Operator Calculus for Noncommuting Operators over Symmetric Fock Space

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Abstract

In this paper we construct an operator calculus over the symmetric Fock space for countable set of noncommuting generators of strongly continuous groups, acting on a Hilbert space. As a symbol class of the calculus we use some algebra of functions of infinitely many variables. This algebra is described as the image of the space of polynomial ultradifferentiable functions under Fourier-Laplace transformation.

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1 Introduction

A functional (or an operator) calculus is a theory that studies how to construct functions depending on operators. An operator calculus is the useful tool, which allows us to use the functional-analytic methods in general spectral analysis and theory of Banach algebras.

There are many different approaches to construct a functional calculus for one operator acting on a Banach space. Most of them are based on some integral representation of a function from a symbol class. For Riesz-Dunford functional calculus (or $H^\infty$-calculus), based on the Cauchy formula, we refer
the reader to the book [3]. Such a functional calculus has applications, in particular, in the spectral theory of elliptic differential equations and maximal regularity of parabolic evolution equations (see e.g. [5, 7]).

$H^\infty$-calculus is a good tool when we work with generators of analytic semigroups. However, such a calculus is not suitable for an arbitrary strongly continuous semigroup. For such operators E. Hille and R. Phillips developed in [4] another method, based on the Laplace transformation. This method is known as the Hille-Phillips functional calculus. It has many helpful applications, in particular, in hydrology (see [1] and the references given there).

The Hille-Phillips functional calculus for functions of several variables was considered, for example, in [8, 10]. The case of functions of infinitely many variables is less studied. We mention the book [11] that is devoted to spectral questions (among them there is a functional calculus) of countable families of self-adjoint operators on a Hilbert space.

In the recent article [12] we construct the Hille-Phillips type functional calculus for commuting generators of strongly continuous (or $(C_0)$) semigroups over a Banach space. The main goal of this article is the construction of such calculus over the symmetric Fock space $\bigoplus_{n \in \mathbb{Z}_+} \mathcal{H}^{\otimes n}$ for countable set of noncommuting generators of $(C_0)$ groups, acting on a Hilbert space $\mathcal{H}$.

2 Notations and preliminaries

In what follows $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of all continuous linear operators from a locally convex space $\mathcal{X}$ in another such space $\mathcal{Y}$, endowed with the topology of uniform convergence on bounded subsets of $\mathcal{X}$. Let $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$. The dual space $\mathcal{X}^\prime := \mathcal{L}(\mathcal{X}, \mathbb{C})$ is endowed with strong topology. The pairing between elements of $\mathcal{X}^\prime$ and $\mathcal{X}$ we denote $\langle \cdot, \cdot \rangle$.

Spaces of functions. Let us fix any real $\beta > 1$. An infinitely differentiable function $\varphi$ is called to be a Gevrey ultradifferentiable if for each segment $[\mu, \nu] \subset \mathbb{R}$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t \in [\mu, \nu]} |\partial^k \varphi(t)| \leq C h^k k^{\beta}$ holds for all $k \in \mathbb{Z}_+$. For a fixed $h > 0$ let us consider the subspace

$$
\mathcal{G}^h_{\beta}[\mu, \nu] := \{ \varphi \in C^\infty : \text{supp} \varphi \subset [\mu, \nu], \| \varphi \|_{\mathcal{G}^h_{\beta}[\mu, \nu]} := \sup_{k \in \mathbb{Z}_+} \sup_{t \in [\mu, \nu]} \frac{|\partial^k \varphi(t)|}{h^k k^{\beta}} < \infty \}.
$$

In [6] it is proved that each subspace $\mathcal{G}^h_{\beta}[\mu, \nu]$ is a Banach space and maps $\mathcal{G}^h_{\beta}[\mu, \nu] \ni \mathcal{G}^l_{\beta}[\mu, \nu]$, with $h < l$, are compact inclusions. Consider the space

$$
\mathcal{G}_{\beta} := \bigcup_{\mu < \nu, h > 0} \mathcal{G}^h_{\beta}[\mu, \nu], \quad \mathcal{G}_{\beta} \simeq \lim \text{ind} \mathcal{G}^h_{\beta}[\mu, \nu],
$$
of Gevrey ultradifferentiable functions with compact supports and endow it with topology of inductive limit with respect to above mentioned compact inclusions. Let $\mathcal{G}_\beta'$ be its dual space of Roumieu ultradistributions.

