Analytical Solutions for Systems of Fredholm IDEs by Using Modified Taylor's Optimization

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Abstract

In this work, a numerical approach based on the modified Taylor's technique for derive the analytical approximate solutions is applied for systems of Fredholm integro-differential equations. The solution is calculated in the form of a rapidly convergent series with easily computable components using symbolic computation software. The results obtained depend on Taylor’s series expansions and they reproduce to the exact solutions when the solutions are polynomials. Numerical examples are presented and discussed quantitatively to illustrate the method. The results show the potentiality, the generality, and the superiority of our algorithm for solving such systems.

Keywords: Integro-differential equations, Taylor’s series expansions, Numerical solutions, Fredholm type

1. Introduction

System of Fredholm integro-differential equations (FIDEs) are very important branch of mathematics, which are arising frequently in many area of pure and applied mathematics, engineering and physics such as gas dynamics, nuclear physics, chemical reactions, atomic structures, atomic calculations, and so forth. Mostly, Fredholm IDEs do not always have solutions, which can be obtained using analytical methods.
The purpose of this work is to extend the applications of the residual power series method to construct the solutions for systems of FIDEs of the following general form:

\[ x_i'(t) = F_i(t, x_1(t), x_2(t), ..., x_n(t)) + \int_{t_0}^{b} K_i(t, s) G_i(x_1(s), x_2(s), ..., x_n(s)) ds, \]  

(1)

with the initial conditions

\[ x_i(t_0) = \alpha_i, \quad i = 1, 2, ..., n, \]  

(2)

where \( t \in [t_0, b], \ t_0, b \in \mathbb{R} \) with \( b > 0 \), \( F_i: [t_0, t_0 + a] \times \mathbb{R}^n \rightarrow \mathbb{R}, \ G_i: \mathbb{R}^n \rightarrow \mathbb{R} \) are analytical functions, \( K_i(t, \tau) \) are continuous arbitrary kernel functions over the domain of interest, and \( x_i(t) \) are unknown functions to be determined.

Mathematical approximate techniques are widely used by scientists and engineers to handle real world problems. A major advantage for numerical techniques is that a numerical answer can be obtained even when a problem has no analytical solution. The main advantage of the RPS method is the simplicity in computing the coefficients of terms of the series solutions by using only the differential operators [1-8]. Moreover, the proposed method can be easily applied in the spaces of higher dimension solutions without any limitation on the nature of the systems. On the other hand, many applications for different problems by using other numerical algorithms can be found in [9-17].

In this paper, we use the RPSM to develop a new numerical method for obtaining smooth approximations to solutions and their derivatives for systems of Fredholm IDEs. This paper is organized as follows. In Section 2, a short description for the RPS is presented. In Section 3, we discuss the problem of the study. In Section 4, we present some numerical results. Finally, conclusions are given in Section 5.

2. The modified Taylor’s technique

In this section, we construct solutions to such systems subject to given initial conditions, by substituting their RPS expansion among their truncated residual functions. From the resulting equations recursion formulas for the computation of the coefficients are derived, while the coefficients in the expansions can be computed recursively by recurrent differentiating of the truncated residual functions by means of the symbolic computation software used [18-25].

The RPS technique is different from the traditional higher order Taylor series approach. The Taylor series approach is computationally expensive for large orders. By using residual error concept, we get series solutions, in practice truncated series solutions [26-33]. To apply the residual power series method, set the counter \( i = 1, 2, ..., n \) and rewrite the system of IDEs (1) and (2) in the form of the following:

\[ x_i'(t) = F_i(t, x_1(t), x_2(t), ..., x_n(t)) + \int_{t_0}^{b} K_i(t, s) G_i(x_1(s), x_2(s), ..., x_n(s)) ds, \]

with the initial conditions

\[ x_i(t_0) = \alpha_i, \]
The RPS technique consists in expressing the solutions of IDE (1) and (2) as a PS expansion about the initial point \( t = t_0 \). To achieve our goal, let the solutions have the form

\[
x_i(t) = \sum_{m=0}^{\infty} x_{i,m}(t), \tag{3}
\]

where \( x_{i,m} \) are terms of approximations and are given as \( x_{i,m}(t) = c_{i,m}(t - t_0)^m \).

