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Operator Calculus for Noncommuting Operators over Symmetric Fock Space

Sergii Sharyn

Vasyl Stefanyk Precarpathian National University
57 Shevchenka str., 76-018, Ivano-Frankivsk, Ukraine

Abstract

In this paper we construct an operator calculus over the symmetric Fock space for countable set of noncommuting generators of strongly continuous groups, acting on a Hilbert space. As a symbol class of the calculus we use some algebra of functions of infinitely many variables. This algebra is described as the image of the space of polynomial ultra-differentiable functions under Fourier-Laplace transformation.

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1 Introduction

A functional (or an operator) calculus is a theory that studies how to construct functions depending on operators. An operator calculus is the useful tool, which allows us to use the functional-analytic methods in general spectral analysis and theory of Banach algebras.

There are many different approaches to construct a functional calculus for one operator acting on a Banach space. Most of them are based on some integral representation of a function from a symbol class. For Riesz-Dunford functional calculus (or H^∞-calculus), based on the Cauchy formula, we refer the reader to the book [3]. Such a functional calculus has applications, in particular, in the spectral theory of elliptic differential equations and maximal regularity of parabolic evolution equations (see e.g. [5, 7]).

H^∞-calculus is good tool when we work with generators of analytic semigroups. However, such a calculus is not suitable for an arbitrary strongly continuous semigroup. For such operators E. Hille and R. Phillips developed in [4] another method, based on the Laplace transformation. This method is known as the Hille-Phillips functional calculus. It has many helpful applications, in particular, in hydrology (see [1] and the references given there).

The Hille-Phillips functional calculus for functions of several variables was considered, for example, in [8, 10]. The case of functions of infinitely many variables is less studied. We mention the book [11] that is devoted to spectral questions (among them there is a functional calculus) of countable families of self-adjoint operators on a Hilbert space.

In the recent article [12] we construct the Hille-Phillips type functional calculus for commuting generators of strongly continuous (or (C_0)) semigroups over a Banach space. The main goal of this article is the construction of such calculus over the symmetric Fock space $\bigoplus_{n\in\mathbb{Z}_+}\mathcal{H}^{\hat{\otimes}n}$ for countable set of noncommuting generators of (C_0) groups, acting on a Hilbert space $\mathcal{H}$.

2 Notations and preliminaries

In what follows $\mathscr{L}(\mathcal{X},\mathcal{Y})$ denotes the space of all continuous linear operators from a locally convex space $\mathcal{X}$ in other such space $\mathcal{Y}$, endowed with the topology of uniform convergence on bounded subsets of $\mathcal{X}$. Let $\mathscr{L}(\mathcal{X}) := \mathscr{L}(\mathcal{X},\mathcal{X})$. The dual space $\mathcal{X}' := \mathscr{L}(\mathcal{X},\mathbb{C})$ is endowed with strong topology. The pairing between elements of $\mathcal{X}'$ and $\mathcal{X}$ we denote $\langle\cdot,\cdot\rangle$.

Spaces of functions. Let us fix any real $\beta > 1$. An infinitely differentiable function φ is called to be a Gevrey ultradifferentiable if for each segment $[\mu,\nu] \subset \mathbb{R}$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t\in[\mu,\nu]} |\partial^k\varphi(t)| \le Ch^k k^{k\beta}$ holds for all $k \in \mathbb{Z}_+$. For a fixed $h > 0$ let us consider the subspace

$$\mathcal{G}^h_\beta[\mu,\nu] := \Big\{\varphi \in C^\infty : \operatorname{supp}\varphi \subset [\mu,\nu], \|\varphi\|_{\mathcal{G}^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+}\sup_{t\in[\mu,\nu]} \frac{|\partial^k\varphi(t)|}{h^k k^{k\beta}} < \infty\Big\}.$$

In [6] it is proved that each subspace $\mathcal{G}^h_\beta[\mu,\nu]$ is a Banach space and maps $\mathcal{G}^h_\beta[\mu,\nu] \hookrightarrow \mathcal{G}^l_\beta[\mu,\nu]$, with $h < l$, are compact inclusions. Consider the space

$$\mathcal{G}_\beta := \bigcup_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad \mathcal{G}_\beta \simeq \liminf_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu],$$

of Gevrey ultradifferentiable functions with compact supports and endow it with topology of inductive limit with respect to above mentioned compact inclusions. Let $\mathcal{G}'_\beta$ be its dual space of Roumieu ultradistributions.

Let $h > 0$ be any positive real and $\mu, \nu \in \mathbb{R}$ be any reals such that $\mu < \nu$. In the space of entire functions of exponential type we consider the subspace $E^h_\beta[\mu,\nu]$ of functions with the finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{z\in\mathbb{C}} \frac{|z^k \psi(z) e^{-H_{[\mu,\nu]}(\eta)}|}{h^k k^{k\beta}}, \qquad \text{where} \quad H_{[\mu,\nu]}(\eta) := \sup_{t\in[\mu,\nu]} t\eta.$$

Each space $E^h_\beta[\mu,\nu]$ is a Banach one, and all maps $E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu']$ with $[\mu,\nu] \subset [\mu',\nu']$, $h < h'$, are compact inclusions. Consider the space

$$E_\beta := \bigcup_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu], \qquad E_\beta \simeq \liminf_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu],$$

and endow it with the topology of inductive limit with respect to above mentioned compact inclusions.

