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Global Well-Posedness of Weak Solutions to the Time-Dependent Ginzburg-Landau Model for Superconductivity in $\mathbb{R}^2$

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Abstract

We prove the global existence and uniqueness of weak solutions to the time dependent Ginzburg-Landau system in superconductivity in $\mathbb{R}^2$ with Lorentz or Coulomb gauge.

Mathematics Subject Classifications: 35A05, 35A40, 35K55, 82D55

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1 Introduction

We consider the existence and uniqueness problem for the Ginzburg-Landau model in superconductivity:

$$\eta\partial_t\psi + i\eta k\phi\psi + \left(\frac{i}{k}\nabla + A\right)^2\psi + (|\psi|^2 - g)\psi = 0, \tag{1.1}$$

$$\partial_t A + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\} = \operatorname{curl} H, \tag{1.2}$$

$$(\psi, A)(\cdot, 0) = (\psi_0, A_0)(\cdot) \ \text{ in } \ \mathbb{R}^2. \tag{1.3}$$

Here, the unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^2$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. Two positive constants η and κ are Ginzburg-Landau constants, g is a positive function that depends on the material as well as on the temperature and other variables, H is the applied magnetic field, and $i := \sqrt{-1}$. $\overline{\psi}$ denotes the complex conjugate of $\psi, \mathrm{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ, and $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers. T is any given positive constant.

It is well-known that the Ginzburg-Landau equations are gauge invariant, namely, if (ψ, A, ϕ) is a solution of (1.1)-(1.4), then $(\psi e^{ik\chi}, A + \nabla\chi, \phi - \partial_t\chi)$ is also a solution for any real-valued smooth function χ. Accordingly, in order to obtain the well-posedness of the problem, we need to impose some gauge condition. From physical point of view, one may usually think of four types of the gauge condition:

(1) Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_\Omega \phi dx = 0$.

(2) Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω.

(3) Lorenz gauge: $\partial_t\phi = -\operatorname{div} A$ in Ω.

(4) Temporal gauge (Weyl gauge): $\phi = 0$ in Ω.

For the initial data $(\psi_0, A_0) \in W_0 := \{(\psi_0, A_0)\big|\psi_0 \in L^\infty \cap H^1, A_0 \in H^1\}$, Chen et al. [2, 3], Du [4], Fan and Ozawa [7], and Tang [12] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb, Lorenz and Lorentz as well as temporal gauges.

For the initial data $\psi_0, A_0 \in L^2$, under the Coulomb or Lorentz gauge, Tang and Wang (2-D) [13], Fan and Jiang (3-D) [6] proved the global existence of weak solutions. Recently, Fan and Ozawa [8] (2-D) and Fan, Gao and Guo [5] (3-D) prove the global existence and uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$.

Here we point out that all the above results [2, 3, 4, 7, 12, 13, 6, 8, 5] require $g = 1$ and in a bounded domain Ω.

When $g = 1$ and $\Omega := \mathbb{R}^3$, there were many studies. For given initial data $\psi_0 \in H^1 \cap L^\infty,\ A_0 \in H^1$, Guo-Yuan [9] proved the global well-posedness of strong solutions to the problem (1.1)-(1.3). For given initial data $\psi_0 \in L^2 \cap L^3,\ A_0 \in L^2$, Rodriguez-Bernal and Wang [11] showed the global existence of weak solutions; furthermore, they also proved the uniqueness of weak solutions if $\psi_0 \in L^2 \cap L^4$ and $A_0 \in L^3$. Fan-Jiang [6] proved the global existence of weak solutions when $\psi_0,\ A_0 \in L^2$. Fan, Gao and Guo [5] obtained uniqueness of weak solutions for $\psi_0,\ A_0 \in L^2 \cap L^3$.

We will assume that

$$g := g(x,t) \in L^p(0,T;L^q) \text{ with } \frac{1}{p} + \frac{1}{q} = 1,\ 1 < p < \infty \text{ and } 1 < q < \infty, \tag{1.4}$$

$$H := H(x,t) \in L^2(0,T;L^2) \tag{1.5}$$

and denote

$$\operatorname{curl} H := \begin{pmatrix} \partial_2 H \\ -\partial_1 H \end{pmatrix}$$

for scalar function H.

