sigma(x)$ are called fuzzy neighborhoods of $x$. It is well-known that each fuzzy neighborhood space gives rise to a fuzzy topological space $(X, \Delta = t(\Sigma))$ via a
fuzzy neighborhood base
A family of fuzzy closure operator described in [16]. For further details, see [16], pp. 169. Furthermore, note that if \((X, U)\) fulfills the following properties:

\[ \forall (x, y) \in X \times X, \nu(x) \leq \nu(y) \]

The pair \((QFU1)\) is called a **fuzzy neighborhood space**. For the level topological space \((X, \Sigma)\) associated with it is a fuzzy neighborhood space, and the fuzzy neighborhood system is given by

\[ \Sigma(x) = \{ \nu_j \land \nu_{j+1} \land \ldots \land \nu_{j+n}, \nu_j \in \Sigma_{j+1}(x), i = 1, 2, \ldots, n; n \in \mathbb{N} \} \]

**Definition 2.3.** [16] A mapping \(f : (X, \Sigma) \rightarrow (Y, \Sigma')\) between fuzzy neighborhood spaces is said to be **continuous at** \(x \in X\) if for any \(\nu' \in \Sigma'(f(x))\), \(f^{-1}(\nu') \subseteq \Sigma(x)\) or equivalently, for any \(\nu' \in \Sigma'(f(x))\) there is a \(\nu \in \Sigma(x)\) such that \(f(\nu) \leq \nu'\).

**Proposition 2.4.** [16] A mapping \(f : (X, \Sigma) \rightarrow (Y, \Sigma')\) between fuzzy neighborhood spaces is continuous at \(x \in X\) if for any \(\nu' \in \Sigma'(f(x))\), and for any \(\epsilon \in I_0\), there is a \(\nu \in \Sigma(x)\) such that \(\nu - \epsilon \leq f^{-1}(\nu')\).

**Theorem 2.5.** [16] If \((X, \Sigma)\) is a fuzzy neighborhood space and \(\alpha \in I_1\), then the level topological space \((X, \nu_{\alpha}(\Sigma))\) has as a neighborhood system the family \((\nu_{\alpha}(\Sigma(x)))_{x \in X}\) where for all \(x \in X\),

\[ \nu_{\alpha}(\Sigma(x)) = \{ \nu^{-1}[\beta, 1][\nu \in \Sigma(x), \beta \in [0, 1 - \alpha]\} \]

where \(\nu^{-1}[\beta, 1] = \{ x \in X | \nu(x) > \beta \}\).

**Theorem 2.6.** [16] If \((X, \tau)\) is a topological space, then the fuzzy topological space \((X, \omega(\tau))\) associated with it is a fuzzy neighborhood space, and the fuzzy neighborhood system is given by \((\Sigma(x))_{x \in X}\), where for all \(x \in X\)

\[ \Sigma(x) = \{ \nu \in I^X | \nu(x) = 1 \text{ and } \nu \text{ is l.s.c in } x \} \]

**Definition 2.7.** [17] A fuzzy quasi-uniformity \(U\) on \(X\) is a subset \(U \subseteq I^X \times X\) fulfilling the following properties:

- (QFU1) \(U\) is a prefilter;
- (QFU2) \(\tilde{U} = \overline{U}\);
- (QFU3) \(\forall \nu \in U\), and \(\forall x \in X\), \(\nu(x, x) = 1\);
- (QFU4) \(\forall \nu \in U\), \(\forall \epsilon > 0\), there exists \(\nu_\epsilon \in U\) such that \(\nu_\epsilon \circ \nu_\epsilon - \epsilon \leq \nu\), where \(\nu_\epsilon \circ \nu_\epsilon'(x, y) = \bigvee_{z \in X} \nu_\epsilon(x, z) \land \nu_\epsilon'(z, y)\) for any \((x, y) \in X \times X\).

The pair \((X, U)\) is called a **fuzzy quasi-uniform space**, while the pair \((X, U)\) is called a **fuzzy uniform space** if in addition to the above conditions, \(U\) satisfies

- (FU) \(\nu^{-1}(x, y) = \nu(y, x)\), for any \((x, y) \in X \times X\) and \(\nu \in U\).
3. Quasi bi-fuzzy neighborhood groups

**Definition 2.8.** [17] A subset \( B \subset I^X \times X \) is called a fuzzy quasi-uniform basis if and only if the following conditions are fulfilled:

- (QFUB1) \( B \) is a prefilterbasis;
- (QFUB2) \( \forall \beta \in B, \beta(x,x) = 1, \forall x \in X \);
- (QFUB3) \( \forall \beta \in B, \forall \epsilon > 0, \) there exists a \( \beta_\epsilon \in B \) such that \( \beta_\epsilon \circ \beta_\epsilon \epsilon \leq \beta \).

