On the Finite Sums of Reciprocal Pell Numbers

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Abstract

In this paper we present several identities related to the finite sums of reciprocal Pell numbers.

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1 Introduction

As is well known, the Fibonacci numbers $F_n$ are generated from the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \ (n \geq 2),$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Recently Ohtsuka and Nakamura [5] reported interesting properties of the Fibonacci numbers and proved Theorem 1.1 below, where $\lfloor \cdot \rfloor$ is the floor function, and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

**Theorem 1.1** For the Fibonacci numbers $F_n$, the following identities hold:

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (1)$$
Following the work of Ohtsuka and Nakamura [5], diverse results in the same direction have been reported in the literature [1], [2], [4], [6–10]. In particular, Wang and Wen [6] investigated the finite sums of reciprocal Fibonacci numbers, and proved Theorem 1.2–Theorem 1.4 below.

**Theorem 1.2** If \( n = 2 \) or \( n \geq 4 \), then

\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1} F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1} F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\] (2)

**Theorem 1.3** If \( m \geq 3 \), then

\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\] (4)

**Theorem 1.4** If \( m \geq 2 \), then

\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1} F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1} F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\] (5)

It is easily seen that the last two theorems considerably extend Theorem 1.1. Choo [2] obtained similar results for the Lucas numbers.

In this paper, we study the sums of reciprocal Pell numbers \( P_n \), which are generated from the recurrence relation

\[ P_n = 2P_{n-1} + P_{n-2} \quad (n \geq 2), \]

with initial conditions \( P_0 = 0, P_1 = 1 \). According to Holliday and Komatsu [4], the infinite sums of reciprocal Pell numbers satisfy the identities given in Theorem 1.5 below.

**Theorem 1.5** For the Pell numbers \( P_n \), the following identities hold:

\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n - P_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ P_n - P_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases}
\] (6)

\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1} P_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ 2P_{n-1} P_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\] (7)

We extend Theorem 1.5 by deriving several identities for the finite sums of reciprocal Pell numbers.
2 Main results

Lemma 2.1 ([3]) Let \( m \geq n \geq 1 \). Then
(a) \( P_{m+n} = P_{m-1}P_n + P_mP_{n+1} \).
(b) \( P_{m+1}P_n - P_mP_{n+1} = (-1)^{n-1}P_{m-n} \).

Using Lemma 2.1, we can easily prove Lemma 2.2 below.

Lemma 2.2 For the Pell numbers, the following inequalities hold.
(a) \((P_n - P_{n-1})(P_{n+1} - P_n)P_n > P_{2n+1} - P_{2n}, \ n \geq 3\).
(b) \(P_{2n+1} - P_{2n} > (P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1), \ n \geq 2\).
(c) \(P_{3n} - P_{3n-1} > (P_n - P_{n-1})P_nP_{n+1}, \ n \geq 1\).
(d) \(2P_{2n-1}P_{2n} > P_n^2P_{n+1}^2, \ n \geq 1\).

Proposition 2.1 If \( n \geq 2 \), then
\[
\sum_{k=n}^{2n} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}}.
\] (8)

Proof. Consider
\[
\frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+1} - P_n} - \frac{1}{P_n} = \frac{X_1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n},
\]
where, by Lemma 2.1(b)
\[
X_1 = 2P_{n-1}P_n - (P_n - P_{n-1})(P_{n+1} - P_n)
= 2P_{n-1}P_n - (P_n - P_{n-1})(P_n + P_{n-1})
= P_{n-1}^2 - P_{n-2}P_n
= (-1)^n.
\]

Then
\[
\sum_{k=n}^{2n} \frac{1}{P_k} = \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} + \sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k}.
\]

If \( n \in \mathbb{N}_e \), then
\[
\sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k} < 0,
\]
and so
\[
\sum_{k=n}^{2n} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} < \frac{1}{P_n - P_{n-1}}.
\]
If \( n \in \mathbb{N}_o \), then
\[
\sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k} \leq \frac{1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n} - \frac{1}{(P_{2n} - P_{2n-1})(P_{2n+1} - P_{2n})P_{2n}}.
\]
Hence
\[
\sum_{k=n}^{2n} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+1} - P_{2n}} + \frac{1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n} - \frac{1}{(P_{2n} - P_{2n-1})(P_{2n+1} - P_{2n})P_{2n}} < \frac{1}{P_n - P_{n-1}},
\]
where the last inequality follows from Lemma 2.2(a), and the proof is completed.