The aim of this paper is to study the well-posedness of the problem (1.1)-(1.3) under the conditions (1.4) and (1.5), we will prove

Theorem 1.1. *Let $\psi_0,\ A_0 \in L^2$ and (1.4) and (1.5) hold true. Then there exists a unique weak solution (ψ, A) of (1.1)-(1.3) with the choice of Lorentz gauge, such that*

$$\psi, A \in W := L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(\mathbb{R}^2 \times (0,T)), \tag{1.6}$$

$$\partial_t \psi, \partial_t A \in W' := \text{ the dual space of } W \tag{1.7}$$

for any $T > 0$.

Remark 1.1. *We can prove a similar result with the choice of Coulomb gauge and thus we omit the details here.*

Remark 1.2. *When the space dimension $d = 3$, the local well-posedness of mild solutions was proved in [1] by the method of Kato [10] when $\psi_0, A_0 \in L^3$. It is an open problem to prove a global result with large data.*

2 Proof of Theorem 1.1

By the results proved in [9], one can prove a similar well-posedness result of strong solutions in $\mathbb{R}^2$. We take $\psi_{0n} \in H^1 \cap L^\infty, A_{0n} \in H^1, g_n \in H^2(\mathbb{R}^2 \times (0,T))$ and $H_n \in H^2(\mathbb{R}^2 \times (0,T))$ such that

$$\|\psi_{0n} - \psi_0\|_{L^2} \to 0,\ \|A_{0n} - A_0\|_{L^2} \to 0,$$
$$\|g_n - g\|_{L^p(0,T;L^q)} \to 0,\ \|H_n - H\|_{L^2(0,T;L^2)} \to 0 \tag{2.1}$$

as $n \to \infty$. Thus we have a unique strong solution ψ_n, A_n with the data $(\psi_{0n}, A_{0n}, g_n, H_n)$. We want to establish a priori estimates (1.6) and (1.7) uniformly with respect to n. Then by the standard compactness argument, we can get $\psi_n \to \psi$ and $A_n \to A$ as $n \to \infty$, thus we conclude that the existence of weak solutions and a prior estimates (1.6) and (1.7). Now we drop the subscript "n" of ψ_n and A_n and do as follows.

Multiplying (1.1) by $\overline{\psi}$, integrating by parts, and then taking the real part, we see that

$$
\begin{aligned}
& \frac{\eta}{2}\frac{d}{dt}\int |\psi|^2 dx + \int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 dx + \int |\psi|^4 dx = \int g|\psi|^2 dx \\
\leq\ & \|g\|_{L^q}\|\psi\|^2_{L^{\frac{2q}{q-1}}} \\
\leq\ & C\|g\|_{L^q}\|\psi\|_{L^2}^{2\left(1-\frac{1}{q}\right)}\|\nabla|\psi|\|_{L^2}^{\frac{2}{q}} \\
\leq\ & \frac{1}{2}\left\|\frac{1}{k}\nabla|\psi|\right\|^2_{L^2} + C\|g\|^p_{L^q}\|\psi\|^2_{L^2} \\
\leq\ & \frac{1}{2}\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2_{L^2} + C\|g\|^p_{L^q}\|\psi\|^2_{L^2},
\end{aligned}
$$

which gives

$$\|\psi\|_{L^\infty(0,T;L^2)} + \|\psi\|_{L^4(0,T;L^4)} \leq C, \tag{2.2}$$

$$\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2(0,T;L^2)} \leq C. \tag{2.3}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\psi\|_{L^{\frac{2q}{q-1}}} \leq C\|\psi\|_{L^2}^{1-\frac{1}{q}}\|\nabla|\psi|\|_{L^2}^{\frac{1}{q}}, \tag{2.4}$$

and the diamagnetic inequality

$$\left|\frac{1}{k}\nabla|\psi|\right| \leq \left|\frac{i}{k}\nabla\psi + \psi A\right|. \tag{2.5}$$