Consider the Fourier-Laplace transformation

$$\widehat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}} e^{-itz} \varphi(t)\, dt, \qquad \varphi \in \mathcal{G}_\beta,\ z \in \mathbb{C}.$$

It is known [13], that $F(\mathcal{G}_\beta) = E_\beta$.

Polynomial ultradifferentiable functions and polynomial ultradistributions. For any locally convex space $\mathcal{X}$, let $\mathcal{X}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, be the symmetric nth tensor degree of $\mathcal{X}$, completed in the projective tensor topology. For any $x \in \mathcal{X}$ we denote $x^{\otimes n} := \underbrace{x \otimes \cdots \otimes x}_{n} \in \mathcal{X}^{\hat{\otimes} n}$, $n \in \mathbb{N}$. Set $\mathcal{X}^{\hat{\otimes} 0} := \mathbb{C}$, $x^{\otimes 0} := 1 \in \mathbb{C}$.

To define the locally convex space $\mathcal{P}({}^n\mathcal{G}'_\beta)$ of n-homogeneous polynomials on $\mathcal{G}'_\beta$ we use the canonical topological linear isomorphism $\mathcal{P}({}^n\mathcal{G}'_\beta) \simeq (\mathcal{G}'^{\hat{\otimes} n}_\beta)'$, described in [2]. We equip $\mathcal{P}({}^n\mathcal{G}'_\beta)$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{G}'_\beta$. Set $\mathcal{P}({}^0\mathcal{G}'_\beta) := \mathbb{C}$. The space $\mathcal{P}(\mathcal{G}'_\beta)$ of all continuous polynomials on $\mathcal{G}'_\beta$ is defined to be the complex linear span of all $\mathcal{P}({}^n\mathcal{G}'_\beta)$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{G}'_\beta)$ mean the strong dual of $\mathcal{P}(\mathcal{G}'_\beta)$.

Elements of the spaces $\mathcal{P}(\mathcal{G}'_\beta)$ and $\mathcal{P}'(\mathcal{G}'_\beta)$ we call the polynomial test ultradifferentiable functions and polynomial ultradistributions, respectively.

Denote $\Gamma(\mathcal{G}_\beta) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin}\, \mathcal{G}^{\hat{\otimes} n}_\beta$ and $\Gamma(\mathcal{G}'_\beta) := \times_{n\in\mathbb{Z}_+} \mathcal{G}'^{\hat{\otimes} n}_\beta$. Note, that we consider the case when the elements of direct sum consist of finite but not fixed number of addends. In what follows elements of the spaces $\Gamma(\mathcal{G}_\beta)$ and $\Gamma(\mathcal{G}'_\beta)$ will be written as $\boldsymbol{p} = \big(p_n\big)$ and $\boldsymbol{u} = \big(u_n\big)$, respectively.

For elements of total subset of the space $\Gamma(\mathcal{G}'_\beta)$ let us define the operation $\big(f^{\otimes n}\big) \circledast \big(g^{\otimes n}\big) := \big((f * g)^{\otimes n}\big)$ and extend it onto whole space by linearity and continuity. It is easy to see, that $\Gamma(\mathcal{G}'_\beta)$ is an algebra with respect to $\circledast$. Since the space $\Gamma(\mathcal{G}_\beta)$ is dense in $\Gamma(\mathcal{G}'_\beta)$ (see [9]), the space $\Gamma(\mathcal{G}_\beta)$ also is an algebra with respect to the operation $\circledast$.

Using the tensor structure of the space $\Gamma(\mathcal{G}_\beta)$, we extend the Fourier-Laplace transformation onto $\Gamma(\mathcal{G}_\beta)$. First, for elements of total subset of the space $\mathcal{G}_\beta^{\hat{\otimes} n}$ we define the operator $\mathcal{F}^{\otimes n} : \varphi^{\otimes n} \longmapsto \widehat{\varphi}^{\otimes n}$, $\mathcal{F}^{\otimes 0} := I_{\mathbb{C}}$, where $\widehat{\varphi}^{\otimes n} := (F\varphi)^{\otimes n}$. Next, we extend the map $\mathcal{F}^{\otimes n}$ onto whole space $\mathcal{G}_\beta^{\hat{\otimes} n}$ by linearity and continuity. As a result we obtain the map $\mathcal{F}^{\otimes n} \in \mathscr{L}\big(\mathcal{G}_\beta^{\hat{\otimes} n}, E_\beta^{\hat{\otimes} n}\big)$. And finally, we define the mapping $\mathcal{F}^{\otimes}$ by the formula

$$\mathcal{F}^{\otimes} := \big(\mathcal{F}^{\otimes n}\big) : \Gamma(\mathcal{G}_\beta) \ni \boldsymbol{p} = \big(p_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{p}} := \big(\widehat{p}_n\big) \in \Gamma(E_\beta) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin} E_\beta^{\hat{\otimes} n},$$

where $p_n \in \mathcal{G}_\beta^{\hat{\otimes} n}$, $\widehat{p}_n := \mathcal{F}^{\otimes n} p_n \in E_\beta^{\hat{\otimes} n}$.