Let $h > 0$ be any positive real and $\mu, \nu \in \mathbb{R}$ be any reals such that $\mu < \nu$. In the space of entire functions of exponential type we consider the subspace $E^h_\beta[\mu, \nu]$ of functions with the finite norm

$$\|\psi\|_{E^h_\beta[\mu, \nu]} := \sup_{k \in \mathbb{Z}_+} \sup_{z \in \mathbb{C}} \frac{|z^k \psi(z) e^{-H_{\mu, \nu}(t)}|}{h^k k^\beta},$$

where $H_{\mu, \nu}(t) := \sup_{t \in [\mu, \nu]} t \eta$.

Each space $E^h_\beta[\mu, \nu]$ is a Banach, and all maps $E^h_\beta[\mu, \nu] \hookrightarrow E^{h'}_\beta[\mu', \nu']$ with $[\mu, \nu] \subset [\mu', \nu']$, $h < h'$, are compact inclusions. Consider the space

$$E_\beta := \bigcup_{\mu < \nu, h > 0} E^h_\beta[\mu, \nu], \quad E_\beta \simeq \lim_{\mu < \nu, h > 0} \text{ind } E^h_\beta[\mu, \nu],$$

and endow it with the topology of inductive limit with respect to above mentioned compact inclusions.

Consider the Fourier-Laplace transformation

$$\hat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}} e^{-itz} \varphi(t) \, dt, \quad \varphi \in \mathcal{G}_\beta, \ z \in \mathbb{C}.$$ 

It is known [13], that $F(\mathcal{G}_\beta) = E_\beta$.

**Polynomial ultradifferentiable functions and polynomial ultradistributions.** For any locally convex space $\mathcal{X}$, let $\mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$, be the symmetric $n$th tensor degree of $\mathcal{X}$, completed in the projective tensor topology. For any $x \in \mathcal{X}$ we denote $x^{\otimes n} := \underbrace{x \otimes \cdots \otimes x}_{n} \in \mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$. Set $\mathcal{X}^{\otimes 0} := \mathbb{C}$, $x^{\otimes 0} := 1 \in \mathbb{C}$.

To define the locally convex space $\mathcal{P}(\mathcal{G}_\beta')$ of $n$-homogeneous polynomials on $\mathcal{G}_\beta'$ we use the canonical topological linear isomorphism $\mathcal{P}(\mathcal{G}_\beta') \simeq (\mathcal{G}_\beta'^{\otimes n})'$, described in [2]. We equip $\mathcal{P}(\mathcal{G}_\beta')$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{G}_\beta'$. Set $\mathcal{P}(\mathcal{G}_\beta') := \mathbb{C}$. The space $\mathcal{P}(\mathcal{G}_\beta')$ of all continuous polynomials on $\mathcal{G}_\beta'$ is defined to be the complex linear span of all $\mathcal{P}(\mathcal{G}_\beta')$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{G}_\beta')$ mean the strong dual of $\mathcal{P}(\mathcal{G}_\beta')$.

Elements of the spaces $\mathcal{P}(\mathcal{G}_\beta')$ and $\mathcal{P}'(\mathcal{G}_\beta')$ we call the polynomial test ultradifferentiable functions and polynomial ultradistributions, respectively.

Denote $\Gamma(\mathcal{G}_\beta) := \bigoplus_{n \in \mathbb{Z}_+} \mathcal{G}_\beta^{\otimes n}$ and $\Gamma(\mathcal{G}_\beta') := \times_{n \in \mathbb{Z}_+} \mathcal{G}_\beta'^{\otimes n}$. Note, that we consider the case when the elements of direct sum consist of finite but not fixed number of addends. In what follows elements of the spaces $\Gamma(\mathcal{G}_\beta)$ and $\Gamma(\mathcal{G}_\beta')$ will be written as $\mathbf{p} = (p_n)$ and $\mathbf{u} = (u_n)$, respectively.
For elements of total subset of the space $\Gamma(\mathcal{G}_\beta')$ let us define the operation 

\[(f^{\otimes n}) \otimes (g^{\otimes n}) := ((f \ast g)^{\otimes n})\]

and extend it onto whole space by linearity and continuity. It is easy to see, that $\Gamma(\mathcal{G}_\beta')$ is an algebra with respect to $\otimes$. Since the space $\Gamma(\mathcal{G}_\beta)$ is dense in $\Gamma(\mathcal{G}_\beta')$ (see [9]), the space $\Gamma(\mathcal{G}_\beta)$ also is an algebra with respect to the operation $\otimes$.