Testing (1.2) by A and using (2.2) and (2.3), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |A|^2dx + \int |\nabla A|^2 dx \\
= \ & \int H \operatorname{curl} A dx - \operatorname{Re}\int \left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi} A dx \\
\leq \ & \|H\|_{L^2}\|\operatorname{curl} A\|_{L^2} + \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2}\|\psi\|_{L^4}\|A\|_{L^4} \\
\leq \ & C\|H\|_{L^2}\|\nabla A\|_{L^2} + C\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2}\|\psi\|_{L^4}\|A\|_{L^2}^{\frac{1}{2}}\|\nabla A\|_{L^2}^{\frac{1}{2}} \\
\leq \ & \frac{1}{2}\|\nabla A\|_{L^2}^2 + C\|H\|_{L^2}^2 + C\|\psi\|_{L^4}^4\|A\|_{L^2}^2 + C\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} + \|A\|_{L^4(0,T;L^4)} \leq C. \tag{2.6}$$

Inequalities (2.2) and (2.6) yield

$$\|\psi A\|_{L^2(0,T;L^2)} \leq C. \tag{2.7}$$

Inequalities (2.3) and (2.7) imply

$$\|\psi\|_{L^2(0,T;H^1)} \leq C. \tag{2.8}$$

Next it is easy to show that (1.7) holds true and thus we omit the details here.

Now we are in a position to prove the uniqueness part. Although the following calculations are rather formal, they are justified by an approximating argument. Let $(\psi_i,\ A_i)\ \ (i = 1, 2)$ be two weak solutions satisfying (1.6) and (1.7). Then

$$\psi := \psi_1 - \psi_2,\ A := A_1 - A_2$$

satisfy the following equations:

$$\begin{aligned}
\eta\partial_t\psi + i\left(\frac{1}{k} - \eta k\right)\psi \operatorname{div} A_1 + i\left(\frac{1}{k} - \eta k\right)\psi_2 \operatorname{div} A - \frac{1}{k^2}\Delta\psi + \frac{2i}{k}A_1\nabla\psi& \\
+\frac{2i}{k}A\nabla\psi_2 + A_1^2\psi_1 - A_2^2\psi_2 + |\psi_1|^2\psi_1 - |\psi_2|^2\psi_2 - g\psi = 0,& \quad (2.9)
\end{aligned}$$

$$\partial_t A - \Delta A + \operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right)\overline{\psi}_1 - \left(\frac{i}{k}\nabla\psi_2 + \psi_2 A_2\right)\overline{\psi}_2\right\} = 0 \tag{2.10}$$

Multiplying (2.9) by $\overline{\psi}$ and taking the real part, we deduce that

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int|\psi|^2dx+\frac{1}{k^2}\int|\nabla\psi|^2dx\le\left|\frac{1}{k}-\eta k\right|\int|\psi_2||\mathrm{div}\,A||\psi|dx\\
&+\frac{2}{k}\int|A_1||\nabla\psi||\psi|dx+\frac{2}{k}\int|\psi_2||\mathrm{div}\,A||\psi|dx+\frac{2}{k}\int|\psi_2||A||\nabla\psi|dx\\
&+\int|\psi_2||A||A_1+A_2||\psi|dx+\int|\psi_1||\psi_1+\psi_2||\psi|^2dx+\int g|\psi|^2dx\\
=&:\sum_{i=1}^{7}I_i. \qquad (2.11)
\end{aligned}$$

We bound each term I_i $(i=1,\cdots,7)$ as follows.