**Proposition 2.2** If \( n \geq 1 \), then
\[
\sum_{k=n}^{2n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1} + 1}. \tag{9}
\]

*Proof* Since (9) holds for \( n = 1 \), let \( n \geq 2 \). Consider
\[
\frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_{n+1} + 1} - \frac{1}{P_n} = \frac{X_2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)P_n},
\]
where \( X_2 = X_1 + P_{n-1} - P_{n+1} - 1 = (-1)^n - 2P_n - 1 \). Here \( X_1 \) is as defined in the proof of Proposition 2.1. Then
\[
\frac{1}{P_n} = \frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_{n+1} + 1} + \frac{2P_n + 1 - (-1)^n}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)P_n} \\
\geq \frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_{n+1} + 1} + \frac{2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)},
\]
and
\[
\sum_{k=n}^{2n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} + \frac{2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)} > \frac{1}{P_n - P_{n-1}},
\]
where the last inequality follows from Lemma 2.2(b). Hence the proof is completed.
From Proposition 2.1 and Proposition 2.2, we obtain the following result.

**Theorem 2.3** If \( n \geq 2 \), then

\[
\left( \sum_{k=n}^{2n} \frac{1}{P_k} \right)^{-1} = P_n - P_{n-1}. \tag{10}
\]

**Proposition 2.4** If \( n \geq 2 \) with \( n \in \mathbb{N}_e \) and \( m \geq n + 1 \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}}. \tag{11}
\]

**Proof.** Consider

\[
\frac{1}{P_n - P_{n-1}} - \frac{1}{P_n} - \frac{1}{P_{n+1} - P_{n+2}} = \frac{Y_1}{(P_n - P_{n-1})P_nP_{n+1}(P_{n+2} - P_{n+1})},
\]

where, by Lemma 2.1,

\[
Y_1 = (P_{n-1}P_{n+1} - P_n^2)P_{n+2} + (P_nP_{n+2} - P_{n+1}^2)P_{n-1} = (-1)^n(P_{n+2} - P_{n-1}),
\]

and so

\[
\frac{1}{P_n} + \frac{1}{P_{n+1}} = \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}} < \frac{P_{n+2} - P_{n-1}}{(P_n - P_{n-1})P_nP_{n+1}(P_{n+2} - P_{n+1})}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_e \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{m+1} - P_m} + \frac{1}{P_m} < \frac{1}{P_n - P_{n-1}}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_o \), then

\[
\sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{m+1} - P_m} < \frac{1}{P_n - P_{n-1}},
\]

and the proof is completed.

From Proposition 2.2 and Proposition 2.4, we have the following result.

**Theorem 2.5** If \( n \geq 2 \) with \( n \in \mathbb{N}_e \) and \( m \geq 2n \), then

\[
\left( \sum_{k=n}^{2n} \frac{1}{P_k} \right)^{-1} = P_n - P_{n-1}. \tag{12}
\]
Proposition 2.6 If \( n \geq 3 \) with \( n \in \mathbb{N}_o \) and \( m \geq n + 1 \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1}.
\]

(13)

