Note on Minimal Möbius-Invariant Subspaces of Analytic and Harmonic Functions on the Unit Disc

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Abstract

We study minimal Möbius-invariant subspaces for some classes of linear topological spaces of analytic and harmonic functions on the unit disc.

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Introduction

Let $D$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $G=\text{Aut}(D)$ be the group of Möbius transformations of the form $\varphi = \lambda \varphi_a$ where $\lambda \in \Pi = \partial D$ and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Let $X$ be a linear topological Möbius-invariant subspace of analytic functions on $D$. That is, $f \in X$ implies $f \circ \varphi \in X$ for every $\varphi \in G$.

Definition:

A closed Möbius-invariant subspace $V$ is said to be minimal if it does not contain a closed non-trivial Möbius-invariant subspace. It follows that for each $f \in V$, $f \neq \text{const}$, the span of $\{f \circ \varphi, \varphi \in G\}$ is dense in $V$.

For the study of semi- normed and normed Möbius-invariant subspaces $X$ see [1].
Main results

Theorem 1.
Let \( X \) be a Möbius-invariant space of analytic functions on \( D \) which satisfies:

1. The polynomials \( P(z) = a_0 + a_1 z + \ldots + a_n z^n \) are dense in \( X \).
2. If \( V \) is a closed Möbius-invariant subspace of \( X \), and \( f \in V \) then the function \( h(z) = \int f(z e^{i\alpha}) \, d\mu(\alpha) \) belongs to \( V \) for each measure \( \mu \) on the subgroup of rotations \( K \).

Then \( X \) is a minimal Möbius-invariant subspace.

Proof:
For \( f \neq \text{const.} \), let \( M(f) \) denote the closed subspace spanned by \( \{ f \circ \varphi, \varphi \in G \} \).

Let \( z_0 \in D \) such that \( f'(z_0) \neq 0 \). If \( \varphi_0 \in G, \varphi_0(0) = z_0 \), then \( h_0'(0) \neq 0 \) where \( h_0 = f \circ \varphi_0 \) and \( h_0 \in M(f) \). Hence \( h_0(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( a_1 \neq 0 \). For each \( \alpha \), \( h_0(z e^{i\alpha}) \in M(f) \) and by (2) \( h_1(z) = \int_0^{2\pi} h_0(ze^{i\alpha}) e^{-i\alpha} d\alpha = 2\pi a_1 z \in M(f) \).

But \( h_2 = h_1 \circ \varphi_2 = 2\pi a_1 \varphi_2 \), for some \( \varphi_2 \in G, \varphi_2(0) \neq 0 \), belongs to \( M(f) \).

Since \( \varphi_2 = \sum_{n=0}^{\infty} a_n^* z^n \) where \( a_n^* \neq 0 \) \( \forall n \). By averaging rotations as in (2) we obtain \( z^k \in M(f) \) \( \forall k \geq 0 \), implying by (1) that \( M(f) = X \).

The following result shows that we don’t need the whole group \( G = \text{Aut}(D) \).

Let \( \Lambda \subseteq \text{Aut}(D) = \{ e^{i\alpha}, \alpha \in \mathbb{R} \} \cup \{ \varphi_n \in G: \varphi_n(0) \to \beta, \varphi_n(0) \neq \beta \) for some \( \beta \in D \} \).

Theorem 2.
If \( X \) satisfies (1) and (2) in Theorem 1 then \( X \) is minimal with respect to \( \Lambda \). That is, for every \( f \in X, f \neq \text{const.} \), the subspace spanned by \( f \circ \varphi, \varphi \in \Lambda \) is dense in \( X \).

Proof:
Let \( f \in X, f \neq \text{const.} \). Since \( f \) is analytic there exists \( n_0 \) such that \( f'(\varphi_{n_0}(0)) \neq 0 \).

For \( h = f \circ \varphi_{n_0} \) we have \( h'(0) \neq 0 \) and the result follows as in the proof of Theorem 1.
The analogue result for harmonic functions is the following:

**Theorem 3.**

Let $X$ be a Möbius-invariant linear topological space over $\mathbb{R}$ of real-valued harmonic functions on $D$.

Suppose $X$ satisfies:

1. The harmonic polynomials $P_n(z) + \overline{P_n(z)}$ are dense in $X$.
2. Same as in Theorem 1.

Then $X$ is a minimal Möbius-invariant subspace.

**Proof:**

Let $f \in X$, $f \neq \text{const}$. Since $f$ is the real part of an analytic function there exists $z_0 \in D$ such that $\frac{\partial f}{\partial z}(z_0, \overline{z}_0) \neq 0$ and $\frac{\partial f}{\partial \overline{z}}(z_0, \overline{z}_0) \neq 0$. If $\varphi \in G$, $\varphi(0) = z_0$ then $h = f \circ \varphi$ belongs to $M(f)$ and satisfies $\frac{\partial h}{\partial z}(0,0) \neq 0$ and $\frac{\partial h}{\partial \overline{z}}(0,0) \neq 0$. It follows that $h(z, \overline{z}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{a}_n \overline{z}^n$ where $a_1 \neq 0$. By taking average of rotations of $h$ with $\mu(\alpha) = e^{i\alpha} + e^{-i\alpha}$ in (2)*, we obtain $a_1 z + \overline{a}_1 \overline{z} \in M(f)$.

Hence $h_1 = a_1 \varphi_0 + \overline{a}_1 \overline{\varphi}_0$ for some $\varphi_0 \in G$, $\varphi_0(0) \neq 0$. But $h_1(z, \overline{z}) = \sum_{n=0}^{\infty} (b_n z^n + \overline{b}_n \overline{z}^n)$ with $b_n \neq 0$ for every $n$. By rotating $h_1$ with $\mu(\alpha) = c e^{i\alpha} + \overline{c} e^{-i\alpha}$ in (2)* we obtain $c b_n z^n + \overline{c} b_n \overline{z}^n \in M(f)$ for every $n$ and $c \in \mathbb{C}$, implying by (1)* that $M(f) = X$.

**Corollary 4.** [2, 2.1 Theorem].

Let $f(e^{i\alpha})$ be a real valued continuous, non-constant function on $\Pi = \partial D$.

Then the span of $f \circ \varphi$, $\varphi \in G$ is dense in $C(\Pi)$.

**Proof:**

Let $F(z, \overline{z})$ be the harmonic extension of $f$ to $D$. Since the disc algebra satisfies (1) and (2) the result follows from Theorem 3.

Similarly we obtain:

**Theorem 5.**

Theorem 3 and Corollary 4 hold when $G$ is replaced by $\Lambda$.

**References**

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