$$\begin{aligned}
I_1+I_3 &\le C\|\nabla A\|_{L^2}\|\psi_2\|_{L^4}\|\psi\|_{L^4}\\
&\le C\|\nabla A\|_{L^2}\|\psi_2\|_{L^4}\|\psi\|_{L^2}^{\frac12}\|\nabla\psi\|_{L^2}^{\frac12}\\
&\le \frac{1}{16}\|\nabla A\|_{L^2}^2+\frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+C\|\psi_2\|_{L^4}^4\|\psi\|_{L^2}^2,\\
I_2 &\le C\|\nabla\psi\|_{L^2}\|A_1\|_{L^4}\|\psi\|_{L^4}\\
&\le \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+C\|A_1\|_{L^4}^4\|\psi\|_{L^2}^2,\\
I_4 &\le C\|\nabla\psi\|_{L^2}\|A\|_{L^4}\|\psi_2\|_{L^4}\\
&\le \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+\frac{1}{16}\|\nabla A\|_{L^2}^2+C\|\psi_2\|_{L^4}^4\|A\|_{L^2}^2,\\
I_5 &\le C\|\psi_2\|_{L^4}\|A_1+A_2\|_{L^4}\|A\|_{L^4}\|\psi\|_{L^4}\\
&\le \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+\frac{1}{16}\|\nabla A\|_{L^2}^2+C(\|A_1\|_{L^4}^4+\|A_2\|_{L^4}^4)\|A\|_{L^2}^2+C\|\psi_2\|_{L^4}^4\|\psi\|_{L^2}^2,\\
I_6 &\le \|\psi_1\|_{L^4}\|\psi_1+\psi_2\|_{L^4}\|\psi\|_{L^4}^2\\
&\le \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+C(\|\psi_1\|_{L^4}^4+\|\psi_2\|_{L^4}^4)\|\psi\|_{L^2}^2,\\
I_7 &\le \|g\|_{L^q}\|\psi\|_{L^{\frac{2q}{q-1}}}^2\le C\|g\|_{L^q}\|\psi\|_{L^2}^{2\left(1-\frac1q\right)}\|\nabla\psi\|_{L^2}^{\frac2q}\\
&\le \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2+C\|g\|_{L^q}^p\|\psi\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.11), we get

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int|\psi|^2dx+\frac{10}{16k^2}\int|\nabla\psi|^2dx\\
\le&\ \frac{3}{16}\|\nabla A\|_{L^2}^2+C(\|A_1\|_{L^4}^4+\|A_2\|_{L^4}^4+\|\psi_1\|_{L^4}^4+\|\psi_2\|_{L^4}^4)(\|\psi\|_{L^2}^2+\|A\|_{L^2}^2)\\
&+C\|g\|_{L^q}^p\|\psi\|_{L^2}^2. \qquad (2.12)
\end{aligned}$$

Testing (2.10) by A, we observe that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int |A|^2dx + \int |\nabla A|^2dx \\ \leq & \ \frac{1}{k}\int |\nabla\psi||\psi_1||A|dx + \frac{1}{k}\int |\nabla\psi||\psi_2||A|dx \\ & +\frac{1}{k}\int |\mathrm{div}\, A||\psi_2||\psi|dx + \int (|\psi_1| + |\psi_2|)|\psi||A_2||A|dx \\ = & \ : I_8 + I_9 + I_{10} + I_{11}. \end{aligned} \tag{2.13}$$

Each term I_i $(i = 8, \cdots, 11)$ is bounded as follows.

$$\begin{aligned} I_8 + I_9 &\leq C\|\nabla\psi\|_{L^2}(\|\psi_1\|_{L^4} + \|\psi_2\|_{L^4})\|A\|_{L^4} \\ &\leq \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2 + \frac{1}{16}\|\nabla A\|_{L^2}^2 + C(\|\psi_1\|_{L^4}^4 + \|\psi_2\|_{L^4}^4)\|A\|_{L^2}^2, \\ I_{10} &\leq C\|\mathrm{div}\, A\|_{L^2}\|\psi_2\|_{L^4}\|\psi\|_{L^4} \\ &\leq \frac{1}{16}\|\nabla A\|_{L^2}^2 + \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2 + C\|\psi_2\|_{L^4}^4\|\psi\|_{L^2}^2, \\ I_{11} &\leq (\|\psi_1\|_{L^4} + \|\psi_2\|_{L^4})\|\psi\|_{L^4}\|A_2\|_{L^4}\|A\|_{L^4} \\ &\leq \frac{1}{16k^2}\|\nabla\psi\|_{L^2}^2 + \frac{1}{16}\|\nabla A\|_{L^2}^2 + C(\|\psi_1\|_{L^4}^4 + \|\psi_2\|_{L^4}^4)\|\psi\|_{L^2}^2 + C\|A_2\|_{L^4}^4\|A\|_{L^2}^2. \end{aligned}$$