If, moreover, for any \( \beta \in B, \beta^{-1} \in B \), then \( B \) is called a fuzzy uniform basis, where \( \beta^{-1}(x,y) = \beta(y,x) \).

**Proposition 3.2.** If \( (X, \cdot, \Sigma = (\Sigma(x))_{x \in X}) \) is a fuzzy neighborhood group, then

\( (X, \cdot, \Sigma^{-1} = \Sigma^{-1}(x))_{x \in X} \) is also a fuzzy neighborhood group, where \( \Sigma^{-1} \) is known as the conjugate fuzzy neighborhood system of \( \Sigma \), as given by \( \Sigma^{-1}(x) = \{ \nu^{-1} \in I^G | \nu \in \Sigma(x) \}^\sim \), \( \forall x \in G \), whence \( \nu^{-1} : X \rightarrow I \), defined by \( \nu^{-1}(x) = \nu(x^{-1}) \).

**Definition 3.3.** A quasi bi-fuzzy neighborhood group is a quadruple \( (G, \cdot, \Sigma, \Sigma^{-1}) \), where \( (G, \cdot, \Sigma) \) is a quasi fuzzy neighborhood group and \( \Sigma^{-1} \) is the conjugate fuzzy neighborhood system of \( \Sigma \).

**Remark 3.4.** If \( (G, \cdot, \Sigma, \Sigma^{-1}) \) is a quasi bi-fuzzy neighborhood group, then the left translation by the element \( x, L_x : G \rightarrow G \) defined by \( L_x(x) = xz \), and the right translation by the element \( x, R_x : G \rightarrow G \) defined by \( R_x(x) = zx \), for any \( x \in G \), both are homeomorphisms of the fuzzy neighborhood space \( (G, \Sigma) \) onto itself. Also, they are homeomorphisms of \( (G, \Sigma^{-1}) \) onto itself, i.e., \( L_x, R_x : (G, \Sigma^{-1}) \rightarrow (G, \Sigma^{-1}) \) are homeomorphisms.

The following Propositions 3.5 and 3.6 follow almost in the similar way as in the Proposition 2.6 [1].

**Proposition 3.5.** Let \( (G, \cdot, \Sigma, \Sigma^{-1}) \) be a quasi bi-fuzzy neighborhood group, \( x \in G \) and \( \nu \in I^G \). Then the following hold:

- (a) \( \nu \in \Sigma(e) \iff L_x(\nu) \in \Sigma(x) \iff R_x(\nu) \in \Sigma(x) \);
- (b) \( \nu \in \Sigma(e) \iff L_x(\nu^{-1}) \in \Sigma^{-1}(x) \iff R_x(\nu^{-1}) \in \Sigma^{-1}(x) \);
- (c) \( \nu \in \Sigma(e) \iff L_x^{-1}(\nu) \in \Sigma(e) \iff R_x^{-1}(\nu) \in \Sigma(e) \).

**Proposition 3.6.** Let \( (G, \cdot, \Sigma, \Sigma^{-1}) \) be a quasi bi-fuzzy neighborhood group, \( x \in G \). Then the following assertions are fulfilled:

- (i) \( \Sigma'(x) = \{ L_x(\nu) | \nu \in \Sigma(e) \}^\sim = \{ R_x(\nu) | \nu \in \Sigma(e) \}^\sim \);
- (ii) \( \Sigma'^{-1}(x) = \{ L_x(\nu^{-1}) | \nu \in \Sigma(e) \}^\sim = \{ R_x(\nu^{-1}) | \nu \in \Sigma(e) \}^\sim \).
Lemma 3.7. [1] Let \((G, \cdot)\) be a group and \(\Sigma\) be a fuzzy neighborhood system on \(G\). Then

(a) The mapping \(r: G \rightarrow G, x \mapsto x^{-1}\) is continuous at \(e \in G\) if and only if for all \(\nu \in \Sigma(e)\) and for all \(\epsilon > 0\) there exists \(\mu \in \Sigma(e)\) such that \(\mu - \epsilon \leq \nu^{-1}\).
(b) The mapping \(m: G \times G \rightarrow G, (x, y) \mapsto xy\) is continuous at \((e, e) \in G \times G\) if and only if for all \(\epsilon > 0\) and for all \(\nu \in \Sigma(e)\) there exists \(\theta \in \Sigma(e)\) such that \(\theta \odot \theta'(z) = \bigvee_{s=t=z} \theta(s) \wedge \theta'(t)\) for any \(\theta, \theta' \in I^G\) and \(z \in G\).