Using the tensor-Laplace transformation onto $\Gamma(\mathcal{G}_\beta)$. First, for elements of total subset of the space $\mathcal{G}_\beta^{\otimes n}$ we define the operator $\mathcal{F}^{\otimes n} : \varphi^{\otimes n} \mapsto \hat{\varphi}^{\otimes n}$, $\mathcal{F}^{\otimes 0} := I_{\mathbb{C}}$, where $\hat{\varphi}^{\otimes n} := (F\varphi)^{\otimes n}$. Next, we extend the map $\mathcal{F}^{\otimes n}$ onto whole space $\mathcal{G}_\beta^{\otimes n}$ by linearity and continuity. As a result we obtain the map $\mathcal{F}^{\otimes n} \in \mathcal{L}(\mathcal{G}_\beta^{\otimes n}, E_{\beta}^{\otimes n})$.

And finally, we define the mapping $\mathcal{F}^{\otimes}$ by the formula

\[
\mathcal{F}^{\otimes} := (\mathcal{F}^{\otimes n}) : \Gamma(\mathcal{G}_\beta) \ni p = (p_n) \mapsto \hat{p} := (\hat{p}_n) \in \Gamma(E_{\beta}) := \bigoplus_{n \in \mathbb{N}_+} E_{\beta}^{\otimes n},
\]

where $p_n \in \mathcal{G}_\beta^{\otimes n}$, $\hat{p}_n := \mathcal{F}^{\otimes n}p_n \in E_{\beta}^{\otimes n}$.

Note, that for each $n \in \mathbb{N}$ an element $\hat{p}_n$ is a symmetric function of $n$ complex variables $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto \hat{p}(z_1, \ldots, z_n) \in \mathbb{C}$, i.e. $\hat{p}_n(z_1, z_2, \ldots, z_n) = \hat{p}_n(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)})$ for every permutation $\sigma$ of $\{1, \ldots, n\}$. It implies that elements of the space $\Gamma(E_{\beta})$ can be considered as functions $\hat{p} : \times_{n \in \mathbb{N}} \mathbb{C} \mapsto \mathbb{C}$ of infinite many variables 

\[
\hat{p} : (z_1, \ldots, z_n, \ldots) \mapsto \hat{p}(z_1, \ldots, z_n, \ldots) = \hat{p}_0 + \sum_{n \in \mathbb{N}} \hat{p}_n(z_{b_n}, \ldots, z_{c_n}), \quad (1)
\]

where $b_n := \frac{n(n-1)}{2} + 1$, $c_n := \frac{n(n+1)}{2}$. But we note that actually each function $\hat{p} \in \Gamma(E_{\beta})$ depends on finite (depending on $\hat{p}$) number of variables, because for each $\hat{p}$ the sequence in the right hand side of (1) is finite.

For any operator $K \in \mathcal{L}(\mathcal{G}_\beta)$ let us define the operator $K^{\otimes} \in \mathcal{L}(\Gamma(\mathcal{G}_\beta))$ by the formula

\[
K^{\otimes} := (K^{\otimes n}) : p = (p_n) \mapsto K^{\otimes}p := (K^{\otimes n}p_n), \quad (2)
\]

where $K^{\otimes 0} := I_{\mathbb{C}}$ is the identity, and the each operator $K^{\otimes n} \in \mathcal{L}(\mathcal{G}_\beta^{\otimes n})$, $n \in \mathbb{N}$, is defined as linear and continuous extension of the following map $\varphi^{\otimes n} \mapsto (K\varphi)^{\otimes n}$, $\varphi \in \mathcal{G}_\beta$.

