A Note on $gw$-Continuity Induced by Generalized $w$-Open Sets in Associated $w$-Spaces

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Abstract

The purpose of this paper is to introduce the notions of $gw$-continuous and $gw^*$-continuous functions induced by $gw$-open sets in associated $w$-spaces, and to study some properties and the relationships among such notions and other continuity.

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1 Introduction

Siwiec [20] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [11]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure,
w-space, $W$-continuity and $W^*$-continuity were investigated in [12]. In [13], the notions of associated $w$-spaces, $WO$-continuity and $WK$-continuity were investigated. Levine [5] introduced the notion of $g$-closed subsets in topological spaces. In fact, the set of all $g$-closed subsets is a kind of weak structure. In the same way, we introduced the notions of $gw$-closed sets [15] and $gw_\tau$-closed sets [16] in weak spaces, and investigated some basic properties of such notions. The notions of $gW$-continuous, $gW^*$-continuous, $gW^*$-irresolute, and $gW^*$-irresolute functions induced by $gw$-open sets introduced in [18], and also the notions of $gw_\tau W$-continuous, $gw_\tau W^*$-continuous, $gw_\tau W^*$-irresolute, and $gw_\tau W^*$-irresolute functions investigated in [18]. The purpose of this note is to introduce the notions of $gw$-continuity and $gw^*$-continuity and to study the relationships among such notions and the other continuity in associated $w$-spaces.

2 Preliminaries

Let $X$ be a nonempty set. A subfamily $w_X$ of the power set $P(X)$ is called a weak structure [12] on $X$ if it satisfies the following:

1. $\emptyset \in w_X$ and $X \in w_X$.
2. For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair $(X, w_X)$ is called a $w$-space on $X$. Then $V \in w_X$ is called a $w$-open set and the complement of a $w$-open set is a $w$-closed set. The collection of all $w$-open sets (resp., $w$-closed sets) in a $w$-space $X$ will be denoted by $W(X)$ (resp., $WC(X)$). We set $W(x) = \{ U \in W(X) : x \in U \}$.

Let $S$ be a subset of a topological space $X$. The closure (resp., interior) of $S$ will be denoted by $clS$ (resp., $intS$). A subset $S$ of $X$ is called a preopen set [9] (resp., $\alpha$-open set [19], semi-open [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S)))$). The complement of a preopen set (resp., $\alpha$-open set, semi-open) is called a preclosed set (resp., $\alpha$-closed set, semi-closed). The family of all preopen sets (resp., $\alpha$-open sets, semi-open sets) in $X$ will be denoted by $PO(X)$ (resp., $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on $X$. And a subset $A$ of $X$ is said to be $g$-closed [5] (resp., $gp$-closed [7], $gs$-closed [1, 3]) if $cl(A)$ (resp., $pCl(A)$, $sCl(A)) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

Then the family $\tau, GO(X), goO(X)$, and $go^*O(X)$, are all weak structures on $X$. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on $X$. A subfamily $m_X$ of the power set $P(X)$ of a nonempty set $X$ is called a minimal structure on $X$ [8] if $\emptyset \in m_X$ and $X \in m_X$. Thus clearly every weak structure is a minimal structure.
For a subset \( A \) of \( X \), the \( w\)-closure of \( A \) and the \( w\)-interior of \( A \) are defined as follows in [12]:

1. \( wC(A) = \cap \{ F : A \subseteq F, X - F \in w_X \} \).
2. \( wI(A) = \cup \{ U : U \subseteq A, U \in w_X \} \).

**Theorem 2.1** ([12]). Let \((X, w_X)\) be a \( w \)-space and \( A \subseteq X \).

1. \( x \in wI(A) \) if and only if there exists an element \( U \in W(x) \) such that \( U \subseteq A \).
2. \( x \in wC(A) \) if and only if \( A \cap V \neq \emptyset \) for all \( V \in W(x) \).
3. If \( A \subseteq B \), then \( wI(A) \subseteq wI(B) \); \( wC(A) \subseteq wC(B) \).
4. \( wC(X - A) = X - wI(A) \); \( wI(X - A) = X - wC(A) \).
5. If \( A \) is \( w \)-closed (resp., \( w \)-open), then \( wC(A) = A \) (resp., \( wI(A) = A \)).