If we choose \( x_{i,0}(t) = x_i(t_0) \) as initial guesses approximations of \( x_i(t) \), then we can calculate \( x_{i,m}(t) \) for \( m = 1, 2, \ldots \) and approximate the solutions \( x_i(t) \) of IDE (1) and (2) by the \( k \)th-truncated series

\[
x_i^k(t) = \sum_{m=0}^{k} c_{i,m}(t - t_0)^m. \tag{4}
\]

Prior to applying the RPS technique, we rewrite IDEs (1) and (2) in the form:

\[
x_i'(t) - F_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - \int_{t_0}^{b} K_i(t, s)G_i(x_1(s), x_2(s), \ldots, x_n(s))ds = 0. \tag{5}
\]

The subsisting of \( k \)th-truncated series \( x_i^k(t) \) into equation (5) leads to the following definition for the \( k \)th residual functions:

\[
\text{Res}_i^k(t) = \sum_{m=2}^{k} m(m-1)c_{i,m}(t - t_0)^{m-2}
- F_i\left(t, \sum_{m=0}^{k} c_{1,m}(t - t_0)^m, \sum_{m=0}^{k} c_{2,m}(t - t_0)^m, \ldots, \sum_{m=0}^{k} c_{n,m}(t - t_0)^m\right)
- \int_{t_0}^{b} K_i(t, \tau)G_i\left(\sum_{m=0}^{k} c_{1,m}(\tau - t_0)^m, \sum_{m=0}^{k} c_{2,m}(\tau - t_0)^m, \ldots, \sum_{m=0}^{k} c_{n,m}(\tau - t_0)^m\right)d\tau, \tag{6}
\]

and the following \( \infty \)th residual functions: \( \text{Res}_i^\infty(t) = \lim_{k \to \infty} \text{Res}_i^k(t) \).

To obtain the 1st-approximate solutions, we put \( k = 1 \) and substitute \( t = t_0 \) into equation (6) and using the fact that \( \text{Res}_i^\infty(t_0) = \text{Res}_i^1(t_0) = 0 \), to conclude the following:

\[
c_{i,1} = F_i(t_0, c_{1,0}, c_{2,0}, \ldots, c_{n,0}) = F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n).
\]

Thus, using 1\(^{\text{st}}\)-truncated series, the 1\(^{\text{st}}\) approximation for (1) and (2) can be written by

\[
x_i^1(t) = x_i(t_0) + F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n)(t - t_0). \tag{7}
\]

Similarly, to find the 2\(^{\text{nd}}\) approximation, we put \( k = 2 \) and \( x_i^2(t) = \sum_{m=0}^{2} c_{i,m}(t - t_0)^m \). Now, differentiate both sides of (7) with respect to \( t \) and substitute \( t = t_0 \), to get
\[
\frac{d}{dt} \text{Res}_i^2(t_0) = 2c_{i,2} - \frac{\partial}{\partial t} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) - \sum_{j=1}^{n} c_{i,1} \frac{\partial}{\partial x_j} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) - K_i(t_0, t_0) G_i(\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{8}
\]

In fact \(\frac{d}{dt} \text{Res}_i^2(t_0) = \frac{d}{dt} \text{Res}_i^\infty(t_0) = 0\). Thus, one can write

\[
c_{i,2} = \frac{1}{2} \left[ \frac{\partial}{\partial t} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + K_i(t_0, t_0) G_i(\alpha_1, \alpha_2, \ldots, \alpha_n) \right]. \tag{9}
\]

Hence, using 2nd-truncated series, the 2nd approximation of IDEs (1) and (2) can be written as

\[
x_i^2(t) = x_i(t_0) + F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n)(t-t_0) + \frac{1}{2} \left[ \frac{\partial}{\partial t} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + K_i(t_0, t_0) G_i(\alpha_1, \alpha_2, \ldots, \alpha_n) \right](t-t_0)^2. \tag{10}
\]

This procedure can be repeated till the arbitrary order coefficients of the RPS solutions of FIDEs (1) and (2) are obtained [34-44].

### 3. Existence and Uniqueness of Solutions

In this section, the RPSM is used to seek the solution for system of FIDEs. The solution is constructed by substituting their RPS expansion among their truncated residual functions. From the resulting equations; recursion formulas for the computation of the coefficients are derived, while the coefficients in the expansions can be computed recursively by recurrent differentiating of the truncated residual functions by means of the symbolic computation software used.

**Theorem 1**: Suppose that \(x_i(t)\) are the exact solutions of IDEs (1) and (2). Then, the approximate solutions obtained by the residual power series technique are just the Taylor expansion of \(x_i(t)\) about \(t = t_0\).