Note, that for each $n \in \mathbb{N}$ an element $\widehat{p}_n$ is a symmetric function of n complex variables $\mathbb{C}^n \ni (z_1, \dots, z_n) \longmapsto \widehat{p}(z_1, \dots, z_n) \in \mathbb{C}$, i.e. $\widehat{p}_n(z_1, z_2, \dots, z_n) = \widehat{p}_n(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)})$ for every permutation σ of $\{1, \dots, n\}$. It implies that elements of the space $\Gamma(E_\beta)$ can be considered as functions $\widehat{\boldsymbol{p}} : \times_{n\in\mathbb{N}} \mathbb{C} \longrightarrow \mathbb{C}$ of infinite many variables

$$\widehat{\boldsymbol{p}} : (z_1, \dots, z_n, \dots) \;\longmapsto\; \widehat{\boldsymbol{p}}(z_1, \dots, z_n, \dots) = \widehat{p}_0 + \sum_{n\in\mathbb{N}} \widehat{p}_n(z_{\mathfrak{b}_n}, \dots, z_{\mathfrak{e}_n}), \tag{1}$$

where $\mathfrak{b}_n := \frac{n(n-1)}{2} + 1$, $\mathfrak{e}_n := \frac{n(n+1)}{2}$. But we note that actually each function $\widehat{\boldsymbol{p}} \in \Gamma(E_\beta)$ depends on finite (depending on $\widehat{\boldsymbol{p}}$) number of variables, because for each $\widehat{\boldsymbol{p}}$ the sequence in the right hand side of (1) is finite.

For any operator $K \in \mathscr{L}(\mathcal{G}_\beta)$ let us define the operator $K^{\otimes} \in \mathscr{L}\big(\Gamma(\mathcal{G}_\beta)\big)$ by the formula

$$K^{\otimes} := \big(K^{\otimes n}\big) : p = (p_n) \quad \longmapsto \quad K^{\otimes} p := \big(K^{\otimes n} p_n\big), \tag{2}$$

where $K^{\otimes 0} := I_{\mathbb{C}}$ is the identity, and the each operator $K^{\otimes n} \in \mathscr{L}(\mathcal{G}_\beta^{\otimes n})$, $n \in \mathbb{N}$, is defined as linear and continuous extension of the following map $\varphi^{\otimes n} \longmapsto (K\varphi)^{\otimes n}$, $\varphi \in \mathcal{G}_\beta$.

For any ultradistribution $f \in \mathcal{G}'_\beta$ and ultradifferentiable function $\varphi \in \mathcal{G}_\beta$ the cross-correlation is the function defined as follows $(f \star \varphi)(s) := \langle f(t), \varphi(t+s)\rangle$. It is easy to see, that the cross-correlation operator $K_f : \varphi \longmapsto f \star \varphi$ belongs to the space $\mathscr{L}(\mathcal{G}_\beta)$ for any ultradistribution $f \in \mathcal{G}'_\beta$.

Using the definition (2) we obtain

$$K_{\boldsymbol{u}}^{\otimes} := \big(K_{u_n}^{\otimes n}\big) \in \mathscr{L}\big(\Gamma(\mathcal{G}_\beta)\big) \quad \text{and} \quad K_{u_n}^{\otimes n} \in \mathscr{L}(\mathcal{G}_\beta^{\otimes n}), \tag{3}$$

where $\boldsymbol{u} := (u_n) \in \Gamma(\mathcal{G}'_+)$ with $u_n \in \mathcal{G}'^{\otimes n}_+$, $n \in \mathbb{Z}_+$.

Let us define the operation, which is the extension of the cross-correlation on the spaces of polynomial ultradifferentiable functions and ultradistributions. For any $\boldsymbol{u} = (u_n) \in \Gamma(\mathcal{G}'_\beta)$ and $\boldsymbol{p} = (p_n) \in \Gamma(\mathcal{G}_\beta)$ their cross-correlation is the element $\boldsymbol{u} \star \boldsymbol{p} := K^{\otimes}_{\boldsymbol{u}} \boldsymbol{p} = \big(K^{\otimes n}_{u_n} p_n\big)$.

Infinite parameter operator groups. Let a countable set of operators $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \dots)$ be defined in a complex Hilbert space $\mathcal{H}$. Note, that we do not assume any commutativity relations. Denote by $\Gamma(\mathcal{H}) := \bigoplus_{n\in\mathbb{Z}_+} \mathcal{H}^{\hat{\otimes} n}$ the symmetric Fock space.