For any ultradistribution $f \in \mathcal{G}_\beta'$ and ultradifferentiable function $\varphi \in \mathcal{G}_\beta$ the cross-correlation is the function defined as follows $(f \ast \varphi)(s) := \langle f(t), \varphi(t + s) \rangle$. It is easy to see, that the cross-correlation operator $K_f : \varphi \mapsto f \ast \varphi$ belongs to the space $\mathcal{L}(\mathcal{G}_\beta')$ for any ultradistribution $f \in \mathcal{G}_\beta'$.

Using the definition (2) we obtain

\[
K_{u}^{\otimes} := (K_{u_n}^{\otimes n}) \in \mathcal{L}(\Gamma(\mathcal{G}_\beta)) \quad \text{and} \quad K_{u_n}^{\otimes n} \in \mathcal{L}(\mathcal{G}_\beta^{\otimes n}), \quad (3)
\]
where \( u := (u_n) \in \Gamma(G'_+^n) \) with \( u_n \in G'_+^n, n \in \mathbb{Z}_+ \).

Let us define the operation, which is the extension of the cross-correlation on the spaces of polynomial ultradifferentiable functions and ultradistributions. For any \( u = (u_n) \in \Gamma(G'_+^n) \) and \( p = (p_n) \in \Gamma(G'_+^n) \) their cross-correlation is the element \( u \star p := K \otimes u \otimes p = (K \otimes u_n \otimes p_n) \).

**Infinite parameter operator groups.** Let a countable set of operators \( A = (A_1, A_2, \ldots, A_n, \ldots) \) be defined in a complex Hilbert space \( H \). Note, that we do not assume any commutativity relations. Denote by \( \Gamma(H) := \bigoplus_{n \in \mathbb{Z}_+^n} H^\otimes n \) the symmetric Fock space.

Suppose, that \( A_j, j \in \mathbb{N}, \) generates an one-parameter \((C_0)\) group (see [4]) \( \mathbb{R} \ni t \mapsto e^{-itA_j} \in \mathcal{L}(H), \) which satisfies the condition

\[
\sup_{t \in \mathbb{R}} \|e^{-itA_j}\|_{\mathcal{L}(H)} \leq 1. \tag{4}
\]

Let us denote \( \mathcal{A}_j := I_H \otimes \cdots \otimes I_H \otimes A_j \otimes I_H \otimes \cdots \in \mathcal{L}(\Gamma(H)), j \in \mathbb{N}, \)
\( A_n := \mathcal{A}_b \otimes \cdots \otimes \mathcal{A}_e, n \in \mathbb{N}. \) Set \( \mathcal{A}_0 := I_H \otimes I_H \otimes \cdots \in \mathcal{L}(\Gamma(H)), A_0 := \mathcal{A}_0 \) by definition.

Instead of the set \( A \) of (noncommuting) operators over Hilbert space \( H \) we consider countable set of commuting operators, acting in the symmetric Fock space \( \Gamma(H) \), namely

\[
A := (A_0, A_1, A_2, \ldots, A_n, \ldots). \tag{5}
\]

It easy to see, that each \( A_n \) generates strong continuous \( n \)-parameter group \( \mathbb{R}^n \ni t \mapsto e^{-itA_n} \in \mathcal{L}(\Gamma(H)), \) where \( e^{-itA_n} := e^{-it\mathcal{A}_b} \otimes \cdots \otimes e^{-it\mathcal{A}_e}, \) and \( e^{-it\mathcal{A}_j} := I_H \otimes \cdots \otimes I_H \otimes e^{-itA_j} \otimes I_H \otimes \cdots, \) \( i = 1, \ldots, n, j \in \mathbb{N}. \) Note, that each one-parameter group \( e^{-itA_j} \) satisfies the condition (4).

