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On the Finite Sums of Reciprocal Pell Numbers

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Abstract

In this paper we present several identities related to the finite sums of reciprocal Pell numbers.

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1 Introduction

As is well known, the Fibonacci numbers F_n are generated from the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \ \ (n \geq 2),$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Recently Ohtsuka and Nakamura [5] reported interesting properties of the Fibonacci numbers and proved Theorem 1.1 below, where $\lfloor \cdot \rfloor$ is the floor function, and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Theorem 1.1 *For the Fibonacci numbers F_n, the following identities hold:*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, \text{ if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, \text{ if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{2}$$

Following the work of Ohtsuka and Nakamura [5], diverse results in the same direction have been reported in the literature [1], [2], [4], [6–10]. In particular, Wang and Wen [6] investigated the finite sums of reciprocal Fibonacci numbers, and proved Theorem 1.2–Theorem 1.4 below.

Theorem 1.2 *If $n = 2$ or $n \geq 4$, then*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_n - F_{n-1}. \tag{3}$$

Theorem 1.3 *If $m \geq 3$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{4}$$

Theorem 1.4 *If $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{5}$$

It is easily seen that the last two theorems considerably extend Theorem 1.1. Choo [2] obtained similar results for the Lucas numbers.

In this paper, we study the sums of reciprocal Pell numbers P_n, which are generated from the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \quad (n \geq 2),$$

with initial conditions $P_0 = 0$, $P_1 = 1$. According to Holliday and Komatsu [4], the infinite sums of reciprocal Pell numbers satisfy the identities given in Theorem 1.5 below.

Theorem 1.5 *For the Pell numbers P_n, the following identities hold:*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n - P_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ P_n - P_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{6}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1}P_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ 2P_{n-1}P_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{7}$$

We extend Theorem 1.5 by deriving several identities for the finite sums of reciprocal Pell numbers.

2 Main results

Lemma 2.1 ([3]) *Let* $m \geq n \geq 1$. *Then*
(a) $P_{m+n} = P_{m-1}P_n + P_m P_{n+1}$.
(b) $P_{m+1}P_n - P_m P_{n+1} = (-1)^{n-1} P_{m-n}$.

Using Lemma 2.1, we can easily prove Lemma 2.2 below.

Lemma 2.2 For the Pell numbers, the following inequalities hold.
(a) $(P_n - P_{n-1})(P_{n+1} - P_n)P_n > P_{2n+1} - P_{2n}, \ \ n \geq 3$.
(b) $P_{2n+1} - P_{2n} > (P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1), \ \ n \geq 2$.
(c) $P_{3n} - P_{3n-1} > (P_n - P_{n-1})P_n P_{n+1}, \ \ n \geq 1$
(d) $2P_{2n-1}P_{2n} > P_n^2 P_{n+1}^2, \ \ n \geq 1$.

Proposition 2.1 *If* $n \geq 2$, *then*

$$\sum_{k=n}^{2n} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}}. \tag{8}$$

Proof. Consider

$$\frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+1} - P_n} - \frac{1}{P_n} = \frac{X_1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n},$$

where, by Lemma 2.1(b)

$$\begin{aligned} X_1 &= 2P_{n-1}P_n - (P_n - P_{n-1})(P_{n+1} - P_n) \\ &= 2P_{n-1}P_n - (P_n - P_{n-1})(P_n + P_{n-1}) \\ &= P_{n-1}^2 - P_{n-2}P_n \\ &= (-1)^n. \end{aligned}$$

Then

$$\sum_{k=n}^{2n} \frac{1}{P_k} = \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} + \sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k}.$$

If $n \in \mathbb{N}_e$, then

$$\sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k} < 0,$$

and so

$$\sum_{k=n}^{2n} \frac{1}{P_n} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} < \frac{1}{P_n - P_{n-1}}.$$

If $n \in \mathbb{N}_o$, then

$$\sum_{k=n}^{2n} \frac{(-1)^{k-1}}{(P_k - P_{k-1})(P_{k+1} - P_k)P_k} < \frac{1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n} - \frac{1}{(P_{2n} - P_{2n-1})(P_{2n+1} - P_{2n})P_{2n}}.$$

Hence

$$\begin{aligned}\sum_{k=n}^{2n} \frac{1}{P_k} &< \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} + \frac{1}{(P_n - P_{n-1})(P_{n+1} - P_n)P_n} \\ &\quad - \frac{1}{(P_{2n} - P_{2n-1})(P_{2n+1} - P_{2n})P_{2n}} \\ &< \frac{1}{P_n - P_{n-1}},\end{aligned}$$

where the last inequality follows from Lemma 2.2(a), and the proof is completed.