Proof. Consider

\[
\frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{n+1}} - \frac{1}{P_{n+2} - P_{n+1} - 1} = Y_2 \quad \text{where} \quad Y_2 = Y_1 + (P_n + P_{n+1})(P_{n+2} - P_{n+1} + P_n - P_{n-1} - 1)
\]

with \( Y_1 \) as defined in the proof of Proposition 2.4. By Lemma 2.1, we have

\[
(P_n + P_{n+1})(P_{n+2} - P_{n+1} + P_n - P_{n-1} - 1) = (P_n - P_{n-1} - 1)P_nP_{n+1}(P_{n+2} - P_{n+1} - 1)
\]

and so

\[
Y_2 = (P_n + P_{n+1})(P_{n+2} - P_{n-1} - 2) + P_{n+1}P_{n+2} - P_{n-1}P_n - P_n - P_{n+1}.
\]

If \( n \in \mathbb{N}_o \), then

\[
Y_2 = P_{n+1}P_{n+2} - P_{n-1}P_n - P_{n+2} - 3P_n + 2,
\]

and so

\[
\frac{1}{P_n} + \frac{1}{P_{n+1}} = \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{n+2} - P_{n+1} - 1} - \frac{1}{P_{n+1}P_{n+2} - P_{n-1}P_n - P_{n+2} - 3P_n + 2} < \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{n+2} - P_{n+1} - 1}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_e \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{m+1} - P_m - 1} < \frac{1}{P_n - P_{n-1} - 1}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_o \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{m+1} - P_m - 1} + \frac{1}{P_m} < \frac{1}{P_n - P_{n-1} - 1},
\]
and the proof is completed.

**Proposition 2.7** If \( n \geq 3 \) with \( n \in \mathbb{N}_o \) and \( m \geq 3n \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}}. 
\]  

**Proof.** From the proof of Proposition 2.4,

\[
\frac{1}{P_n - P_{n-1}} - \frac{1}{P_n} - \frac{1}{P_{n+1}} = \frac{(-1)^n(P_{n+2} - P_{n-1})}{(P_n - P_{n-1})P_nP_{n+1}(P_{n+2} - P_{n+1})}, 
\]

and so

\[
\frac{1}{P_n} + \frac{1}{P_{n+1}} = \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}} + \frac{P_{n+2} - P_{n-1}}{(P_n - P_{n-1})P_nP_{n+1}(P_{n+2} - P_{n+1})}. 
\]

Then

\[
\sum_{k=n}^{3n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{3n} - P_{3n-1}} + \frac{1}{(P_n - P_{n-1})P_nP_{n+1} + \frac{1}{P_{3n}} > \frac{1}{P_n - P_{n-1}}, 
\]

where the last inequality is a result of Lemma 2.2(c). Hence

\[
\sum_{k=n}^{m} \frac{1}{P_k} \geq \sum_{k=n}^{3n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}}, 
\]

and the proof is completed.

**Theorem 2.8** If \( n \geq 1 \) with \( n \in \mathbb{N}_o \) and \( m \geq 3n \), then

\[
\left( \frac{1}{\sum_{k=n}^{m} P_k} \right)^{-1} = P_n - P_{n-1} - 1. \quad (15) 
\]

**Proof.** (15) clearly holds for \( n = 1 \). For the cases for \( n \geq 3 \) with \( n \in \mathbb{N}_o \), (15) follows from Proposition 2.6 and Proposition 2.7.