Operator \( A_n \) and group \( e^{-itA_n} \) are defined on whole Fock space \( \Gamma(H) \), but they do not act as identity only on \( H^\otimes n. \) So, without restriction of generality we can write \( A_n \in \mathcal{L}(H^\otimes n), e^{-itA_n} \in \mathcal{L}(H^\otimes n), n \in \mathbb{Z}_+. \)

Let \( \mathcal{G} \) be the set of countable systems of operators of view (5). For all \( n \in \mathbb{N} \) let \( \mathcal{G}_n \) be a set of collections of operators of view \( A_n = \mathcal{A}_b \otimes \cdots \otimes \mathcal{A}_e. \) Set \( \mathcal{G}_0 := \{ \mathcal{A}_0 \} \) by definition.
3 Functional calculus for countable set of operators

For all $n \in \mathbb{Z}_+$ let us define the set $\mathcal{H}_n := \{ \tilde{p}_n : \mathcal{G} \to \mathcal{L}(\mathcal{H}^\otimes n) : p_n \in G_\beta^\otimes n \}$, which consist of functions of operator argument

$$\tilde{p}_n(A_n) := \int_{\mathbb{R}^n} e^{-it_1 \alpha_{kn}} \cdots e^{-it_n \alpha_{kn}} p_n(t_1, \ldots, t_n) \, dt_1 \ldots dt_n. \quad (6)$$

Set $\tilde{p}_0 : \mathcal{G} \ni A_0 \mapsto \tilde{p}_0(A_0) := p_0 I_C \in \mathcal{L}(C)$ by definition.

Define the map

$$\mathcal{F} := (\mathcal{F}_n) : \Gamma(G_\beta) \ni \mathcal{p} = (p_n) \longmapsto \tilde{\mathcal{p}} := \sum_{n \in \mathbb{Z}_+} \tilde{p}_n \in \mathcal{H}, \quad (7)$$

where $\mathcal{H} := \sum_{n \in \mathbb{Z}_+} \mathcal{H}_n$. Condition (4) and [4, Theorem 15.2.1] imply, that all mappings $\mathcal{F}_n : p_n \mapsto \tilde{p}_n$, $n \in \mathbb{Z}_+$, are isomorphisms.

Note, that $H := \{ \tilde{\mathcal{p}} : \mathcal{G} \to \mathcal{L}(\Gamma(\mathcal{H})) : \mathcal{p} \in \Gamma(G_\beta) \}$ is an algebra of functions with pointwise multiplication $(\tilde{\mathcal{p}} \cdot \tilde{\mathcal{q}})(A) := \tilde{\mathcal{p}}(A) \circ \tilde{\mathcal{q}}(A)$.

**Remark 1.** The mapping $\mathcal{F} : \Gamma(G_\beta) \rightarrow \mathcal{H}$ acts as homomorphism of algebra $\{ \Gamma(G_\beta), \otimes \}$ into algebra $\{ \mathcal{H}, \}$. On the other hand, results of the article [13] imply that there exists a homomorphism $F^\otimes : \Gamma(G_\beta) \rightarrow \Gamma(E_\beta)$. Therefore, the map $\mathcal{F} \circ (F^\otimes)^{-1} : \Gamma(E_\beta) \rightarrow \mathcal{H}$ we may treat as “elementary” functional calculus. In other words, we understand the operator $\tilde{\mathcal{p}}(A) := \sum_n \tilde{p}_n(A_n) \in \mathcal{L}(\Gamma(\mathcal{H}))$ as a “value” of a function $\tilde{\mathcal{p}}$ of infinite many variables (see (1)) at a countable system $A = (A_0, A_1, A_2, \ldots, A_n, \ldots) \in \mathcal{G}$ of operators (see (5)).

Consider the one-parameter semigroup $T^\otimes : \mathbb{R} \ni s \mapsto T^\otimes_s \in \mathcal{L}(\mathcal{H})$ on the space $\mathcal{H}$, where

$$T^\otimes_s := (T^\otimes_s)^n) : \tilde{\mathcal{p}} = \sum_{n \in \mathbb{Z}_+} \tilde{p}_n \longmapsto \tilde{T}^\otimes_s \tilde{\mathcal{p}} := \sum_{n \in \mathbb{Z}_+} \tilde{T}^\otimes_s \tilde{p}_n. \quad \text{The function } \tilde{T}^\otimes_s \tilde{p}_n \in \tilde{\mathcal{H}}_n \text{ is defined to be the map } \tilde{T}^\otimes_s \tilde{p}_n : \mathcal{G} \ni A_n \mapsto \tilde{T}^\otimes_s \tilde{p}_n(A_n) \in \mathcal{L}(\mathcal{H}^\otimes n), \text{ where }$$