Theorem 3.8. Let \((G, \cdot, \Sigma, \Sigma^{(-1)})\) be a quasi bi-fuzzy neighborhood group. Then

\((G, \cdot, \Sigma^*)\) is a fuzzy neighborhood group, where for any \(x \in G\),

\[\Sigma^*(x) = \Sigma(x) \vee \Sigma^{-1}(x) = \{\xi \wedge \xi^{-1} | \xi \in \Sigma(x), \xi^{-1} \in \Sigma^{-1}(x)\}^{\sim} .\]

Proof. It follows at once from the Proposition 2.2 that \(\Sigma^*\) is indeed a fuzzy neighborhood system. Need to check the continuity conditions of \(m\) and \(r\).

For, let \(x, y \in G, \nu^* \in \Sigma^*(xy)\) and \(\epsilon > 0\). Then there exists \(\nu \in \Sigma(xy)\) such that \(\nu \wedge \nu^{-1} - \frac{\epsilon}{2} \leq \nu^*\). Consequently, there are \(\nu_1 \in \Sigma(x), \nu_2 \in \Sigma(y)\) such that \(\nu_1 \odot \nu_2 - \frac{\epsilon}{2} \leq \nu\). Also, there are \(\mu_1 \in \Sigma^{-1}(x)\) and \(\mu_2 \in \Sigma^{-1}(y)\) such that \(\mu_1 \odot \mu_2 - \frac{\epsilon}{2} \leq \nu^{-1}\).

Put \(\xi_1 = \nu_1 \wedge \mu_1\) and \(\xi_2 = \nu_2 \wedge \mu_2\). Hence, one obtains:

\[\xi_1 \odot \xi_2 = (\nu_1 \wedge \mu_1) \odot (\nu_2 \wedge \mu_2)\]

\[= (\nu_1 \odot \nu_2) \wedge (\mu_1 \odot \mu_2)\]

\[\leq (\nu + \frac{\epsilon}{2}) \wedge (\nu^{-1} + \frac{\epsilon}{2})\]

\[\leq (\nu \wedge \nu^{-1}) + \frac{\epsilon}{2} \leq \nu^* + \epsilon, \text{ i.e. } \xi_1 \odot \xi_2 \leq \nu^* + \epsilon,\]

showing that the mapping \(m: (G \times G, \Sigma^* \times \Sigma^*) \rightarrow (G, \Sigma^*), (x, y) \mapsto xy\) is continuous.

Finally, to show the continuity of inversion map \(r: (G, \Sigma^*) \rightarrow (G, \Sigma^*), x \mapsto x^{-1}\), we let \(x \in X\) and \(\nu^* \in \Sigma^*(x^{-1})\) and \(\epsilon > 0\). Then there is a \(\eta \in \Sigma(e)\) such that \((x^{-1} \odot \eta) \wedge (x^{-1} \odot \eta^{-1}) - \epsilon \leq \nu^*\). Put \(\rho = \eta \odot x \wedge \eta^{-1} \odot x\), then \(r(\rho) = (x^{-1} \odot \eta) \wedge (x^{-1} \odot \eta^{-1}) \leq \nu^* + \epsilon\). This completes the proof that the triple \((G, \cdot, \Sigma^*)\) is a fuzzy neighborhood group.

Proposition 3.9. [1] The quadruple \((G, \cdot, \tau, \tau^{-1})\) is a quasi bi-topological group if and only if \((G, \cdot, \omega(\tau), \omega(\tau^{-1}))\) is a quasi bi-fuzzy neighborhood group.

Proposition 3.10. [3] A mapping \(f: (X, \Sigma) \rightarrow (Y, \Sigma')\) between fuzzy neighborhood spaces is continuous if and only if \(f: (X, \iota_\alpha(\Sigma)) \rightarrow (Y, \iota_\alpha(\Sigma'))\) is continuous between \(\alpha\)-level topologies, for all \(0 < \alpha < 1\).

The following proposition follows almost the same way as in [3, 4].