Suppose, that $\mathbf{A}_j$, $j \in \mathbb{N}$, generates an one-parameter (C_0) group (see [4]) $\mathbb{R} \ni t \longmapsto e^{-\mathrm{i}t\mathbf{A}_j} \in \mathscr{L}(\mathcal{H})$, which satisfies the condition

$$\sup_{t\in\mathbb{R}} \|e^{-\mathrm{i}t\mathbf{A}_j}\|_{\mathscr{L}(\mathcal{H})} \leq 1. \tag{4}$$

Let us denote $\mathscr{A}_j := \underbrace{I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}}}_{j} \otimes \mathbf{A}_j \otimes I_{\mathcal{H}} \otimes \cdots \in \mathscr{L}(\Gamma(\mathcal{H}))$, $j \in \mathbb{N}$, $A_n := \mathscr{A}_{\mathfrak{b}_n} \otimes \cdots \otimes \mathscr{A}_{\mathfrak{e}_n}$, $n \in \mathbb{N}$. Set $\mathscr{A}_0 := I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \cdots \in \mathscr{L}(\Gamma(\mathcal{H}))$, $A_0 := \mathscr{A}_0$ by definition.

Instead of the set $\mathbf{A}$ of (noncommuting) operators over Hilbert space $\mathcal{H}$ we consider countable set of commuting operators, acting in the symmetric Fock space $\Gamma(\mathcal{H})$, namely

$$A := (A_0, A_1, A_2, \dots, A_n, \dots). \tag{5}$$

It easy to see, that each A_n generates strong continuous n-parameter group $\mathbb{R}^n \ni t \longmapsto e^{-\mathrm{i}tA_n} \in \mathscr{L}(\Gamma(\mathcal{H}))$, where $e^{-\mathrm{i}tA_n} := e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}} \otimes \cdots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}_n}}$ and $e^{-\mathrm{i}t_i\mathscr{A}_j} := \underbrace{I_{\mathcal{H}} \otimes \ldots \otimes I_{\mathcal{H}}}_{j} \otimes e^{-\mathrm{i}t_i\mathbf{A}_j} \otimes I_{\mathcal{H}} \otimes \ldots$, $i = 1, \dots, n$, $j \in \mathbb{N}$. Note, that each one-parameter group $e^{-\mathrm{i}t_i\mathbf{A}_j}$ satisfies the condition (4).

Operator A_n and group $e^{-\mathrm{i}tA_n}$ are defined on whole Fock space $\Gamma(\mathcal{H})$, but they do not act as identity only on $\mathcal{H}^{\hat{\otimes} n}$. So, without restriction of generality we can write $A_n \in \mathscr{L}(\mathcal{H}^{\hat{\otimes} n})$, $e^{-\mathrm{i}tA_n} \in \mathscr{L}(\mathcal{H}^{\hat{\otimes} n})$, $n \in \mathbb{Z}_+$.

Let $\mathscr{G}$ be the set of countable systems of operators of view (5). For all $n \in N$ let $\mathscr{G}_n$ be a set of collections of operators of view $A_n = \mathscr{A}_{\mathfrak{b}_n} \otimes \cdots \otimes \mathscr{A}_{\mathfrak{e}_n}$. Set $\mathscr{G}_0 := \{\mathscr{A}_0\}$ by definition.

3 Functional calculus for countable set of operators

For all $n \in \mathbb{Z}_+$ let us define the set $\widetilde{\mathcal{H}}_n := \{\widetilde{p}_n : \mathscr{G}_n \longrightarrow \mathscr{L}(\mathcal{H}^{\hat{\otimes} n}) : p_n \in \mathcal{G}_\beta^{\hat{\otimes} n}\}$, which consist of functions of operator argument

$$\widetilde{p}_n(A_n) := \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1 \mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n \mathscr{A}_{\mathfrak{e}_n}} p_n(t_1, \dots, t_n)\, dt_1 \dots dt_n. \tag{6}$$

Set $\widetilde{p}_0 \colon \mathscr{G}_0 \ni A_0 \longmapsto \widetilde{p}_0(A_0) := p_0 I_{\mathbb{C}} \in \mathscr{L}(\mathbb{C})$ by definition.

Define the map

$$\mathcal{F} := (\mathcal{F}_n) \colon \Gamma(\mathcal{G}_\beta) \ni \boldsymbol{p} = \big(p_n\big) \quad \longmapsto \quad \widetilde{\boldsymbol{p}} := \sum_{n \in \mathbb{Z}_+} \widetilde{p}_n \in \widetilde{\mathcal{H}}, \tag{7}$$

where $\widetilde{\mathcal{H}} := \sum_{n \in \mathbb{Z}_+} \widetilde{\mathcal{H}}_n$. Condition (4) and [4, Theorem 15.2.1] imply, that all mappings $\mathcal{F}_n : p_n \longmapsto \widetilde{p}_n$, $n \in \mathbb{Z}_+$, are isomorphisms.