**Proof.** Set the counter \(i = 1, 2, \ldots, n\) and assume that the approximate solutions of IDE (1) and (2) are as follows:

\[
\bar{x}_i(t) = c_{i,0} + c_{i,1}(t-t_0) + c_{i,2}(t-t_0)^2 + \ldots. \tag{11}
\]

To prove the theorem, it is enough to show that the coefficients \(c_{i,m}\) in equation (11) take the form

\[
c_{i,m} = \frac{1}{m!} x_i^{(m)}(t_0), \tag{12}
\]
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for each $m = 0, 1, \ldots$, where $x_i(t)$ are the exact solutions of IDE (1) and (2). Clear that for $m = 0$ the initial conditions of equation (2) give

$$c_{i,0} = \alpha_i.$$  \hfill(13)

Moreover, for $m = 1$, substitute $t = t_0$ into equation (1), we obtain

$$x'_i(t_0) = F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n).$$  \hfill(14)

From equations (11) and (13), we can write

$$\tilde{x}_i(t) = \alpha_i + c_{i,1}(t - t_0) + c_{i,2}(t - t_0)^2 + \cdots,$$  \hfill(15)

by substituting equation (15) into equation (1) and then setting $t = t_0$, we get

$$c_{i,1} = F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) = x'_i(t_0).$$  \hfill(16)

Further, for $m = 2$, differentiating both sides of equation (1) with respect to $t$, we obtain

$$x''_i(t) = \frac{\partial}{\partial t} F_i(t, x_1(t), x_2(t), \ldots, x_n(t))$$

$$+ \sum_{j=1}^{n} x'_j(t) \frac{\partial}{\partial x_j} F_i(t, x_1(t), x_2(t), \ldots, x_n(t))$$

$$+ K_i(t, t) G_i(x_1(t), x_2(t), \ldots, x_n(t)),$$  \hfill(17)

by substituting $t = t_0$ in equation (17), we can conclude that

$$x''_i(t_0) = \frac{\partial}{\partial t} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$+ \sum_{j=1}^{n} x'_j(t_0) \frac{\partial}{\partial x_j} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + K_i(t_0, t_0) G_i(\alpha_1, \alpha_2, \ldots, \alpha_n).$$  \hfill(18)

According equations (15) and (16), the approximation of (1) and (2) can be written by:

$$\tilde{x}_i(t) = \alpha_i + x'_i(t_0)(t - t_0) + c_{i,2}(t - t_0)^2 + \cdots,$$  \hfill(19)

by substituting equation (19) into equation (17) and setting $t = t_0$, we obtain

$$2c_{i,2} = \frac{\partial}{\partial t} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$+ \sum_{j=1}^{n} x'_j(t_0) \frac{\partial}{\partial x_j} F_i(t_0, \alpha_1, \alpha_2, \ldots, \alpha_n) + K_i(t_0, t_0) G_i(\alpha_1, \alpha_2, \ldots, \alpha_n).$$  \hfill(20)
Finally, by comparing equations (18) and (20), we can conclude the following values
\[ c_{i,2} = \frac{1}{2} x_i''(t_0), \]  
(21)

By continuing the above procedure, we can easily prove equation (12) for \( m = 3, 4, \ldots \).

**Corollary 3.1**: If some of \( x_i(t), i = 1, 2, \ldots, n \) is a polynomial, then the RPS technique will give the exact solution.

It will be convenient to have a notation for the error in the approximation \( x_i(t) \approx x^k_i(t) \). Accuracy refers to how closely a computed or measured value agrees with the true value. To show this accuracy for the present method, we define the residual error by
\[ \text{Res}_k(t) = \left( x^k_i(t) \right)'' - F_i \left( t, x^k_i(t), x^k_2(t), \ldots, x^k_n(t) \right) \]
\[ - \int_{t_0}^{t} K_i(t, \tau) G_i \left( x^k_i(\tau), x^k_2(\tau), \ldots, x^k_n(\tau) \right) d\tau. \]
where \( x \in [t_0, b] \), \( x^k_i(t) \) is the \( k \)-th order approximation of \( x(t) \) obtained by the RPSM.

**4. Numerical Examples**

In most real-life situations, the differential equation that models the problem is too complicated to solve exactly, and there is a practical need to approximate the solution.