Proposition 3.11. The quadruple \((G, \cdot, \Sigma, \Sigma^{(-1)})\) is a quasi bi-fuzzy neighborhood group if and only if for all \(0 < \alpha < 1\), the \(\alpha\) level spaces \((G, \cdot, \iota_\alpha(\Sigma), \iota_\alpha(\Sigma^{(-1)}))\) are quasi bi-topological groups.

We present below conditions under which a fuzzy neighborhood system on a group generates a quasi bi-fuzzy neighborhood group, and conversely, conditions under which one has a structure of quasi bi-fuzzy neighborhood system on a given group. The proof of which can be extracted from Theorems 2.18 and 2.19 [1].
Theorem 3.12. Let \((G, \cdot)\) be a group and \(\Sigma\) a fuzzy neighborhood system on \(G\). Then the quadruple \((G, \cdot, \Sigma, \Sigma^{-1})\) is a quasi bi-fuzzy neighborhood group if and only if the following properties are fulfilled:

(a) \(\forall x \in G, \Sigma(x) = \{\mathcal{L}_x(\nu) | \nu \in \Sigma(e)\}\), where \(\mathcal{L}_x\) is a left translation by the element \(x\);

(b) \(\forall \nu \in \Sigma(e), \forall \epsilon > 0\) there exists \(\theta \in \Sigma(e)\) such that \(\theta \odot \theta - \epsilon \leq \nu\);

(c) \(\forall \nu \in \Sigma(e), \forall \epsilon > 0\) and \(\forall z \in G\) there exists \(\theta \in \Sigma(e)\) such that \(z \odot \theta \odot z^{-1} - \epsilon \leq \nu\).

Theorem 3.13. Let \((G, \cdot)\) be a group and \(\mathfrak{F}\) a family of fuzzy subsets of \(G\) such that the following hold:

(a) \(\mathfrak{F}\) is a prefilterbasis such that \(\nu(e) = 1, \forall \nu \in \mathfrak{F}\);

(b) \(\forall \nu \in \mathfrak{F}\) and \(\forall \epsilon > 0\) there exists \(\theta \in \mathfrak{F}\) such that \(\theta \odot \theta - \epsilon \leq \nu\);

(c) \(\forall \nu \in \mathfrak{F}, \forall \epsilon > 0\) and \(\forall x \in G\) there exists \(\theta \in \mathfrak{F}\) such that \(x \odot \theta^{-1} - \epsilon \leq \nu\).

Then there exists a unique fuzzy neighborhood system \(\Sigma\) such that \(\mathfrak{F}\) is a basis for the fuzzy neighborhood system \(\Sigma\) at \(e\) and \((G, \cdot, \Sigma, \Sigma^{-1})\) is a quasi bi-fuzzy neighborhood group.

Definition 3.14. [5, 10, 21] If \(\mathbb{R}^+\) denotes the set of all nonnegative real numbers, then a nonnegative fuzzy real number \(\xi\) is a descending, left continuous real map: \(\xi: \mathbb{R}^+ \rightarrow I,\) with \(\xi(0) = 1\), and infimum 0. The set of all nonnegative fuzzy real numbers is denoted by \(\mathfrak{R}(I)\).

Definition 3.15. [10, 21] A fuzzy quasi pseudo-metric (quasi pseudo-probabilistic metric) is a mapping \(d: X \times X \rightarrow \mathfrak{R}(I)\) satisfying the following two conditions:

(QPM1) \(d(x, x) = 0, \forall x \in X,\) where \(\tilde{0}(t) = 1,\) if \(t = 0\) and \(\tilde{0}(t) = 0\) if \(t > 0\);

(QPM2) \(d(x, y) \leq d(x, z) \oplus d(z, y), \forall x, y, z \in X,\) where
\[
[d(x, z) \oplus d(z, y)](\nu) = \sum_{s+t=v} d(x, z)(s) \wedge d(z, y)(t).
\]

The pair \((X, d)\) is called fuzzy quasi pseudo-metric space. If, moreover, \(d\) satisfies (PM3) \(d(x, y) = d(y, x), \forall x, y \in X,\) then the pair \((X, d)\) is a called fuzzy pseudo-metric space; and if \(d\) satisfies (M) \(\forall x \neq y \Rightarrow d(x, y) > 0,\) then the pair is called fuzzy (probabilistic) metric space.