Note, that $\widetilde{\mathcal{H}} := \{\widetilde{\boldsymbol{p}} : \mathscr{G} \longrightarrow \mathscr{L}(\Gamma(\mathcal{H})) : \boldsymbol{p} \in \Gamma(\mathcal{G}_\beta)\}$ is an algebra of functions with pointwise multiplication $(\widetilde{\boldsymbol{p}} \cdot \widetilde{\boldsymbol{q}})(A) := \widetilde{\boldsymbol{p}}(A) \circ \widetilde{\boldsymbol{q}}(A)$.

Remark 1. *The mapping $\mathcal{F} \colon \Gamma(\mathcal{G}_\beta) \longrightarrow \widetilde{\mathcal{H}}$ acts as homomorphism of algebra $\big\{\Gamma(\mathcal{G}_\beta), \circledast\big\}$ into algebra $\{\widetilde{\mathcal{H}}, \cdot\}$. On the other hand, results of the article [13] imply that there exists a homomorphism $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$. Therefore, the map $\mathcal{F} \circ (F^{\otimes})^{-1} \colon \Gamma(E_\beta) \longrightarrow \widetilde{\mathcal{H}}$ we may treat as "elementary" functional calculus. In other words, we understand the operator $\widetilde{\boldsymbol{p}}(A) = \sum_n \widetilde{p}_n(A_n) \in \mathscr{L}(\Gamma(\mathcal{H}))$ as a "value" of a function $\widehat{\boldsymbol{p}}$ of infinite many variables (see* (1)*) at a countable system $A = (A_0, A_1, A_2, \dots, A_n, \dots) \in \mathscr{G}$ of operators (see* (5)*).*

Consider the one-parameter semigroup $\widetilde{T}^{\otimes} : \mathbb{R} \ni s \longmapsto \widetilde{T}_s^{\otimes} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$ on the space $\widetilde{\mathcal{H}}$, where

$$\widetilde{T}_s^{\otimes} := \big(\widetilde{T}_s^{\otimes n}\big) \colon \widetilde{\boldsymbol{p}} = \sum_{n \in \mathbb{Z}_+} \widetilde{p}_n \quad \longmapsto \quad \widetilde{T}_s^{\otimes} \widetilde{\boldsymbol{p}} := \sum_{n \in \mathbb{Z}_+} \widetilde{T}_s^{\otimes n} \widetilde{p}_n.$$

The function $\widetilde{T}_s^{\otimes n} \widetilde{p}_n \in \widetilde{\mathcal{H}}_n$ is defined to be the map $\widetilde{T}_s^{\otimes n} \widetilde{p}_n : \mathscr{G}_n \ni A_n \longmapsto \widetilde{T}_s^{\otimes n} \widetilde{p}_n(A_n) \in \mathscr{L}(\mathcal{H}^{\hat{\otimes} n})$, where

$$\widetilde{T}_s^{\otimes n} \widetilde{p}_n(A_n) := \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1 \mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n \mathscr{A}_{\mathfrak{e}_n}} p_n(t_1 + s, \dots, t_n + s)\, dt_1 \dots dt_n.$$

Here the function $\widetilde{p}_n$ of operator argument is defined by (6).

Using Bochner's integral properties (see [4, 3.7]), we obtain that for any $\boldsymbol{p} = \big(p_n\big) \in \Gamma(\mathcal{G}_\beta)$ with $p_n = \varphi^{\otimes n} \in \mathcal{G}_\beta^{\hat{\otimes} n}$, $\varphi \in \mathcal{G}_\beta$, the following equalities

$$\begin{aligned}
\widetilde{T_s^{\otimes}\boldsymbol{p}}(A) &= \mathcal{F}\big[\big(T_s^{\otimes n} p_n\big)\big](A) = \mathcal{F}\big[\big((T_s\varphi)^{\otimes n}\big)\big](A) \\
&= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}_n}} (T_s\varphi)^{\otimes n}(t_1,\dots,t_n)\, dt_1\dots dt_n \\
&= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}_n}} \bigtimes_{k=1}^{n} \varphi(t_k+s)\, dt_1\dots dt_n \\
&= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}_n}} p_n(t_1+s,\dots,t_n+s)\, dt_1\dots dt_n \\
&= \widetilde{p}_0(A_0) + \sum_{n\in\mathbb{N}} \widetilde{T}_s^{\otimes n}\widetilde{p}_n(A_n) = \widetilde{T}_s^{\otimes}\widetilde{\boldsymbol{p}}(A),
\end{aligned}$$

hold for all $s \in \mathbb{R}$ and $A = \big(A_n\big) \in \mathscr{G}$.

Hence, the operator $\widetilde{T}_s^{\otimes}$ can be represented as follows $\widetilde{T}_s^{\otimes} = \mathcal{F} \circ T_s^{\otimes} \circ \mathcal{F}^{-1}$. Continuity of the mappings $T_s^{\otimes}$ and $\mathcal{F}$ as well as openness of $\mathcal{F}$ imply that the group $\widetilde{T}^{\otimes} : \mathbb{R} \ni s \longmapsto \widetilde{T}_s^{\otimes} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$ has the (C_0) property.