Proposition 2.2 *If $n \geq 1$, then*

$$\sum_{k=n}^{2n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1} + 1}. \tag{9}$$

Proof Since (9) holds for $n = 1$, let $n \geq 2$. Consider

$$\frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_n + 1} - \frac{1}{P_n} = \frac{X_2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)P_n},$$

where $X_2 = X_1 + P_{n-1} - P_{n+1} - 1 = (-1)^n - 2P_n - 1$. Here X_1 is as defined in the proof of Proposition 2.1. Then

$$\begin{aligned}\frac{1}{P_n} &= \frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_n + 1} + \frac{2P_n + 1 - (-1)^n}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)P_n} \\ &\geq \frac{1}{P_n - P_{n-1} + 1} - \frac{1}{P_{n+1} - P_n + 1} + \frac{2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)},\end{aligned}$$

and

$$\sum_{k=n}^{2n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{2n+1} - P_{2n}} + \frac{2}{(P_n - P_{n-1} + 1)(P_{n+1} - P_n + 1)} > \frac{1}{P_n - P_{n-1}},$$

where the last inequality follows from Lemma 2.2(b). Hence the proof is completed.

From Proposition 2.1 and Proposition 2.2, we obtain the following result.

Theorem 2.3 *If $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1}. \tag{10}$$

Proposition 2.4 *If $n \geq 2$ with $n \in \mathbb{N}_e$ and $m \geq n+1$, then*

$$\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}}. \tag{11}$$

Proof. Consider

$$\frac{1}{P_n - P_{n-1}} - \frac{1}{P_n} - \frac{1}{P_{n+1}} - \frac{1}{P_{n+2} - P_{n+1}} = \frac{Y_1}{(P_n - P_{n-1})P_n P_{n+1}(P_{n+2} - P_{n+1})},$$

where, by Lemma 2.1,

$$Y_1 = (P_{n-1}P_{n+1} - P_n^2)P_{n+2} + (P_n P_{n+2} - P_{n+1}^2)P_{n-1} = (-1)^n(P_{n+2} - P_{n-1}),$$

and so

$$\begin{aligned} \frac{1}{P_n} + \frac{1}{P_{n+1}} &= \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}} - \frac{P_{n+2} - P_{n-1}}{(P_n - P_{n-1})P_n P_{n+1}(P_{n+2} - P_{n+1})} \\ &< \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}}. \end{aligned}$$

If $m \geq n+1$ and $m \in \mathbb{N}_e$, then

$$\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_m - P_{m-1}} + \frac{1}{P_m} < \frac{1}{P_n - P_{n-1}}.$$

If $m \geq n+1$ and $m \in \mathbb{N}_o$, then

$$\sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{m+1} - P_m} < \frac{1}{P_n - P_{n-1}},$$

and the proof is completed.

From Proposition 2.2 and Proposition 2.4, we have the following result.

Theorem 2.5 *If $n \geq 2$ with $n \in \mathbb{N}_e$ and $m \geq 2n$, then*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1}. \tag{12}$$

Proposition 2.6 *If $n \geq 3$ with $n \in \mathbb{N}_o$ and $m \geq n+1$, then*

$$\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1}. \tag{13}$$

Proof. Consider

$$\begin{aligned} & \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_n} - \frac{1}{P_{n+1}} - \frac{1}{P_{n+2} - P_{n+1} - 1} \\ = \quad & \frac{Y_2}{(P_n - P_{n-1} - 1)P_n P_{n+1}(P_{n+2} - P_{n+1} - 1)}, \end{aligned}$$

where $Y_2 = Y_1 + (P_n + P_{n+1})(P_{n+2} - P_{n+1} + P_n - P_{n-1} - 1)$ with Y_1 as defined in the proof of Proposition 2.4. By Lemma 2.1, we have

$$\begin{aligned} & (P_n + P_{n+1})(P_{n+2} - P_{n+1} + P_n - P_{n-1} - 1) \\ = \quad & P_n P_{n+2} - P_{n+1}^2 + P_n^2 - P_{n-1}P_{n+1} + P_{n+1}P_{n+2} - P_{n-1}P_n - P_n - P_{n+1} \\ = \quad & 2(-1)^{n-1} + P_{n+1}P_{n+2} - P_{n-1}P_n - P_n - P_{n+1}, \end{aligned}$$

and so

$$Y_2 = (-1)^n (P_{n+2} - P_{n-1} - 2) + P_{n+1}P_{n+2} - P_{n-1}P_n - P_n - P_{n+1}.$$

