Common Fixed Point and Best Approximation

Results for Family of $C_q$ - Commuting Mappings in

Convex Metric Space

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Abstract

In the present paper we establish some common fixed point and best approximation results for family of $C_q$ - commuting mappings in the setting of convex metric space, which unify and generalize various known results. We also furnish some suitable examples in support of the proved results.

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1. Introduction

Since the existence of common fixed point theory, various extensions of commuting mappings viz. weakly commuting mappings, compatible mappings, weakly compatible mappings, R- subweakly mappings have been appeared in the literature. In 2006, proceeding in the same direction, Al-Thagafi and Shahzad [3] defined the notion of $C_q$ - commuting mappings and proved common fixed point
results for these mappings. After that several authors utilized the concept of $C_q$–commuting mappings and extended the results of Al-Thagafi and Shahzad [3] in several possible ways.

The aim of the present paper is to establish some common fixed point and best approximation results for family of $C_q$–commuting mappings in convex metric space. The results proved in this paper generalise and extend the related results to the family of mappings.

2. Preliminaries

We now give some known definitions and standard notations which will be needed in the sequel:

**Definition 2.1**[14]. Let $(X, d)$ be a metric space. A continuous mapping $W: X \times X \times [0, 1] \rightarrow X$ is said to be a convex structure on $X$, if for all $x, y \in X$ and $\lambda \in [0, 1]$, the following condition is satisfied:

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \text{ for all } u \in X.$$  

A metric space $X$ together with this convex structure is called a convex metric space.

**Example 2.2** [14]. Banach space and each of its convex subsets are simple examples of convex metric spaces with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

**Definition 2.3** [4]. The subset $M$ of a convex metric space $(X, d)$ is said to be $q$-starshaped if there exists $q \in M$ such that $W(q, x, \lambda) \in M$ for all $x \in M$ and $\lambda \in [0, 1]$. In other words, the set $M$ is called $q$-starshaped with $q \in M$, if the segment $[q, x] = \{W(q, x, \lambda) : 0 \leq \lambda \leq 1\}$ joining $q$ to $x$, is contained in $M$ for all $x \in M$.

**Definition 2.4.** A convex metric space $X$ is said to satisfy property (I) [4] if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(W(x, z, \lambda), W(y, z, \lambda)) \leq \lambda d(x, y),$$

where $z$ is arbitrary but fixed point of $X$.

**Definition 2.5.** A continuous function $T$ from a closed convex subset $M$ of a convex metric space $X$ into itself is said to be affine on $M$ if $T(W(x, y, \lambda)) = W(Tx, Ty, \lambda)$ for all $x, y \in M$ and $\lambda \in (0, 1) \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers.

**Definition 2.6.** Let $(X, d)$ be a metric space and $S, T: X \rightarrow X$. A point $x \in X$ is called:

1. fixed point of $T$ if $Tx = x$,
2. coincidence point of the pair $(S, T)$ if $Tx = Sx$,
3. common fixed point of the pair $(S, T)$ if $x = Tx = Sx$. 
We shall denote by \( \text{Fix}(T) \) and \( C(S, T) \), the set of all fixed points of \( T \) and the set of all coincidence points of \( S \) and \( T \) respectively.

**Definition 2.7.** Let \( (X, d) \) be a metric space and \( M \) be a nonempty subset of \( X \). Let \( S, T : X \to X \). Then the mappings \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points, i.e., if \( STx = TSx \) whenever \( Sx = Tx \).

**Definition 2.8 [3].** Let \( (X, d) \) be a convex metric space, \( M \) be a \( q \)-starshaped subset of \( X \) with \( q \in \text{Fix}(S) \) and \( S, T \) be self mappings on \( X \). Then \( S \) and \( T \) are said to be \( C_q \)-commuting mappings if \( STx = TSx \) for all \( x \in C_q(S, T) \), where \( C_q(S, T) = \bigcup \{ C(S, T_k) : 0 \leq k \leq 1 \} \) and \( T_k(x) = W(Tx, q, k) \).

**Definition 2.9.** Let \( M \) be a closed subset of a metric space \( (X, d) \) and \( x \in X \). If there exists an element \( y_0 \) in \( M \) such that \( d(x, y_0) = d(x, M) \), then \( y_0 \) is called best approximation to \( x \) out of \( M \). We denote by \( P_M(x) \), the set of all best approximation to \( x \) out of \( M \).