The commutant of the group $\widetilde{T}^{\otimes}$ is defined to be the set

$$\big[\widetilde{T}^{\otimes}\big]^c := \big\{\widetilde{T} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big) : \widetilde{T} \circ \widetilde{T}_s^{\otimes} = \widetilde{T}_s^{\otimes} \circ \widetilde{T}, \forall s \in \mathbb{R}\big\}.$$

Define the mapping

$$\mathcal{Q} := \big(\mathcal{Q}_n\big) : \Gamma(\mathcal{G}'_\beta) \ni \boldsymbol{u} = \big(u_n\big) \quad \longmapsto \quad \mathcal{Q}_{\boldsymbol{u}} := \sum_{n\in\mathbb{Z}_+} \mathcal{Q}_{u_n} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big), \tag{8}$$

where $u_n := f^{\otimes n} \in \mathcal{G}'^{\hat{\otimes} n}_\beta$, $f \in \mathcal{G}'_\beta$. Here $\mathcal{Q}_{u_n} \in \mathscr{L}\big(\widetilde{\mathcal{H}}_n\big)$, $n \in \mathbb{Z}_+$, is defined by the following formulas: $(\mathcal{Q}_{u_0}\widetilde{p}_0)(A_0) := I_{\mathbb{C}}$ and $\mathcal{Q}_{u_n} : \widetilde{p}_n \longmapsto \mathcal{Q}_{u_n}\widetilde{p}_n$, $n \in \mathbb{N}$, where

$$(\mathcal{Q}_{u_n}\widetilde{p}_n)(A_n) := \int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}} \otimes \dots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}_n}} K_f^{\otimes n} p_n(t_1,\dots,t_n)\, dt_1\dots dt_n.$$

Here the function $\widetilde{p}_n$ of operator argument is defined by (6), and the operator $K_f^{\otimes n}$ is defined by (2) and (3).

Theorem 3.1. *The map* $\mathcal{Q}$*, defined by* (8)*, is an algebraic isomorphism of the algebra* $\big\{\Gamma(\mathcal{G}'_\beta), \circledast\big\}$ *and the subalgebra in the commutant* $\big[\widetilde{T}^{\otimes}\big]^c$ *of operators of view* $\widetilde{K}^{\otimes} = \mathcal{F} \circ K^{\otimes} \circ \mathcal{F}^{-1} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$*, where* $K \in \mathscr{L}(\mathcal{G}_\beta)$*. In particular, the equality* $\mathcal{Q}_{\boldsymbol{u}\circledast\boldsymbol{v}} = \mathcal{Q}_{\boldsymbol{u}} \circ \mathcal{Q}_{\boldsymbol{v}}$ *holds for all* $\boldsymbol{u}, \boldsymbol{v} \in \Gamma(\mathcal{G}'_\beta)$ *and* $\mathcal{Q}_{\boldsymbol{\delta}}$ *is the identity in* $\mathscr{L}\big(\widetilde{\mathcal{H}}\big)$*, where* $\boldsymbol{\delta} = \big(\delta^{\otimes n}\big)$.

Proof. Let $\boldsymbol{u} = \big(u_n\big) \in \Gamma(\mathcal{G}'_\beta)$ and $\boldsymbol{p} = \big(p_n\big) \in \Gamma(\mathcal{G}_\beta)$, where $u_n = f^{\otimes n}$, $f \in \mathcal{G}'_\beta$. The following equalities are valid

$$\begin{aligned}(\mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}})(A) &= \sum_{n\in\mathbb{Z}_+}(\mathcal{Q}_{u_n}\widetilde{p}_n)(A_n)\\ &= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}}\int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}}\otimes\cdots\otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{c}_n}} K_f^{\otimes n}p_n(t_1,\dots,t_n)\,dt_1\dots dt_n \qquad (9)\\ &= \mathcal{F}\big[K_f^{\otimes n}p_n\big](A) = \widetilde{K_{\boldsymbol{u}}^{\otimes}\boldsymbol{p}}(A), \qquad K_{\boldsymbol{u}}^{\otimes} = \big(K_f^{\otimes n}\big),\end{aligned}$$

for all $A = \big(A_n\big) \in \mathscr{G}$. It follows that the map $\mathcal{Q}$ can be represented in the form $\mathcal{Q}_{\boldsymbol{u}} = \mathcal{F}\circ K_{\boldsymbol{u}}^{\otimes}\circ\mathcal{F}^{-1}$. Continuity of the mappings $K_{\boldsymbol{u}}^{\otimes}$ and $\mathcal{F}$ as well as openness of $\mathcal{F}$ imply that $\mathcal{Q}_{\boldsymbol{u}} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$ for all $\boldsymbol{u} \in \Gamma(\mathcal{G}'_\beta)$. Therefore, the equalities