Definition 3.16. [21] If \((X, d)\) is a fuzzy quasi pseudo-metric space, then the fuzzy open ball \(B(x; r) \in I^X,\) with center at \(x \in X\) and radius \(r > 0,\) is the fuzzy subset of \(X\) given by
\[
B(x; r)(y) = \mathbb{L}_r[d(x, y)](1 - d(x, y)(r)),\]
where for each \(r > 0, \mathbb{L}_r: \mathfrak{R}(I) \rightarrow I,\) is defined by \(\mathbb{L}_r(\eta) = 1 - \eta(r),\) for all \(\eta \in \mathfrak{R}(I)\).

Theorem 3.17. [21, 17, 10] Let \((X, d)\) be a fuzzy pseudo-metric space. Then the family
\[
\mathcal{V}_d = \{\psi_r \in I^{X \times X} | r > 0\}
\]
is a fuzzy uniform basis in \(X,\) where \(\psi_r(x, y) = \mathbb{L}_r[d(x, y)], r > 0, x, y \in X.\)

The fuzzy uniformity \(\mathcal{U}_d = \mathcal{V}_d\) is called the fuzzy metric uniformity induced by \(d.\)
Since every fuzzy uniform space gives rise to a fuzzy neighborhood space \([17]\), it is shown in \([21]\) that the pair \((X, t(d))\) is a fuzzy pseudo-metric neighborhood space, where the fuzzy neighborhood topology is obtained as \(t(U_d)\) having the fuzzy neighborhood basis given by \(\mathbb{B}_d = (\mathbb{B}_d(x))_{x \in X}\), where \(\mathbb{B}_d(x) = \{B(x : r) : r > 0\}\).

**Theorem 3.18.** [5] A mapping \(f : (X, t(d)) \rightarrow (Y, t(d'))\) between fuzzy quasi pseudo-metric spaces is continuous at \(x \in X\) if and only if for all \(\epsilon > 0\) and for all \(\delta > 0\) there exists a \(\gamma = \gamma_{x, \epsilon, \delta} > 0\) such that \(d'(f(x), f(z)) < d(x, z)(\gamma) + \delta\), for all \(z \in X\).

**Definition 3.19.** Let \((G, \cdot)\) be a group and \(d\) a fuzzy quasi pseudo-metric on \(G\). Then \(d\) is called left invariant if for all \(a, x, y \in G\), \(d(ax, ay) = d(x, y)\); right invariant if \(d(xa, yb) = d(x, y)\). Note that if \(d\) is left or right invariant, then so is \(d^{-1}\).

**Definition 3.20.** [3] Let \((G, \cdot)\) be a group. A mapping \(p : (G, \cdot) \rightarrow \mathbb{R}^*(I)\) is called a fuzzy quasi absolute valued mapping if the following holds:

- (AQV1) \(p(e) = 0\);
- (AQV2) \(p(xy) \leq p(x) \oplus p(y), \forall x, y \in G\);
- (CC) \(\forall z \in G, \forall \epsilon > 0 \text{ and } \forall \delta \in I_0 \exists \gamma = \gamma_{z, \epsilon, \delta} > 0 \text{ such that } p(zz^{-1})(\epsilon) \leq p(x)(\gamma) + \delta, \forall x \in X\).

**Theorem 3.21.** Let \((G, \cdot)\) be a group and \(p\) be a fuzzy quasi absolute valued map on \(G\). Then the mapping \(d : G \times G \rightarrow \mathbb{R}^*(I)\) defined by \(d(x, y) = p(x^{-1}y)\) is a fuzzy quasi pseudo-metric on \(G\) such that the quadruple \((G, \cdot, t(d), t(d^{-1}))\) is a quasi bi-fuzzy neighborhood group. Conversely, let \((G, \cdot)\) be a group and \(d\) be a left (or right) invariant fuzzy quasi pseudo-metric on \(G\) such that \((G, \cdot, t(d), t(d^{-1}))\) is a quasi bi-fuzzy neighborhood group. Then the mapping \(p : G \rightarrow \mathbb{R}^*(I)\) defined by \(p(x) = d(e, x)\) is a fuzzy quasi absolute valued map.