If $n \in \mathbb{N}_o$, then

$$Y_2 = P_{n+1}P_{n+2} - P_{n-1}P_n - P_{n+2} - 3P_n + 2,$$

and so

$$\begin{aligned} \frac{1}{P_n} + \frac{1}{P_{n+1}} \quad = \quad & \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{n+2} - P_{n+1} - 1} \\ & - \frac{P_{n+1}P_{n+2} - P_{n-1}P_n - P_{n+2} - 3P_n + 2}{(P_n - P_{n-1} - 1)P_n P_{n+1}(P_{n+2} - P_{n+1} - 1)} \\ < \quad & \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{n+2} - P_{n+1} - 1}. \end{aligned}$$

If $m \geq n+1$ and $m \in \mathbb{N}_e$, then

$$\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_{m+1} - P_m - 1} < \frac{1}{P_n - P_{n-1} - 1}.$$

If $m \geq n+1$ and $m \in \mathbb{N}_o$, then

$$\sum_{k=n}^{m} \frac{1}{P_k} < \frac{1}{P_n - P_{n-1} - 1} - \frac{1}{P_m - P_{m-1} - 1} + \frac{1}{P_m} < \frac{1}{P_n - P_{n-1}},$$

and the proof is completed.

Proposition 2.7 *If $n \geq 3$ with $n \in \mathbb{N}_o$ and $m \geq 3n$, then*

$$\sum_{k=n}^{m} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}}. \tag{14}$$

Proof. From the proof of Proposition 2.4,

$$\frac{1}{P_n - P_{n-1}} - \frac{1}{P_n} - \frac{1}{P_{n+1}} - \frac{1}{P_{n+2} - P_{n+1}} = \frac{(-1)^n (P_{n+2} - P_{n-1})}{(P_n - P_{n-1}) P_n P_{n+1} (P_{n+2} - P_{n+1})},$$

and so

$$\begin{aligned} \frac{1}{P_n} + \frac{1}{P_{n+1}} &= \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}} + \frac{P_{n+2} - P_{n-1}}{(P_n - P_{n-1}) P_n P_{n+1} (P_{n+2} - P_{n+1})} \\ &> \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{n+2} - P_{n+1}} + \frac{1}{(P_n - P_{n-1}) P_n P_{n+1}}. \end{aligned}$$

Then

$$\sum_{k=n}^{3n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}} - \frac{1}{P_{3n} - P_{3n-1}} + \frac{1}{(P_n - P_{n-1}) P_n P_{n+1}} + \frac{1}{P_{3n}} > \frac{1}{P_n - P_{n-1}},$$

where the last inequality is a result of Lemma 2.2(c). Hence

$$\sum_{k=n}^{m} \frac{1}{P_k} \geq \sum_{k=n}^{3n} \frac{1}{P_k} > \frac{1}{P_n - P_{n-1}},$$

and the proof is completed.

Theorem 2.8 *If $n \geq 1$ with $n \in \mathbb{N}_o$ and $m \geq 3n$, then*

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1} - 1. \tag{15}$$

Proof. (15) clearly holds for $n = 1$. For the cases for $n \geq 3$ with $n \in \mathbb{N}_o$, (15) follows from Proposition 2.6 and Proposition 2.7.

Proposition 2.9 *If $n \geq 2$ with $n \in \mathbb{N}_e$ and $m \geq n + 1$, then*

$$\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1} P_n - 1}. \tag{16}$$

Proof. Consider

$$\frac{1}{2P_{n-1}P_n-1}-\frac{1}{P_n^2}-\frac{1}{P_{n+1}^2}-\frac{1}{2P_{n+1}P_{n+2}-1}=\frac{(P_n^2+P_{n+1}^2)Z_1}{(2P_{n-1}P_n-1)P_n^2P_{n+1}^2(2P_{n+1}P_{n+2}-1)},$$

where, by the identity $P_{n+1}P_{n+2}-P_{n-1}P_n=2(P_n^2+P_{n+1}^2)$,

$$Z_1=4P_nP_{n+1}(P_nP_{n+1}-P_{n-1}P_{n+2})+2(P_{n-1}P_n+P_{n+1}P_{n+2})-1.$$

Using Lemma 2.1(b), we have $P_nP_{n+1}-P_{n-1}P_{n+2}=2(-1)^{n-1}$, and

$$\begin{aligned}P_{n-1}P_n+P_{n+1}P_{n+2} &= (P_{n+1}-2P_n)P_n+P_{n+1}(2P_{n+1}+P_n)\\ &= 6P_nP_{n+1}+2(P_{n-1}P_{n+1}-P_n^2)\\ &= 6P_nP_{n+1}+2(-1)^n.\end{aligned}$$