$$\begin{aligned}(\mathcal{Q}_{\boldsymbol{u}}\widetilde{T}_s^{\otimes}\widetilde{\boldsymbol{p}})(A) &= \sum_{n\in\mathbb{Z}_+}(\mathcal{Q}_{u_n}\widetilde{T}_s^{\otimes n}\widetilde{p}_n)(A_n)\\ &= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}}\int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}}\otimes\cdots\otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{c}_n}} K_f^{\otimes n}p_n(t_1+s,\dots,t_n+s)\,dt\\ &= I_{\mathbb{C}} + \sum_{n\in\mathbb{N}}\widetilde{T}_s^{\otimes n}\int_{\mathbb{R}^n} e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}_n}}\otimes\cdots\otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{c}_n}} K_f^{\otimes n}p_n(t_1,\dots,t_n)\,dt\\ &= \sum_{n\in\mathbb{Z}_+}(\widetilde{T}_s^{\otimes n}\mathcal{Q}_{u_n}\widetilde{p}_n)(A_n) = (\widetilde{T}_s^{\otimes}\mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}})(A), \qquad dt := dt_1\dots dt_n,\end{aligned}$$

hold for all $s \in \mathbb{R}$, $\widetilde{\boldsymbol{p}} = \sum_{n\in\mathbb{Z}_+}\widetilde{p}_n \in \widetilde{\mathcal{H}}$ and $A := \big(A_n\big) \in \mathscr{G}$. Hence, for all $\boldsymbol{u} \in \Gamma(\mathcal{G}'_\beta)$ the operator $\mathcal{Q}_{\boldsymbol{u}}$ belongs to the commutant $\big[\widetilde{T}^{\otimes}\big]^c$.

Conversely, let the operator $\widetilde{K}^{\otimes} = \mathcal{F}\circ K^{\otimes}\circ\mathcal{F}^{-1} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$ with $K \in \mathscr{L}(\mathcal{G}_\beta)$ belongs to the commutant $\big[\widetilde{T}^{\otimes}\big]^c$. Then

$$\begin{aligned}\mathcal{F}\circ K^{\otimes}\circ T_s^{\otimes}\circ\mathcal{F}^{-1} &= \mathcal{F}\circ K^{\otimes}\circ\mathcal{F}^{-1}\circ\mathcal{F}\circ T_s^{\otimes}\circ\mathcal{F}^{-1} = \widetilde{K}^{\otimes}\circ\widetilde{T}_s^{\otimes} = \widetilde{T}_s^{\otimes}\circ\widetilde{K}^{\otimes}\\ &= \mathcal{F}\circ T_s^{\otimes}\circ\mathcal{F}^{-1}\circ\mathcal{F}\circ K^{\otimes}\circ\mathcal{F}^{-1} = \mathcal{F}\circ T_s^{\otimes}\circ K^{\otimes}\circ\mathcal{F}^{-1}.\end{aligned}$$

Therefore the operator $K^{\otimes}$ belongs to the commutant of the semigroup $T_s^{\otimes}$.

Let us define the ultradistribution $f_0 \in \mathcal{G}'_\beta$ as follows $\langle f_0,\varphi\rangle := (K\varphi)(0)$ for any $\varphi \in \mathcal{G}_\beta$. It is easy to see, that $(f_0\star\varphi)(s) = \langle f_0, T_s\varphi\rangle = (KT_s\varphi)(0) = (K\varphi)(s)$. Therefore for the elements $\boldsymbol{w} := (1, f_0, \dots, f_0^{\otimes n}, \dots)$ the following equalities $K_{\boldsymbol{w}}^{\otimes}\boldsymbol{p} = \big((f_0\star\varphi)^{\otimes n}\big) = \big((K\varphi)^{\otimes n}\big) = \big(K^{\otimes n}\varphi^{\otimes n}\big) = K^{\otimes}\boldsymbol{p}$, hold for any $\boldsymbol{p} := (\varphi^{\otimes n})$, $\varphi \in \mathcal{G}_\beta$. Hence, $K^{\otimes} = K_{\boldsymbol{w}}^{\otimes}$ and $\widetilde{K}^{\otimes} = \widetilde{K}_{\boldsymbol{w}}^{\otimes}$.