**Proof.** Clearly \(d\) is a fuzzy quasi pseudo-metric on \(G\), and hence gives rise to a conjugate fuzzy quasi-pseudo metric \(d^{-1}\) defined by \(d^{-1}(x, y) = d(y, x)\). Upon using the Theorem 3.12(a) and (c) in conjunction with (CC), one can obtain that the quadruple \((G, \cdot, t(d), t(d^{-1}))\) is a quasi bi-fuzzy neighborhood group (see Proposition 5.7 and Theorem 5.8[3] for detailed clarifications). To agree with the converse, we apply the fact that \(d^{-1}(x, y) = d(y, x)\), for any \(x, y \in G\), and the left invariant of \(d\) as well as \(d^{-1}\) together with the property (QPM2). In fact, \(p(e) = d(e, e) = 0\), which is (AQV1) and for any \(x, y \in G\), \(p(xy) = d(e, xy) = d^{-1}(xy, e) = d^{-1}(y, x^{-1}) = d(x^{-1}, y) \leq d(x^{-1}, e) \oplus d(e, y) = d(e, x) \oplus d(e, y) = p(x) \oplus p(y)\), which proves (AQV2). Finally, (CC) follows from the fact that the quadruple \((G, \cdot, t(d), t(d^{-1}))\) is already a bi-fuzzy neighborhood group.

**Theorem 3.22.** Let \((G, \cdot, \Sigma, \Sigma^{(-1)})\) be a quasi bi-fuzzy neighborhood group and let \(d\) be a left (resp. right) invariant quasi fuzzy pseudo-metric on \(G\) such that \(\Sigma = t(d)\). Then \(d\) induces the left invariant fuzzy quasi-uniformity \(\mathbb{U}_L\) (resp. right invariant fuzzy quasi-uniformity \(\mathbb{U}_R\)) for \((G, \cdot, \Sigma, \Sigma^{(-1)})\).
Proof. That $\mathcal{U}(d) = \mathcal{U}_L$ follows from Theorem 5.11[3], while $\mathcal{U}^{-1}(d) = \mathcal{U}^{-1}_L$ follows same way as in Theorem 5.11[3]. In fact, for any $(x, y) \in X \times X,$ 
\[ \psi_r^{-1}(x, y) = \psi_r(y, x) = L_r[d(y, x)] \]
= $L_r[d(e, y^{-1}x)] = \psi_r < e > (y^{-1}x)$
= $(\psi_r < e >)^{-1}(x^{-1}y) = ((\psi_r < e >)^{-1})_L(x, y),$ 
whence one obtains: $\mathcal{U}^{-1}(d) = \{(\psi_r < e >)^{-1} \mid \psi_r < e > \in \mathcal{B}(e)\} = \mathcal{U}^{-1}_L.$ \hfill \Box

4. FUZZY QUASI-UNIFORMIZABILITY OF QUASI BI-FUZZY NEIGHBORHOOD GROUPS

If $(G, \cdot, \Sigma, \Sigma'(−1))$ is a quasi bi-fuzzy neighborhood group, then it has three fuzzy quasi-uniformities: $\mathcal{U}_L$ (the left quasi-uniformity), $\mathcal{U}_R$ (the right quasi-uniformity) and $\mathcal{U}_B$ (the two-sided quasi-uniformity), which are defined as:

$\mathcal{U}_L = \{\nu_L \mid \nu \in \Sigma(e)\}^\sim, \mathcal{U}_R = \{\nu_R \mid \nu \in \Sigma(e)\}^\sim$ and $\mathcal{U}_B = \{\nu_L \land \nu_R \mid \nu \in \Sigma(e)\}^\sim$,

whence $\nu_L : G \times G \rightarrow I, (x, y) \mapsto \nu_L(x, y) = \nu(x^{-1}y)$ and $\nu_R : G \times G \rightarrow I, (x, y) \mapsto \nu_R(x, y) = \nu(yx^{-1})$.

Definition 4.1. A quasi bi-fuzzy neighborhood space $(X, \Sigma, \Sigma'(−1))$ is called fuzzy quasi-uniformizable if and only if there exists a fuzzy quasi-uniformity $\mathcal{U}$ on $X$ such that $\Sigma = t(\mathcal{U})$ and $\Sigma'(−1) = t(\mathcal{U}^{-1})$, where $\mathcal{U}^{-1} = \{\nu^{-1} \mid \nu \in \mathcal{U}\}$, and $\nu^{-1}(x, y) = \nu(y, x)$, for any $(x, y) \in X \times X$.

Theorem 4.2. Each quasi bi-fuzzy neighborhood group is fuzzy quasi-uniformizable.