In 2003, Singh and Tomar [13] gave the following result which is needed to prove our main result:

**Theorem A:** Let \( \{ A_i \} \), \( i = 1, 2, \ldots \), \( S \) and \( T \) be selfmaps of a metric space \( (X, d) \). If one of \( SX, TX \) or \( A_1X \) is a complete subspace of \( X \) such that

(i) \( A_1X \subseteq TX \) and \( A_1X \subseteq SX \),
(ii) \( d(A_1x, A_2y) < \varphi(M_{12}(x, y)) \) whenever \( M_{12}(x, y) > 0 \),
(iii) \( d(A_1x, A_1y) < \varphi(M_{11}(x, y)) \) whenever \( M_{11}(x, y) \neq 0 \),

where \( M_{11}(x, y) = \max\{d(Sx, Ty), d(A_1x, Sx), d(A_1y, Ty), \frac{d(A_1xTy) + d(A_1ySx)}{2} \} \) and \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an upper semicontinuous function such that \( \varphi(t) < t \) for each \( t > 0 \). Then,

(I) \( A_1 \) and \( S \) have a coincidence,
(II) \( A_1 \) and \( T \) have a coincidence,
(III) \( A_1 \) and \( S \) have a common fixed point provided that they are weakly compatible,
(IV) \( A_1 \) and \( T \) have a common fixed point provided that they are weakly compatible,
(V) \( A_1, S \) and \( T \), for each \( i \), have a unique common fixed point provided that (III) and (IV) both are true.

### 3. Main Results

Now, we prove our main result which extends and improves the results of Abbas [1], Al-Thagafi [2], Al-Thagafi and Shahzad [3], Habiniak’s [5], Hussain and Rhoades [6], O’Regan and Shahzad [8], Rhoades and Saliga [9], Singh and Tomar [13] and Shahzad [12].
Theorem 3.1: Let $M$ be a nonempty q-starshaped subset of a convex metric space $(X, d)$ with property (I) and $S$ and $T$ be affine as well as continuous self mappings of $M$. Let $\{A_i\}$, $i = 1, 2, \ldots$, be a family of self continuous mappings of $M$ satisfying the following for each $i$:

(i) $A_1(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$,
(ii) $(A_1, S)$ and $(A_i, T)$ are $C_q$ – commuting mappings with $q \in \text{Fix}(S) \cap \text{Fix}(T)$,
(iii) $d(A_1 x, A_2 y) < \varphi(M_{12} (x, y))$ whenever $M_{12} (x, y) > 0$,
(iv) $d(A_1 x, A_i y) < \varphi(M_{1i} (x, y))$ whenever $M_{1i} (x, y) \neq 0$, where

\[ M_{1i}(x, y) = \max \{d(Sx, Ty), \text{dist}(Sx, Y_q A_{1i} x), d(Ty, Y_q A_{1i} y)\}, \]

\[ \frac{1}{2}[\text{dist}(Sx, Y_q A_{1i} y) + \text{dist}(Ty, Y_q A_{1i} x)] \]

and $\varphi: R_+ \to R_+$ be an upper semicontinuous function such that $\varphi(t) < t$ for each $t > 0$. Then, $A_1, S$ and $T$ have a unique common fixed point for each $i$, provided that $\text{cl}(A_i(M))$ is compact.

Proof: For each $i \in \mathbb{N}$, define $A^n_i: M \to M$ by

\[ A^n_i x = W(A_i x, q, \lambda_n) \text{ for all } x \in M \text{ and for each } n \geq 1, \text{ where } \lambda_n \in (0, 1) \text{ with } \lim_{n \to \infty} \lambda_n = 1. \]

Then, $A^n_i$ is a self-map of $M$ for each $i \in \mathbb{N}$ and $n \geq 1$. Firstly, we prove $A^n_i(M) \subseteq T(M)$.