$$\tilde{T}^\otimes_s \tilde{p}_n(A_n) := \int_{\mathbb{R}^n} e^{-it_1 \alpha_{kn}} \cdots e^{-it_n \alpha_{kn}} p_n(t_1 + s, \ldots, t_n + s) \, dt_1 \ldots dt_n. \quad \text{Here the function } \tilde{p}_n \text{ of operator argument is defined by (6).}$$
Using Bochner’s integral properties (see [4, 3.7]), we obtain that for any \( p = (p_n) \in \Gamma(\mathcal{G}_\beta) \) with \( p_n = \varphi^{\otimes n} \in \mathcal{G}_\beta^{\otimes n} \), \( \varphi \in \mathcal{G}_\beta \), the following equalities

\[
\tilde{T}_s^{\otimes p}(A) = \mathcal{F}[((T_s \varphi)^{\otimes n})](A) = \mathcal{F}[(T_s \varphi)^{\otimes n}] = I_C + \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^n} e^{-it_1 \alpha_{t_1}} \cdots e^{-it_n \alpha_{t_n}} (T_s \varphi)^{\otimes n} (t_1, \ldots, t_n) \, dt_1 \cdots dt_n
\]

\[
= I_C + \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^n} e^{-it_1 \alpha_{t_1}} \cdots e^{-it_n \alpha_{t_n}} \varphi(t_k + s) \, dt_1 \cdots dt_n
\]

\[
= I_C + \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^n} e^{-it_1 \alpha_{t_1}} \cdots e^{-it_n \alpha_{t_n}} p_n(t_1 + s, \ldots, t_n + s) \, dt_1 \cdots dt_n
\]

\[
= \tilde{p}_0(A_0) + \sum_{n \in \mathbb{N}} \tilde{T}_s^{\otimes p} = \tilde{T}_s^{\otimes p}(A)
\]

hold for all \( s \in \mathbb{R} \) and \( A = (A_n) \in \mathcal{G} \).

Hence, the operator \( \tilde{T}_s^{\otimes p} \) can be represented as follows: \( \tilde{T}_s^{\otimes p} = \mathcal{F} \circ T_s^{\otimes} \circ \mathcal{F}^{-1} \). Continuity of the mappings \( T_s^{\otimes} \) and \( \mathcal{F} \) as well as openness of \( \mathcal{F} \) imply that the group \( \tilde{T}^{\otimes}: \mathbb{R} \ni s \mapsto \tilde{T}_s^{\otimes} \in \mathcal{L}(\tilde{\mathcal{H}}) \) has the \((C_0)\) property.

The commutant of the group \( \tilde{T}^{\otimes} \) is defined to be the set

\[
[\tilde{T}^{\otimes}]^c := \{ \tilde{T} \in \mathcal{L}(\tilde{\mathcal{H}}): \tilde{T} \circ \tilde{T}_s^{\otimes} = \tilde{T}_s^{\otimes} \circ \tilde{T}, \forall s \in \mathbb{R} \}.
\]

Define the mapping

\[
\mathcal{Q} := (\mathcal{Q}_n) : \Gamma(\mathcal{G}_\beta) \ni u = (u_n) \mapsto \mathcal{Q}_u := \sum_{n \in \mathbb{Z}^+} \mathcal{Q}_{u_n} \in \mathcal{L}(\tilde{\mathcal{H}}), \quad (8)
\]

where \( u_n := f^{\otimes n} \in \mathcal{G}_\beta^{\otimes n} \), \( f \in \mathcal{G}_\beta \). Here \( \mathcal{Q}_{u_n} \in \mathcal{L}(\tilde{\mathcal{H}}_n), n \in \mathbb{Z}^+ \), is defined by the following formulas:

\[
(\mathcal{Q}_{u_n} \tilde{p}_n)(A_0) := I_C \text{ and } \mathcal{Q}_{u_n} : \tilde{p}_n \mapsto \mathcal{Q}_{u_n} \tilde{p}_n, n \in \mathbb{N},
\]

where

\[
(\mathcal{Q}_{u_n} \tilde{p}_n)(A_n) := \int_{\mathbb{R}^n} e^{-it_1 \alpha_{t_1}} \cdots e^{-it_n \alpha_{t_n}} K_f^{\otimes n} p_n(t_1, \ldots, t_n) \, dt_1 \cdots dt_n.
\]

Here the function \( \tilde{p}_n \) of operator argument is defined by (6), and the operator \( K_f^{\otimes n} \) is defined by (2) and (3).