**Example**: Consider the following nonlinear system of second-order IDEs:

\[ x'_1(t) = f_1(t) + \sin(x_1(t) + x_2(t) + x_3(t)) + \int_0^1 e^{t} x_1(\tau) x_2(\tau) x_3(\tau) d\tau, \]
\[ x'_2(t) = f_2(t) + \exp(x_1(t) x_2(t) x_3(t)) + \int_0^1 \sinh x_1(\tau) \cosh x_3(\tau) d\tau \]  
(22)
\[ x'_3(t) = f_3(t) + x_1(t) + x_2(t) + x_3(t) + \int_0^1 (\tau - t) (x^2_1(\tau) + x^2_2(\tau) + x^2_3(\tau)) c \]
with the initial conditions
\[ x_1(0) = 0, x_2(0) = 1, x_3(0) = -1, \]  
(23)
where \( f_1(t) \) and \( f_2(t) \) are chosen whereas the exact solutions are \( x_1(t) = \sin(t), x_2(t) = \exp(t) \) and \( x_3(t) = -1 + t + t^2 \).

As we mentioned earlier, if we select the initial guesses approximations as \( x_{1,0}(t) = 1, x_{1,1}(t) = 0, x_{2,0}(t) = 1, \) and \( x_{2,1}(t) = -1, \) then the Taylor series expansions of solutions for equations (3.28) and (3.29) are as follows:
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\[ x_1(t) = 0 + \sum_{m=1}^{\infty} c_{1,m} t^m, \quad x_2(t) = 1 + \sum_{m=1}^{\infty} c_{2,m} t^m, \quad x_3(t) = -1 + \sum_{m=1}^{\infty} c_{3,m} t^m, \]  

(24)

in which the \( k \)th residual functions are given as

\[ \text{Res}_1^k(t) = \sum_{m=1}^{k} m c_{1,m} t^{m-1} - f_1(t) \]

\[ - \sin \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right) \left( 1 + \sum_{m=1}^{\infty} c_{2,m} t^m \right) \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right) \]

\[ - \int_0^1 e^t \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right) \left( 1 + \sum_{m=1}^{\infty} c_{2,m} t^m \right) \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right) dt, \]

(25)

\[ \text{Res}_2^k(t) = \sum_{m=1}^{k} m c_{2,m} t^{m-1} - f_2(t) \]

\[ - \exp \left( \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right) \left( 1 + \sum_{m=1}^{\infty} c_{2,m} t^m \right) \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right) \right) \]

\[ - \int_0^1 \sinh \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right) \cosh \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right) dt, \]

(26)

\[ \text{Res}_3^k(t) = \sum_{m=1}^{k} m c_{3,m} t^{m-1} - f_3(t) \]

\[ - \left( \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right) + \left( 1 + \sum_{m=1}^{\infty} c_{2,m} t^m \right) + \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right) \right) \]

\[ - \int_0^t \left( r - t \right) \left( \sum_{m=1}^{\infty} c_{1,m} t^m \right)^2 + \left( 1 + \sum_{m=1}^{\infty} c_{2,m} t^m \right)^2 \]

\[ + \left( -1 + \sum_{m=1}^{\infty} c_{3,m} t^m \right)^2 dt, \]

(27)

Anyhow, when \( N = 10 \) is used throughout the computations; the following are the first few terms of residual power series approximation of equations (22) and (23):

\[ x_1(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880}, \]

\[ x_2(t) = 1 + t + \frac{2}{6} + \frac{t^3}{120} + \frac{t^5}{720} + \frac{t^7}{5040} + \frac{t^8}{40320} + \frac{t^9}{362880}, \]

\[ x_3(t) = -1 + t + t^3. \]

(28)

If we collect the above results, then the exact solutions of equations (22) and (23) have the general form which are coinciding with the exact solutions,
\[
x_1(t) = \sin t, x_2(t) = e^t, x_3(t) = t(1 + t^2) - 1.
\] (29)

5. Conclusion

The main goal has been achieved by introducing RPS technique to solve this class of systems of FIDEs. We can conclude that the RPS technique is powerful and efficient technique in finding approximate solution for nonlinear systems of FIDEs. The proposed algorithm produced a rapidly convergent series with easily computable components using symbolic computation software. The results obtained by the RPS technique are very effective and convenient in nonlinear cases with less computational work and time. This confirms our belief that the efficiency of RPS technique gives it much wider applicability for general classes of nonlinear problems.

References


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