Then

$$Z_1=8(-1)^{n-1}P_nP_{n+1}+12P_nP_{n+1}+4(-1)^n-1.$$

If $n\in\mathbb{N}_e$, then $Z_1=4P_nP_{n+1}+3$, and

$$\begin{aligned}\frac{1}{P_n^2}+\frac{1}{P_{n+1}^2} &= \frac{1}{2P_{n-1}P_n-1}-\frac{1}{2P_{n+1}P_{n+2}-1}-\frac{(P_n^2+P_{n+1}^2)(4P_nP_{n+1}+3)}{(2P_{n-1}P_n-1)P_n^2P_{n+1}^2(2P_{n+1}P_{n+2}-1)}\\ &< \frac{1}{2P_{n-1}P_n-1}-\frac{1}{2P_{n+1}P_{n+2}-1}.\end{aligned}$$

If $m\geq n+1$ and $m\in\mathbb{N}_e$, then

$$\sum_{k=n}^{m}\frac{1}{P_k^2}<\frac{1}{2P_{n-1}P_n-1}-\frac{1}{2P_{m-1}P_m-1}+\frac{1}{P_m^2}<\frac{1}{2P_{n-1}P_n-1}.$$

If $m\geq n+1$ and $m\in\mathbb{N}_o$, then

$$\sum_{k=n}^{m}\frac{1}{P_k^2}<\frac{1}{2P_{n-1}P_n-1}-\frac{1}{2P_mP_{m+1}-1}<\frac{1}{2P_{n-1}P_n-1},$$

and the proof is completed.

Proposition 2.10 *If $n\geq 2$ with $n\in\mathbb{N}_e$ and $m\geq 2n$, then*

$$\sum_{k=n}^{m}\frac{1}{P_k^2}>\frac{1}{2P_{n-1}P_n}. \tag{17}$$

Proof. Consider

$$\begin{aligned}\frac{1}{2P_{n-1}P_n}-\frac{1}{P_n^2}-\frac{1}{P_{n+1}^2}-\frac{1}{2P_{n+1}P_{n+2}} &= \frac{(P_nP_{n+1}-P_{n-1}P_{n+2})(P_n^2+P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}^2P_{n+2}}\\ &= \frac{2(-1)^{n-1}(P_n^2+P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}^2P_{n+2}}.\end{aligned}$$

If $n \in \mathbb{N}_e$, then

$$\begin{aligned}\frac{1}{P_n^2}+\frac{1}{P_{n+1}^2} &= \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_{n+1}P_{n+2}}+\frac{2(P_n^2+P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}^2P_{n+2}} \\ &> \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_{n+1}P_{n+2}}+\frac{2}{P_n^2P_{n+1}^2},\end{aligned}$$

and

$$\sum_{k=n}^{2n}\frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_{2n-1}P_{2n}}+\frac{1}{P_{2n}^2}+\frac{2}{P_n^2P_{n+1}^2} > 0,$$

where the last inequality follows from Lemma 2.2(d). Hence, if $m \geq 2n$, then

$$\sum_{k=n}^{m}\frac{1}{P_k^2} \geq \sum_{k=n}^{2n}\frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n},$$

and the proof is completed.

From Proposition 2.9 and Proposition 2.10, we obtain the following result.

Theorem 2.11 *If $n \geq 2$ with $n \in \mathbb{N}_e$ and $m \geq 2n$, then*

$$\left\lfloor \left(\sum_{k=n}^{m}\frac{1}{P_k^2}\right)^{-1} \right\rfloor = 2P_{n-1}P_n - 1. \tag{18}$$

Proposition 2.12 *If $n \geq 3$ with $n \in \mathbb{N}_o$ and $m \geq n+1$, then*

$$\sum_{k=n}^{m}\frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n}. \tag{19}$$

Proof. From the proof of Proposition 2.10,

$$\frac{1}{2P_{n-1}P_n}-\frac{1}{P_n^2}-\frac{1}{P_{n+1}^2}-\frac{1}{2P_{n+1}P_{n+2}} = \frac{2(-1)^{n-1}(P_n^2+P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}^2P_{n+2}}.$$