**Theorem 3.1.** The map \( \mathcal{Q} \), defined by (8), is an algebraic isomorphism of the algebra \( \{ \Gamma(\mathcal{G}_\beta), \otimes \} \) and the subalgebra in the commutant \( [\tilde{T}^{\otimes}]^c \) of operators of view \( \tilde{K}^{\otimes} = \mathcal{F} \circ K^{\otimes} \circ \mathcal{F}^{-1} \in \mathcal{L}(\tilde{\mathcal{H}}) \), where \( K \in \mathcal{L}(\mathcal{G}_\beta) \). In particular, the equality \( \mathcal{Q}_{u \otimes v} = \mathcal{Q}_u \circ \mathcal{Q}_v \) holds for all \( u, v \in \Gamma(\mathcal{G}_\beta) \) and \( \mathcal{Q}_\delta \) is the identity in \( \mathcal{L}(\tilde{\mathcal{H}}) \), where \( \delta = (\delta^{\otimes n}) \).
Proof. Let $u = (u_n) \in \Gamma(G'_\beta)$ and $p = (p_n) \in \Gamma(G_\beta)$, where $u_n = f^{\otimes n}$, $f \in G'_\beta$. The following equalities are valid

$$(Q_u \tilde{p})(A) = \sum_{n \in \mathbb{Z}_+} (Q_{u_n} \tilde{p}_n)(A_n)$$

$$= I_C + \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^n} e^{-it_1 \partial_{x_n}} \otimes \ldots \otimes e^{-it_n \partial_{x_n}} K_f^{\otimes n} p_n(t_1, \ldots, t_n) \, dt_1 \ldots dt_n \quad (9)$$

$${\mathcal{F}}[K_f^{\otimes n} p_n](A) = \tilde{K}_u \tilde{p}(A), \quad K_u = (K_f^{\otimes n}),$$

for all $A = (A_n) \in \mathcal{G}$. It follows that the map $Q$ can be represented in the form $Q_u = \mathcal{F} \circ K_u^{\otimes} \circ \mathcal{F}^{-1}$. Continuity of the mappings $K_u^{\otimes}$ and $\mathcal{F}$ as well as openness of $\mathcal{F}$ imply that $Q_u \in \mathcal{L}(\hat{\mathcal{H}})$ for all $u \in \Gamma(G'_\beta)$. Therefore, the equalities

$$(Q_u \tilde{t}_s \tilde{p})(A) = \sum_{n \in \mathbb{Z}_+} (Q_{u_n} \tilde{t}_s \tilde{p}_n)(A_n)$$

$$= I_C + \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^n} e^{-it_1 \partial_{x_n}} \otimes \ldots \otimes e^{-it_n \partial_{x_n}} K_f^{\otimes n} p_n(t_1 + s, \ldots, t_n + s) \, dt$$

$$= I_C + \sum_{n \in \mathbb{N}} \tilde{t}_s^{\otimes n} \int_{\mathbb{R}^n} e^{-it_1 \partial_{x_n}} \otimes \ldots \otimes e^{-it_n \partial_{x_n}} K_f^{\otimes n} p_n(t_1, \ldots, t_n) \, dt$$

$$= \sum_{n \in \mathbb{Z}_+} (\tilde{t}_s^{\otimes n} Q_{u_n} \tilde{p}_n)(A_n) = (\tilde{t}_s^{\otimes} Q_u \tilde{p})(A), \quad dt := dt_1 \ldots dt_n,$$

hold for all $s \in \mathbb{R}$, $\tilde{p} = \sum_{n \in \mathbb{Z}_+} \tilde{p}_n \in \hat{\mathcal{H}}$ and $A := (A_n) \in \mathcal{G}$. Hence, for all $u \in \Gamma(G'_\beta)$ the operator $Q_u$ belongs to the commutant $[\hat{T}^{\otimes}]^c$.