**Proposition 2.9** If \( n \geq 2 \) with \( n \in \mathbb{N}_e \) and \( m \geq n + 1 \), then

\[
\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n - 1}. \quad (16) 
\]
Proof. Consider
\[
\frac{1}{2P_{n-1}P_n} - \frac{1}{P_n^2} - \frac{1}{P_{n+1}^2} - \frac{1}{2P_{n+1}P_{n+2} - 1} = \frac{(P_n^2 + P_{n+1}^2)Z_1}{(2P_{n-1}P_n - 1)P_n^2 P_{n+1}^2 (2P_{n+1}P_{n+2} - 1)},
\]
where, by the identity \(P_{n+1}P_{n+2} - P_{n-1}P_n = 2(P_n^2 + P_{n+1}^2),\)
\[Z_1 = 4P_nP_{n+1}(P_nP_{n+1} - P_{n-1}P_n) + 2(P_{n-1}P_n + P_{n+1}P_{n+2}) - 1.
\]
Using Lemma 2.1(b), we have \(P_nP_{n+1} - P_{n-1}P_n = 2(-1)^{n-1},\) and
\[
P_{n-1}P_n + P_{n+1}P_{n+2} = (P_{n+1} - 2P_n)P_n + P_{n+1}(2P_n + P_n) = 6P_nP_{n+1} + 2(P_{n-1}P_n - P_n^2) = 6P_nP_{n+1} + 2(-1)^n.
\]
Then
\[Z_1 = 8(-1)^{n-1}P_nP_{n+1} + 12P_nP_{n+1} + 4(-1)^n - 1.
\]
If \(n \in \mathbb{N}_e,\) then \(Z_1 = 4P_nP_{n+1} + 3,\) and
\[
\frac{1}{P_n^2} + \frac{1}{P_{n+1}^2} = \frac{1}{2P_{n-1}P_n - 1} - \frac{1}{2P_{n+1}P_{n+2} - 1} - \frac{(P_n^2 + P_{n+1}^2)(4P_nP_{n+1} + 3)}{(2P_{n-1}P_n - 1)P_n^2 P_{n+1}^2 (2P_{n+1}P_{n+2} - 1)} < \frac{1}{2P_{n-1}P_n - 1} - \frac{1}{2P_{n+1}P_{n+2} - 1}.
\]
If \(m \geq n + 1\) and \(m \in \mathbb{N}_e,\) then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n - 1} - \frac{1}{2P_{n+1}P_{n+1} - 1} < \frac{1}{2P_{n-1}P_n - 1},
\]
and the proof is completed.

Proposition 2.10 If \(n \geq 2\) with \(n \in \mathbb{N}_e\) and \(m \geq 2n,\) then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n}.
\]
Proof. Consider
\[
\frac{1}{2P_{n-1}P_n} - \frac{1}{P_n^2} - \frac{1}{P_{n+1}^2} - \frac{1}{2P_{n+1}P_{n+2}} = \frac{(P_nP_{n+1} - P_{n-1}P_{n+2})(P_n^2 + P_{n+1}^2)}{P_{n-1}P_n^2 P_{n+1}^2 P_{n+2}} = 2(-1)^{n-1}(P_n^2 + P_{n+1}^2)\frac{1}{P_{n-1}P_n^2 P_{n+1}^2 P_{n+2}}.
\]
If $n \in \mathbb{N}_e$, then
\[
\frac{1}{P_n^2} + \frac{1}{P_{n+1}^2} = \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{n+1}P_{n+2}} + \frac{2(P_n^2 + P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}P_{n+2}} > \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{n+1}P_{n+2}} + \frac{2}{P_n^2P_{n+1}},
\]
and
\[
\sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{2n-1}P_{2n}} + \frac{1}{P_{2n}^2} + \frac{2}{P_n^2P_{2n+1}} > 0,
\]
where the last inequality follows from Lemma 2.2(d). Hence, if $m \geq 2n$, then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} \geq \sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n},
\]
and the proof is completed.

From Proposition 2.9 and Proposition 2.10, we obtain the following result.

**Theorem 2.11** If $n \geq 2$ with $n \in \mathbb{N}_e$ and $m \geq 2n$, then
\[
\left\lfloor \left( \sum_{k=n}^{m} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = 2P_{n-1}P_n - 1. \tag{18}
\]

**Proposition 2.12** If $n \geq 3$ with $n \in \mathbb{N}_o$ and $m \geq n + 1$, then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n}. \tag{19}
\]