Proof. Let $(G, \cdot, \Sigma, \Sigma'(−1))$ be a quasi bi-fuzzy neighborhood group. Since for any $x \in G, \mathcal{U}_L(x) = \{\nu_L < x > \mid \nu \in \Sigma(e)\}^\sim = \{\mathcal{L}_x(\nu) \mid \nu \in \Sigma(e)\}^\sim = \Sigma(x)$, it follows from Theorem 3.3[1] in conjunction with Theorem 3.12(a) that $t(\mathcal{U}_L) = \Sigma$. Hence it suffices to prove that $t(\mathcal{U}^{-1}) = \Sigma'(−1)$. Note that for any $x \in G$ and $\nu \in \Sigma(e)$, we have $\nu_L^{-1} < x > = x \circ \nu^{-1} = \mathcal{L}_x(\nu^{-1})$. Hence it follows that $t(\mathcal{U}_L^{-1}) = \Sigma'(−1)$. Thus we have shown that $t(\mathcal{U}_L) = \Sigma$ and $t(\mathcal{U}_L^{-1}) = \Sigma'(−1)$ proving that $(G, \cdot, \Sigma, \Sigma'(−1))$ is fuzzy quasi-uniformizable.

Similarly, considering right fuzzy quasi-uniformity $\mathcal{U}_R$, one can easily conclude that $t(\mathcal{U}_R) = \Sigma$ and $t(\mathcal{U}_R^{-1}) = \Sigma'(−1)$. \hfill \Box

Remark 4.3. From the Theorem 3.8, it follows that $\Sigma$ and $\Sigma^{-1}$ generates a fuzzy neighborhood group $(G, \cdot, \Sigma^*)$, where $\Sigma^* = \Sigma \lor \Sigma'(−1)$. If $\mathcal{U}_L^\gamma, \mathcal{U}_R^\gamma$ and $\mathcal{U}_B^\gamma$ denote respectively left, right and both-sided fuzzy uniformities for the fuzzy neighborhood group $(G, \cdot, \Sigma^*)$, then we obtain the following

Proposition 4.4. Let $(G, \cdot, \Sigma, \Sigma'(−1))$ be a quasi bi-fuzzy neighborhood group. Then $\mathcal{U}_L^\gamma = \mathcal{U}_L, \mathcal{U}_R^\gamma = \mathcal{U}_R$ and $\mathcal{U}_B^\gamma = \mathcal{U}_B^\gamma$.

Proof. Let $\nu \in \Sigma(e)$. Then since $(\nu \land \nu^{-1})_L = \nu_L \land \nu_L^{-1}$, we get $\mathcal{U}_L^\gamma = \mathcal{U}_L \lor \mathcal{U}_L^{-1} = \mathcal{U}_L$. Similarly, one can obtain: $\mathcal{U}_R^\gamma = \mathcal{U}_R \lor \mathcal{U}_R^{-1} = \mathcal{U}_R$, and $\mathcal{U}_B^\gamma = \mathcal{U}_B \lor \mathcal{U}_B^{-1} = \mathcal{U}_B$. \hfill \Box
Following a classical notion of weakly locally symmetric quasi-uniform space as studied in [7] and [9] (see below the Definition 4.5), we generalize this notion for fuzzy quasi-uniform spaces to characterize fuzzy neighborhood group.

**Definition 4.5.** [7, 9] A quasi-uniform space \((X, \mathcal{U})\) is called weakly locally symmetric provided if \(x \in X\) and \(U \in \mathcal{U}\), there is a symmetric entourage \(V \in \mathcal{U}\) such that \(V[x] \subset U[x]\).

If \(X\) is any set, \(\mu \in I^X\) and \(\nu \in I^{X \times X}\), then the section of \(\nu\) over \(\mu\) is defined by \(\nu < \mu > (x) = \bigvee_{y \in X} \mu(y) \land \nu(x, y)\). So, if \(\mu = 1_{\{x\}} (= x)\), then the section of \(\nu\) over \(\mu\) is given by \(\nu < x >\).

**Definition 4.6.** A fuzzy quasi-uniform space \((X, \mathcal{U})\) is called weakly locally symmetric if and only if for all \(x \in X\), for all \(\nu \in \mathcal{U}\) and for all \(\epsilon > 0\) there exists a symmetric \(\varrho \in \mathcal{U}\) such that \(\varrho < x > - \epsilon \leq \nu < x >\).

The next theorem shows that the above definition fulfills Lowen’s good extension criteria [15].

**Theorem 4.7.** A quasi-uniform space \((X, \mathcal{U})\) is weakly locally symmetric if and only if the fuzzy quasi-uniform space \((X, \omega_u(\mathcal{U}))\) is weakly locally symmetric.