For this, let $y \in A^n_i(M)$. Then, $y = A^n_i x$, for some $x \in M$.

\[ \Rightarrow \quad y = A^n_i x = W(A_i x, q, \lambda_n) = W(Tz, q, \lambda_n), \text{ for some } z \in M \]

\[ \Rightarrow \quad y \in T(M) \quad \text{as } T \text{ is } q\text{-affine and } M \text{ is } q\text{-starshaped}. \]

Hence, $A^n_i(M) \subseteq T(M)$ for each $n \geq 1$. Similarly, it can be shown that for each $i \in \mathbb{N} - \{1\}$ and each $n \geq 1$, $A^n_i(M) \subseteq S(M)$, as $S$ is $q\text{-affine}$ and $M$ is $q\text{-starshaped}$. Now, since $A_1$ and $S$ are $C_q$ – commuting, $q \in \text{Fix}(S)$ and $S$ is affine, therefore, for each $x \in C(S, A^n_1) \subseteq C_q(S, A_1)$.

$S A^n_1 x = S(W(A_1 x, q, \lambda_n)) = W(SA_1 x, q, \lambda_n) = W(A_1 Sx, q, \lambda_n) = A^n_1 Sx$.

Thus, $(A^n_1, S)$ is weakly compatible for all $n$. Similarly, we can prove that the pair $(A^n_i, T)$ is weakly compatible for each $i \in \mathbb{N} - \{1\}$ and each $n \geq 1$. Also, we have

\[ d(A^n_i x, A^n_i y) = d(W(A_1 x, q, \lambda_n), W(A_i y, q, \lambda_n)) \leq \lambda_n d(A_1 x, A_i y) \]

\[ \leq \lambda_n \max \{d(Sx, Ty), \text{dist}(Sx, Y_q A_{1i} x), d(Ty, Y_q A_{1i} y), \}

\[ \frac{1}{2}[\text{dist}(Sx, Y_q A_{1i} y) + \text{dist}(Ty, Y_q A_{1i} x)] \]

\[ \leq \lambda_n \max \{d(Sx, Ty), \text{dist}(Sx, A^n_i x), \text{dist}(Ty, A^n_i y), \}

\[ \frac{1}{2}[\text{dist}(Sx, A^n_i y) + \text{dist}(Ty, A^n_i x)] \]
for each $x, y \in M$ and $0 < \lambda_n < 1$.

Now, since $\text{cl}(A_i(M))$ is compact, $\text{cl}(A_i^n(M))$ is also compact for each $i$. By Theorem A, there exists $x_n \in M$ such that $x_n = Sx_n = Tx_n = A_i^n x_n$ for each $i \in \mathbb{N}$ and for each $n \geq 1$. The compactness of $\text{cl}(A_i(M))$ implies that there exists a subsequence $\{A_i x_k\}$ of $\{A_i x_n\}$ such that $A_i x_k \to y$ as $k \to \infty$. Then the definition of $A_i^k x_k$ and convexity structure on $X$ implies that $x_k \to y$. Hence, by the continuity of $S$, $T$ and all $A_i (i \in \mathbb{N})$, we have $y$ is common fixed point of $S$, $T$ and all $A_i$. Thus, $\text{Fix}(S) \cap \text{Fix}(T) \cap (\bigcap_{i \in \mathbb{N}} \text{Fix}(A_i)) \neq \emptyset$.

Now we give an example in support of Theorem 3.1:

**Example 3.2.** Let $X = \mathbb{R}$ be usual metric space and $M = [0, 1]$. Define $A_1$, $A_i$, $S$, $T$: $M \to M$ as

$$
A_1(x) = 1 \quad \text{for all } x \in M,
$$

$$
A_i(x) = \frac{x+i}{i+1} \quad \text{for each } i \in \mathbb{N} \text{ and for all } x \in M,
$$

$$
S(x) = \frac{x+1}{2}, \quad \text{for all } x \in M,
$$

$$
T(x) = x, \quad \text{for all } x \in M.
$$

Here $A_1(M) = \{1\}$, $T(M) = [0,1]$, $S(M) = \left[\frac{1}{2}, 1\right]$ and $A_i(M) = \left[\frac{i}{i+1}, 1\right]$ for each $i \in \mathbb{N}$, so that $A_1(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i$. Besides $\text{cl}(A_i(M))$ is compact and the pairs of mappings $\{A_1, S\}$ and $\{A_i, T\}$ are $\mathcal{C}_q$–commuting for each $i \in \mathbb{N}$ and also the maps $S$ and $T$ are $q$-affine for $q = 1$. Further, the mappings $S$, $T$ and $A_i (i \in \mathbb{N})$ satisfy the above inequality. Hence all the conditions of Theorem 3.1 are satisfied. Therefore $S$, $T$ and all $A_i (i \in \mathbb{N})$ have a common fixed point and $x = 1$ is such a unique common fixed point.