**Proof.** From the proof of Proposition 2.10,
\[
\frac{1}{2P_{n-1}P_n} - \frac{1}{P_n^2} - \frac{1}{P_{n+1}^2} - \frac{1}{2P_{n+1}P_{n+2}} = \frac{2(-1)^{n-1}(P_n^2 + P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}P_{n+2}}.
\]
If $n \in \mathbb{N}_o$, then
\[
\frac{1}{P_n^2} + \frac{1}{P_{n+1}^2} = \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{n+1}P_{n+2}} - \frac{2(P_n^2 + P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}P_{n+2}} < \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{n+1}P_{n+2}}.
\]
If $m \geq n + 1$ and $m \in \mathbb{N}_e$, then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_mP_{m+1}} < \frac{1}{2P_{n-1}P_n}.
\]
If \( m \geq n + 1 \) and \( m \in \mathbb{N}_o \), then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{m-1}P_m} + \frac{1}{P_m^2} \leq \frac{1}{2P_{n-1}P_n},
\]
and the proof is completed.

**Proposition 2.13** If \( n \geq 1 \) with \( n \in \mathbb{N}_o \) and \( m \geq 2n \), then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} \geq \frac{1}{2P_{n-1}P_n + 1}.
\]

**Proof.** Consider
\[
\frac{1}{2P_{n-1}P_n + 1} \sum_{k=n}^{m} \frac{1}{P_k^2} = \sum_{k=n}^{m} \frac{(P_n^2 + P_{n+1}^2)Z_2}{2P_{n-1}P_n + 1 + (2P_{n-1}P_n + 1)P_n^2P_{n+1}(2P_{n+1}P_{n+2}+1)};
\]
where, by the identity \( P_{n+1}P_{n+2} - P_{n-1}P_n = 2(P_n^2 + P_{n+1}^2) \),
\[
Z_2 = 4P_nP_{n+1}(P_nP_{n+1} - P_{n-1}P_{n+2}) - 2(P_{n-1}P_n + P_{n+1}P_{n+2}) - 1
= 8(-1)^{n-1}P_nP_{n+1} - 12P_nP_{n+1} - 4(-1)^n - 1.
\]

If \( n \in \mathbb{N}_o \), then \( Z_2 = -4P_nP_{n+1} + 3 \), and
\[
\frac{1}{P_n^2} + \frac{1}{P_{n+1}^2} = \frac{1}{2P_{n-1}P_n + 1} - \frac{1}{2P_{n+1}P_{n+2} + 1} + \frac{(P_n^2 + P_{n+1}^2)(4P_nP_{n+1} - 3)}{(2P_{n-1}P_n + 1)(2P_{n+1}P_{n+2}+1)}.
\]
Hence
\[
\sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n + 1} - \frac{1}{2P_{2n}P_{2n+1}} + \frac{(P_n^2 + P_{n+1}^2)(2P_{n+1}P_{n+2}+1)}{P_{n+1}^2P_{n+2}^2(2P_{n+1}P_{n+2}+1)}.
\]
From Lemma 2.1(a), we have \( P_{2n} = P_{n-1}P_n + P_nP_{n+1} \) and \( P_{2n+1} = P_n^2 + P_{n+1}^2 \). Using these identities, it can be shown that, for \( n \geq 1 \)
\[
2P_{2n}P_{2n+1}(P_n^2 + P_{n+1}^2) > P_n^2P_{n+1}(2P_{n+1}P_{n+2} + 1).
\]
If \( m \geq 2n \), then
\[
\sum_{k=n}^{m} \frac{1}{P_k^2} \geq \sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n + 1},
\]
and the proof is completed.
Theorem 2.14 If $n \geq 1$ with $n \in \mathbb{N}_0$ and $m \geq 2n$, then

$$\left\lfloor \left( \sum_{k=n}^{m} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = 2P_{n-1}P_n.$$  \hspace{1cm} (21)

Proof. (21) clearly holds for $n = 1$. For the cases where $n \geq 3$ with $n \in \mathbb{N}_0$ and $m \geq 2n$, the result follows from Proposition 2.12 and Proposition 2.13.

References


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