Inserting the above estimates into (2.13), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int |A|^2dx + \frac{13}{16}\int |\nabla A|^2dx \\ \leq & \ \frac{3}{16k^2}\|\nabla\psi\|_{L^2}^2 + C(\|A_1\|_{L^4}^4 + \|A_2\|_{L^4}^4 + \|\psi_1\|_{L^4}^4 + \|\psi_2\|_{L^4}^4)(\|\psi\|_{L^2}^2 + \|A\|_{L^2}^2). \end{aligned} \tag{2.14}$$

Summing up (2.12) and (2.14) and using the Gronwall inequality, we conclude that

$$\|\psi\|_{L^2} = \|A\|_{L^2} = 0$$

and thus $\psi_1 = \psi_2,\ A_1 = A_2$.

This completes the proof. □

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References

[1] T. Akiyama, H. Kasai, M. Tsutsumi, On the existence of the solution of the time dependent Ginzburg-Landau equations in $\mathbb{R}^3$, *Funkcialaj Ekvacioj*, **43** (2000), 255-270.

[2] Z. M. Chen, C. Elliott, Q. Tang, Justification of a two-dimensional evolutionary Ginzburg-Landau superconductivity model, *ESAIM: Mathematical Modelling and Numerical Analysis*, **32** (1998), no. 1, 25-50. https://doi.org/10.1051/m2an/1998320100251

[3] Z. M. Chen, K. H. Hoffman, J. Liang, On a non-stationary Ginzburg-Landau superconductivity model, *Math. Methods Appl. Sci.*, **16** (1993), no. 12, 855-875. https://doi.org/10.1002/mma.1670161203

[4] Q. Du, Global existence and uniqueness of solutions of the time dependent Ginzburg-Landau model for superconductivity, *Applicable Anal.*, **53** (1994), no. 1-2, 1-17. https://doi.org/10.1080/00036819408840240

[5] J. Fan, H. Gao, B. Guo, Uniqueness of weak solutions to the 3D Ginzburg-Landau superconductivity model, *Int. Math. Res. Notices*, **2015** (2015), no. 5, 1239-1246. https://doi.org/10.1093/imrn/rnt253

[6] J. Fan, S. Jiang, Global existence of weak solutions of a time-dependent 3-D Ginzburg-Landau model for superconductivity, *Appl. Math. Letters*, **16** (2003), no. 3, 435-440. https://doi.org/10.1016/s0893-9659(03)80069-1

[7] J. Fan, T. Ozawa, Global strong solutions of the time-dependent Ginzburg-Landau model for superconductivity with a new gauge, *Int. J. Math. Anal.*, **6** (2012), no. 3, 1679-1684.

[8] J. Fan, T. Ozawa, Uniqueness of weak solutions to the Ginzburg-Landau model for superconductivity, *Zeit. Angew. Math. Phys.*, **63** (2012), no. 3, 453-459. https://doi.org/10.1007/s00033-011-0164-x

[9] B. Guo and G. Yuan, Cauchy problem for the Ginzburg-Landau equation for the superconductivity model, *Proc. Royal Society Edinburgh: Section A Mathematics*, **127** (1997), 1181-1192. https://doi.org/10.1017/s0308210500027001

[10] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, *Math. Z.*, **187** (1984), 481-490. https://doi.org/10.1007/bf01174182

[11] A. Rodriguez-Bernal and B. Wang, Cauchy problem for the time-dependent Ginzburg-Landau model of superconductivity, *Proc. Royal Society Edinburgh: Section A Mathematics*, **130** (2000), 1383-1404. https://doi.org/10.1017/s0308210500000731

[12] Q. Tang, On an evolutionary system of Ginzburg-Landau equations with fixed total magnetic flux, *Communications in Partial Differential Equations*, **20** (1995), no. 1-2, 1-36. https://doi.org/10.1080/03605309508821085

[13] Q. Tang, S. Wang, Time dependent Ginzburg-Landau equations of superconductivity, *Physica D: Nonlinear Phenomena*, **88** (1995), no. 3-4, 139-166. https://doi.org/10.1016/0167-2789(95)00195-a

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