The equality $K_{\boldsymbol{u}\circledast\boldsymbol{v}}^{\otimes} = K_{\boldsymbol{u}}^{\otimes}\circ K_{\boldsymbol{v}}^{\otimes}$ implies

$$\begin{aligned}\mathcal{Q}_{\boldsymbol{u}\circledast\boldsymbol{v}} = \mathcal{F}\circ K_{\boldsymbol{u}\circledast\boldsymbol{v}}^{\otimes}\circ\mathcal{F}^{-1} &= \mathcal{F}\circ K_{\boldsymbol{u}}^{\otimes}\circ K_{\boldsymbol{v}}^{\otimes}\circ\mathcal{F}^{-1}\\ &= \mathcal{F}\circ K_{\boldsymbol{u}}^{\otimes}\circ\mathcal{F}^{-1}\circ\mathcal{F}\circ K_{\boldsymbol{v}}^{\otimes}\circ\mathcal{F}^{-1} = \mathcal{Q}_{\boldsymbol{u}}\circ\mathcal{Q}_{\boldsymbol{v}}.\end{aligned}$$

Since $\boldsymbol{\delta} = \left(\delta^{\otimes n}\right)$ is the unit element in the algebra $\left\{\Gamma(\mathcal{G}'_\beta), \circledast\right\}$, we obtain the equalities $\mathcal{Q}_{\boldsymbol{\delta}} \circ \mathcal{Q}_{\boldsymbol{u}} = \mathcal{Q}_{\boldsymbol{\delta}\circledast\boldsymbol{u}} = \mathcal{Q}_{\boldsymbol{u}} = \mathcal{Q}_{\boldsymbol{u}\circledast\boldsymbol{\delta}} = \mathcal{Q}_{\boldsymbol{u}} \circ \mathcal{Q}_{\boldsymbol{\delta}}$, i.e. $\mathcal{Q}_{\boldsymbol{\delta}} \in \mathscr{L}\big(\widetilde{\mathcal{H}}\big)$ is the identity operator. □

Remark 2. *The map* $\Gamma(\mathcal{G}'_\beta) \ni \boldsymbol{u} \longmapsto \mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{H}}$ *is a homomorphism of the algebra* $\Gamma(\mathcal{G}'_\beta)$ *and the algebra of operator valued functions, defined on* $\mathscr{G}$. *It easy to see (see formulas* (7) *and* (8)*), that the function* $\mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}}$ *of operator argument can be represented as* $\mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}} = \widetilde{\boldsymbol{u}\star\boldsymbol{p}}$. *From* (9) *it follows, that the operator* $\mathcal{Q}_{\boldsymbol{u}}\widetilde{\boldsymbol{p}}(A) = \widetilde{\boldsymbol{u}\star\boldsymbol{p}}(A) \in \mathscr{L}(\Gamma(\mathcal{H}))$ *we can understand as a "value" of a function* $\widehat{\boldsymbol{u}\star\boldsymbol{p}} \in \Gamma(E_\beta)$ *of infinite many variables at a countable system* $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots)$ *of generators of one-parameter* (C_0) *contraction groups.*

References

[1] B. Baeumer, M. Haase, M. Kovács, Unbounded functional calculus for bounded groups with applications, *J. Evol. Eqv.,* **9** (2009), no. 1, 171 - 195. https://doi.org/10.1007/s00028-009-0012-z

[2] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces,* Springer, Berlin-Göttingen-Heidelberg, 1999. https://doi.org/10.1007/978-1-4471-0869-6

[3] M. Haase, *The Functional Calculus for Sectorial Operators,* Birkhäuser-Verlag, Basel, 2006. https://doi.org/10.1007/3-7643-7698-8

[4] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups,* AMS Coll. Publ., vol. XXXI, New York, 1957.

[5] N. Kalton, P. Kunstmann, L. Weis, Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators, *Math. Ann.,* **336** (2006), no. 4, 747 - 801. https://doi.org/10.1007/s00208-005-0742-3

[6] H. Komatsu, *An Introduction to the Theory of Generalized Functions,* University Publ., Tokyo, 2000.

[7] C. Kriegler, Functional calculus and dilation for C_0-groups of polynomial growth, *Semigroup Forum,* **84** (2012), 393 - 433. https://doi.org/10.1007/s00233-012-9393-3

[8] O.V. Lopushansky, S.V. Sharyn, Generalized Hille-Phillips type functional calculus for multiparameter semigroups, *Siberian Math. J.,* **55** (2014), no. 1, 105 - 117. https://doi.org/10.1134/s0037446614010133

[9] O.V. Lopushansky, S.V. Sharyn, Polynomial ultradistributions on cone $\mathbb{R}^d_+$, *Topology,* **48** (2009), no. 2-4, 80 - 90. https://doi.org/10.1016/j.top.2009.11.005

[10] A.R. Mirotin, On some properties of the multidimensional Bochner-Phillips functional calculus, *Siberian Math. J.,* **52** (2011), no. 6, 1032 - 1041. https://doi.org/10.1134/s0037446611060085

[11] Yu.S. Samoilenko, *Spectral Theory of Families of Self-Adjoint Operators,* Kluwer Academic Publ., Dordrecht, 1991. https://doi.org/10.1007/978-94-011-3806-2

[12] S. Sharyn, Joint functional calculus in algebra of polynomial tempered distributions, *Methods Funct. Anal. Topology,* **22** (2016), no. 1, 62 - 73.

[13] S.V. Sharyn, Paley–Wiener-type theorem for polynomial ultradifferentiable functions, *Carpathian Math. Publ.,* **7** (2015), no. 2, 271 - 279. https://doi.org/10.15330/cmp.7.2.271-279

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