**Proof.** Let \((X, \mathcal{U})\) be a weakly locally symmetric quasi-uniform space. It follows from the Theorem 3.1 [17] that \((X, \omega_u(\mathcal{U}))\) is a fuzzy quasi-uniform space. We only need to show that it is weakly locally symmetric. For, let \(\epsilon \in I_0\), \(x \in X\) and \(\nu \in \omega_u(\mathcal{U})\). Put \(\delta = 1 - \epsilon \in I_1\). Then \(\nu^{-1}[\delta, 1] \in \mathcal{U}\). So, there exists a symmetric entourage \(V \in \mathcal{U}\) such that \(V[x] \subset \nu^{-1}[\delta, 1][x]\). Set \(\eta = 1_V\), then \(\eta \in \omega_u(\mathcal{U})\), and \(\eta < x > - \epsilon = 1_V < x > - \epsilon = 1_V[x] - \epsilon \leq 1_{\nu^{-1}[\delta, 1][x]} - \epsilon\), implying that \(\eta < x > - \epsilon \leq \nu < x >\), where \(\eta\) is a symmetric entourage. Conversely, assume that \((X, \omega_u(\mathcal{U}))\) is weakly locally symmetric. To show \((X, \mathcal{U})\) is weakly locally symmetric, let \(U \in \mathcal{U}\) and \(x \in X\). Then \(1_U \in \omega_u(\mathcal{U})\). Thus for any \(\epsilon > 0\) there is a symmetric \(\rho \in \omega_u(\mathcal{U})\) such that \(\rho < x > - \epsilon \leq 1_U < x >\), i.e., \(\rho < x > - \epsilon \leq 1_U[x]\). Now if we choose \(\delta \in I_0\) such that \(0 < \epsilon + \delta < 1\), then upon using symmetry of \(\rho\), one obtains: \(\rho^{-1}[\epsilon + \delta, 1][x] \subset U[x]\), whence \(\rho^{-1}[\epsilon + \delta, 1] \in \mathcal{U}\) is a symmetric entourage. \(\square\)

**Lemma 4.8.** [2] If \((G, \cdot)\) is a group and \(\Sigma\) is a fuzzy neighborhood system on \(G\) such that \(x \mapsto yx\) and \(x \mapsto xy\) are continuous for all \(y \in G\), and such that the the inversion mapping \(r: x \mapsto x^{-1}\) is continuous at the identity, then \(x \mapsto x^{-1}\) is continuous at each \(x \in G\).

**Theorem 4.9.** Let \((G, \cdot, \Sigma)\) be a quasi fuzzy neighborhood group. Then \((G, \cdot, \Sigma)\) is a fuzzy neighborhood group if and only if any one of \(\mathcal{U}_L\), \(\mathcal{U}_R\) or \(\mathcal{U}_B\) is weakly locally symmetric.

**Proof.** If \((G, \cdot, t(\Sigma))\) is a fuzzy neighborhood group, then in view of Theorem 3.3[1], it follows that \(\mathcal{U}_L\), \(\mathcal{U}_R\) and \(\mathcal{U}_B\) are fuzzy uniformities and then by exploiting Theorem 5.1[16] one can show that the preceding fuzzy uniformity, left, right or both sided is a weakly locally symmetric. We prove the converse
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for $U_e$; assume it is weakly locally symmetric. We show that the inversion map $r: x \mapsto x^{-1}$ is continuous at $e \in G$. Let $\nu \in \Sigma(e)$ and $\epsilon > 0$. Then there exists a $\xi \in \Sigma(e)$ and a symmetric entourage $\mu$ such that $\mu < e > -\frac{\epsilon}{2} \leq \nu$ and $\xi - \frac{\epsilon}{2} \leq \mu$.

Then for any $y \in G$, $\xi(y) = \xi_L(y^{-1}, e) \leq \mu(e, y^{-1}) + \frac{\epsilon}{2} = \mu < e > (y^{-1}) + \frac{\epsilon}{2} \leq \nu(y^{-1}) + \epsilon = r^{-1}(\nu)(y) + \epsilon$, i.e. $\xi - \epsilon \leq r^{-1}(\nu)$, showing that the inversion is continuous at $e \in G$. Hence the the continuity of the inversion mapping at an arbitrary point follows from Lemma 4.8 above and this ensures that the triple $(G, \cdot, \Sigma)$ is a fuzzy neighborhood group.

□

References

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