If $n \in \mathbb{N}_o$, then

$$\begin{aligned}\frac{1}{P_n^2}+\frac{1}{P_{n+1}^2} &= \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_{n+1}P_{n+2}}-\frac{2(P_n^2+P_{n+1}^2)}{P_{n-1}P_n^2P_{n+1}^2P_{n+2}} \\ &< \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_{n+1}P_{n+2}}.\end{aligned}$$

If $m \geq n+1$ and $m \in \mathbb{N}_e$, then

$$\sum_{k=n}^{m}\frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n}-\frac{1}{2P_mP_{m+1}} < \frac{1}{2P_{n-1}P_n}.$$

If $m \geq n+1$ and $m \in \mathbb{N}_o$, then

$$\sum_{k=n}^{m} \frac{1}{P_k^2} < \frac{1}{2P_{n-1}P_n} - \frac{1}{2P_{m-1}P_m} + \frac{1}{P_m^2} < \frac{1}{2P_{n-1}P_n},$$

and the proof is completed.

Proposition 2.13 *If $n \geq 1$ with $n \in \mathbb{N}_o$ and $m \geq 2n$, then*

$$\sum_{k=n}^{m} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n + 1}. \tag{20}$$

Proof. Consider

$$\frac{1}{2P_{n-1}P_n+1} - \frac{1}{P_n^2} - \frac{1}{P_{n+1}^2} - \frac{1}{2P_{n+1}P_{n+2}+1} = \frac{(P_n^2+P_{n+1}^2)Z_2}{(2P_{n-1}P_n+1)P_n^2P_{n+1}^2(2P_{n+1}P_{n+2}+1)},$$

where, by the identity $P_{n+1}P_{n+2} - P_{n-1}P_n = 2(P_n^2 + P_{n+1}^2)$,

$$\begin{aligned} Z_2 &= 4P_nP_{n+1}(P_nP_{n+1} - P_{n-1}P_{n+2}) - 2(P_{n-1}P_n + P_{n+1}P_{n+2}) - 1 \\ &= 8(-1)^{n-1}P_nP_{n+1} - 12P_nP_{n+1} - 4(-1)^n - 1. \end{aligned}$$

If $n \in \mathbb{N}_0$, then $Z_2 = -4P_nP_{n+1} + 3$, and

$$\begin{aligned} \frac{1}{P_n^2} + \frac{1}{P_{n+1}^2} &= \frac{1}{2P_{n-1}P_n + 1} - \frac{1}{2P_{n+1}P_{n+2} + 1} + \frac{(P_n^2+P_{n+1}^2)(4P_nP_{n+1}-3)}{(2P_{n-1}P_n+1)P_n^2P_{n+1}^2(2P_{n+1}P_{n+2}+1)} \\ &> \frac{1}{2P_{n-1}P_n + 1} - \frac{1}{2P_{n+1}P_{n+2} + 1} + \frac{(P_n^2 + P_{n+1}^2)}{P_n^2P_{n+1}^2(2P_{n+1}P_{n+2} + 1)}. \end{aligned}$$

Hence

$$\sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n + 1} - \frac{1}{2P_{2n}P_{2n+1}} + \frac{(P_n^2 + P_{n+1}^2)}{P_n^2P_{n+1}^2(2P_{n+1}P_{n+2} + 1)}.$$

From Lemma 2.1(a), we have $P_{2n} = P_{n-1}P_n + P_nP_{n+1}$ and $P_{2n+1} = P_n^2 + P_{n+1}^2$. Using these identities, it can be shown that, for $n \geq 1$

$$2P_{2n}P_{2n+1}(P_n^2 + P_{n+1}^2) > P_n^2P_{n+1}^2(2P_{n+1}P_{n+2} + 1).$$

If $m \geq 2n$, then

$$\sum_{k=n}^{m} \frac{1}{P_k^2} \geq \sum_{k=n}^{2n} \frac{1}{P_k^2} > \frac{1}{2P_{n-1}P_n + 1},$$

and the proof is completed.

Theorem 2.14 *If $n \geq 1$ with $n \in \mathbb{N}_o$ and $m \geq 2n$, then*

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = 2P_{n-1}P_n. \tag{21}$$

Proof. (21) clearly holds for $n = 1$. For the cases where $n \geq 3$ with $n \in \mathbb{N}_o$ and $m \geq 2n$, the result follows from Proposition 2.12 and Proposition 2.13.

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