Conversely, let the operator $\tilde{K}^{\otimes} = \mathcal{F} \circ K^{\otimes} \circ \mathcal{F}^{-1} \in \mathcal{L}(\hat{\mathcal{H}})$ with $K \in \mathcal{L}(G'_\beta)$ belongs to the commutant $[\hat{T}^{\otimes}]^c$. Then

$${\mathcal{F}} \circ K^{\otimes} \circ T_s^{\otimes} \circ \mathcal{F}^{-1} = {\mathcal{F}} \circ K^{\otimes} \circ \mathcal{F}^{-1} \circ {\mathcal{F}} \circ T_s^{\otimes} \circ \mathcal{F}^{-1} = \tilde{K}^{\otimes} \circ \tilde{T}_s^{\otimes} = \tilde{T}_s^{\otimes} \circ \tilde{K}^{\otimes}$$

$$= {\mathcal{F}} \circ T_s^{\otimes} \circ \mathcal{F}^{-1} \circ {\mathcal{F}} \circ K^{\otimes} \circ \mathcal{F}^{-1} = {\mathcal{F}} \circ T_s^{\otimes} \circ K^{\otimes} \circ \mathcal{F}^{-1}.$$

Therefore the operator $K^{\otimes}$ belongs to the commutant of the semigroup $T_s^{\otimes}$.

Let us define the ultradistribution $f_0 \in G'_\beta$ as follows $\langle f_0, \varphi \rangle := (K\varphi)(0)$ for any $\varphi \in G_\beta$. It is easy to see, that $\langle f_0 \ast \varphi \rangle(s) = (f_0, T_s \varphi) = (K T_s \varphi)(0) = (K \varphi)(s)$. Therefore for the elements $w := (1, f_0, f_0^{\otimes n}, \ldots)$ the following equalities $K^{\otimes} p = ((f_0 \ast \varphi)^{\otimes n}) = ((K \varphi)^{\otimes n}) = (K^{\otimes n} \varphi^{\otimes n}) = K^{\otimes} p$, hold for any $p := (\varphi^{\otimes n})$, $\varphi \in G_\beta$. Hence, $K^{\otimes} = K^{\otimes} w$ and $\tilde{K}^{\otimes} = \tilde{K}^{\otimes} w$.

The equality $K_{u \otimes v} = K^{\otimes} u \otimes K^{\otimes} v$ implies

$$Q_{u \otimes v} = {\mathcal{F}} \circ K_{u \otimes v}^{\otimes} \circ \mathcal{F}^{-1} = {\mathcal{F}} \circ K^{\otimes} u \otimes K^{\otimes} v \circ \mathcal{F}^{-1}$$

$$= {\mathcal{F}} \circ K^{\otimes} u \circ \mathcal{F}^{-1} \circ {\mathcal{F}} \circ K^{\otimes} v \circ \mathcal{F}^{-1} = Q_u \circ Q_v.$$
Since \( \delta = (\delta \otimes^n) \) is the unit element in the algebra \( \{ \Gamma(\mathcal{G}_\beta'), \otimes \} \), we obtain the equalities \( Q_\delta \circ Q_u = Q_\delta \otimes u = Q_u = Q_u \otimes \delta = Q_u \circ Q_\delta \), i.e. \( Q_\delta \in \mathcal{L}(\hat{\mathcal{H}}) \) is the identity operator.

**Remark 2.** The map \( \Gamma(\mathcal{G}_\beta') \ni u \mapsto Q_u \hat{\mathcal{P}} \in \hat{\mathcal{H}} \) is a homomorphism of the algebra \( \Gamma(\mathcal{G}_\beta') \) and the algebra of operator valued functions, defined on \( \mathcal{G} \). It easy to see (see formulas (7) and (8)), that the function \( Q_u \hat{\mathcal{P}} \) of operator argument can be represented as \( Q_u \hat{\mathcal{P}} = \hat{u} \times \hat{\mathcal{P}} \). From (9) it follows, that the operator \( Q_u \hat{\mathcal{P}}(A) = u \times \hat{\mathcal{P}}(A) \in \mathcal{L}(\Gamma(\mathcal{H})) \) we can understand as a “value” of a function \( \hat{u} \times \hat{\mathcal{P}} \in \Gamma(\mathcal{E}_\beta) \) of infinite many variables at a countable system \( (A_1, A_2, \ldots, A_n, \ldots) \) of generators of one-parameter \( (C_0) \) contraction groups.

**References**


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