**Remark 3.3:** If we take $A_i = A$ for each $i \in \mathbb{N}$, in Theorem 3.1, then we obtain the following corollary, which generalise Theorem I of Abbas [1].

**Corollary 3.4:** Let $M$ be a nonempty $q$-starshaped subset of a convex metric space $(X, d)$ with property (I) and $S$ and $T$ be affine as well as continuous self mappings of $M$. Let $A$ be self continuous mappings of $M$ with $A(M) \subseteq S(M) \cap T(M)$. Suppose that $(A, S)$ and $(A, T)$ are $\mathcal{C}_q$ – commuting mappings with $q \in \text{Fix}(S) \cap \text{Fix}(T)$ and satisfy

$$
d(Ax, Ay) < M(x, y) \quad \text{whenever } M(x, y) > 0,
$$

$$
M(x, y) = \max\{d(Sx, Ty), \text{dist}(Sx, Y_q^{Ax}), \text{dist}(Ty, Y_q^{Ay}), \frac{1}{2}\text{dist}(Sx, Y_q^{Ay}) + \text{dist}(Ty, Y_q^{Ax})\}.
$$

Then, $A$, $S$ and $T$ have a unique common fixed point provided that $\text{cl}(A(M))$ is compact.
Remark 3.5: If we take $S = T$ and $A_i = A$, for each $i \in \mathbb{N}$, in Theorem 3.1, then we obtain the following corollary, which generalise Theorem 2.1 of Al-Thagafi and Shahzad [3].

Corollary 3.6: Let $M$ be a nonempty $q$-starshaped subset of a convex metric space $(X, d)$ with property (I) and $T$ be affine as well as continuous self mappings of $M$ with $q \in \text{Fix}(T)$. Let $A$ be continuous mappings of $M$ with $A(M) \subseteq T(M)$. Suppose that $(A, T)$ is $C_q$–commuting mapping and satisfy

\[
d(Ax, Ay) < M(x, y) \text{ whenever } M(x, y) > 0,
\]

\[
M(x, y) = \max\{d(Tx, Ty), \text{dist}(Tx, Y_q^{Ax}), \text{dist}(Ty, Y_q^{Ay}), \frac{1}{2}[\text{dist}(Tx, Y_q^{Ay}) + \text{dist}(Ty, Y_q^{Ax})]\}
\]

Then, $A$ and $T$ have a unique common fixed point provided that $\text{cl}(A(M))$ is compact.

4. Best Approximation Results for family of $C_q$ - Commuting Mappings

Now, we establish best approximation results for $C_q$ - commuting mappings, which properly contains Theorem 3.1, improves and extends the results of [10, Theorem 3], [6, Theorem 2.6], [5, Theorem 8], [7, Theorem 4] and [11, Theorem 6].

Theorem 4.1: Let $M$ be a subset of a normed space $X$ and let $S$, $T$, $A_i: X \to X$ be mappings for each $i \in \mathbb{N}$ such that $u \in \text{Fix}(S) \cap \text{Fix}(T) \cap (\cap_{i \in \mathbb{N}} \text{Fix}(A_i))$ for some $u \in X$ and for each $i \in \mathbb{N}$, $A_i(\partial M \cap M) \subseteq M$. Further, suppose that $P_M(u)$ is closed and $q$-starshaped with $q \in \text{Fix}(S) \cap \text{Fix}(T)$, $S$ and $T$ are $q$-affine as well as continuous on $P_M(u)$ and $S(P_M(u)) = P_M(u) = T(P_M(u))$ and $d(A_1 x, A_1 u) \leq d(A_1 x, A_1 u)$. Moreover, if for each $i \in \mathbb{N}$, $(A_1, S)$ and $(A_i, T)$ are $C_q$ – commuting mappings and satisfy for all $x \in P_M(u) \cup \{u\}

\[
d(A_1 x, A_i y) \leq \begin{cases} d(Sx, Tu), & \text{if } y = u \\ \varphi(M_{1i}(x, y)), & \text{if } y \in P_M(u) \end{cases}
\]

where

\[
M_{1i}(x, y) = \max\{d(Sx, Ty), \text{dist}(Sx, Y_q^{A_1 x}), \text{dist}(Ty, Y_q^{A_1 y}), \frac{1}{2}[\text{dist}(Sx, Y_q^{A_1 y}) + \text{dist}(Ty, Y_q^{A_1 x})]\}
\]

Then, $P_M(u) \cap \text{Fix}(S) \cap \text{Fix}(T) \cap (\cap_{i \in \mathbb{N}} \text{Fix}(A_i)) \neq \emptyset$, provided that $\text{cl}(A_i(P_M(u)))$ is compact and $A_i$’s are continuous on $P_M(u)$.

Proof: First, we show that $A_i$’s are self maps on $P_M(u)$. For this, let $x \in P_M(u)$, then for any $k \in (0, 1)$, we have

\[
d(W(u, x, k), u) \leq kd(u, u) + (1-k) d(x, u) = (1-k)d(x, u) < \text{dist}(u, M).
\]
This implies that \( W(u, x, k) \notin M \) for any \( k \in (0, 1] \). It follows that the open line segment \( \{ W(u, x, k) : 0 < k < 1 \} \) and the set \( M \) are disjoint. Thus \( x \) is not in the interior of \( M \) and so \( x \in \partial M \cap M \). Since \( A_i(\partial M \cap M) \subset M \), \( A_iX \) must be in \( M \) for each \( n \in \mathbb{N} \).

Also, since \( S(P_M(u)) = P_M(u) \), therefore, \( Sx \in P_M(u) \). Now from the given contractive condition, we obtain
\[
d(A_1x, u) = d(A_1x, A_1u) \leq d(Sx, Tu) = d(Sx, u) = d(u, M).
\]
Thus \( A_1x \in P_M(u) \).

Again, \( d(A_1x, u) = d(A_1x, A_1u) \leq d(A_1x, A_1u) \leq d(Sx, Tu) = d(Sx, u) = d(u, M) \).

Thus, \( A_1x \in P_M(u) \), for each \( i \in \mathbb{N} \) and hence \( P_M(u) \) is \( A_i \)-invariant for each \( i \in \mathbb{N} \).

As \( S(P_M(u)) = P_M(u) = T(P_M(u)) \), therefore \( A_1(P_M(u)) \subseteq T(P_M(u)) \) and \( A_i(P_M(u)) \subseteq S(P_M(u)) \). Thus all the conditions of Theorem 3.1 are satisfied and hence \( P_M(u) \cap Fix(S) \cap Fix(T) \cap (\cap_{i \in \mathbb{N}} Fix(A_i)) \neq \emptyset \).

**Remark 4.2:** If we take \( A_i = A \) for each \( i \in \mathbb{N} \), in Theorem 4.1, then we obtain the following corollary, which generalise Theorem 5 of Abbass [1].

**Corollary 4.3:** Let \( M \) be a subset of a normed space \( X \) and let \( S, T, A : X \rightarrow X \) be mappings such that \( u \in Fix(S) \cap Fix(T) \cap Fix(A) \) for some \( u \in X \), \( A(\partial M \cap M) \subseteq M \). Further, suppose that \( P_M(u) \) is closed and \( q \)-starshaped with \( q \in Fix(S) \cap Fix(T) \), \( S \) and \( T \) are \( q \)-affine as well as continuous on \( P_M(u) \) and \( S(P_M(u)) = P_M(u) = T(P_M(u)) \). Moreover, if \( (A, S) \) and \( (A, T) \) are \( C_q \)– commuting mappings and for all \( x \in P_M(u) \cup \{ u \} \)
\[
d(Ax, Ay) \leq \begin{cases} 
   d(Sx, Tu), & \text{if } y = u \\
   M(x, y), & \text{if } y \in P_M(u) 
\end{cases}
\]
where
\[
M(x, y) = \max\{d(Sx, Ty), d(Ty, Y_q^{Ax}), d(Sx, Y_q^{Ay})\},
\]
\[
\frac{1}{2} [\text{dist}(Sx, Y_q^{Ay}) + \text{dist}(Ty, Y_q^{Ax})]
\]

Then, \( P_M(u) \cap Fix(S) \cap Fix(T) \cap Fix(A) \neq \emptyset \), provided that \( \text{cl}(A(P_M(u))) \) is compact and \( A \) is continuous on \( P_M(u) \).

**References**


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