Let \((X, w_X)\) be a \( w \)-space and \( A \subseteq X \). Then \( A \) is called a generalized \( w \)-closed set (simply, a \( gw \)-closed set) [15] if \( wC(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( w \)-open. Then the union of two \( gw \)-closed sets is a \( gw \)-closed set, but the intersection of two \( gw \)-closed sets is not always \( gw \)-closed. The family of all \( w \)-closed sets (resp., \( gw \)-closed sets, \( gw \)-open sets) in \( X \) will be denoted by \( WC(X) \) (resp., \( GW(X) \)). We set \( gW(x) = \{ U \in GW(X) : x \in U \} \). \( A \) is called a generalized \( w \)-open set (simply, a \( gw \)-open set) if \( X - A \) is \( gw \)-closed. Then \( A \) is \( gw \)-open if and only if \( F \subseteq wI(A) \) whenever \( F \subseteq A \) and \( F \) is \( w \)-closed. For a subset \( A \) of \( X \), \( gw \)-closure of \( A \) and \( gw \)-interior [15] of \( A \) are defined as the following:

1. \( gwC(A) = \cap \{ F : A \subseteq F, F \text{ is } gw \text{-closed} \} \).
2. \( gwI(A) = \cup \{ U : U \subseteq A, U \text{ is } gw \text{-open} \} \).

**Theorem 2.2** ([15]). Let \((X, w_X)\) be a \( w \)-space and \( A \subseteq X \).

1. If \( A \) is \( gw \)-open (\( gw \)-closed), then \( gwI(A) = A \) (\( gwC(A) = A \)).
2. If \( A \subseteq B \), then \( gwI(A) \subseteq gwI(B) \); \( gwC(A) \subseteq gwC(B) \).
3. \( gwC(X - A) = X - gwI(A) \); \( gwI(X - A) = X - gwC(A) \).
4. \( x \in gwI(A) \) iff there exists a \( gw \)-open set \( U \) containing \( x \) such that \( U \subseteq A \).
5. \( x \in gwC(A) \) iff \( A \cap V \neq \emptyset \) for all \( gw \)-open set \( V \) containing \( x \).

### 3 Main Results

First, we recall that: Let \( X \) be a nonempty set and let \((X, \tau)\) be a topological space. A subfamily \( w \) of the power set \( P(X) \) is called an associated weak structure (simply, \( w_\tau \)) [13] on \( X \) if \( \tau \subseteq w \) and \( w \) is a weak structure. Then the pair \((X, w_\tau)\) is called an associated \( w \)-space with \( \tau \).
Definition 3.1. Let $f : X \rightarrow Y$ be a function in two associated $w$-spaces. Then $f$ is said to be

1. $gw$-continuous if for $x \in X$ and for each open set $V$ containing $f(x)$, there is a $gw$-open set $U$ containing $x$ such that $f(U) \subseteq V$;

2. $gw^*$-continuous if for every open set $V$ in $Y$, $f^{-1}(V)$ is a $gw$-open set in $X$.

Obviously we obtain the following theorem:

Theorem 3.2. Every $gw^*$-continuous function is $gw$-continuous.

The following example supports that the converse of the above theorem is not true in general.

Example 3.3. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a, c\}, X\}$ and an associated $w$-structure $w = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ in $X$. Then for the power set $P(X)$ of $X$, $GW(X) = P(X) - \{\{b, c, d\}, \{b, d\}\}$ is the set of all $gw$-open sets. Consider a function $f : (X, w) \rightarrow (X, w)$ defined by $f(a) = b; f(b) = a; f(c) = d; f(d) = c$. Then $f$ is $gw$-continuous. For an open set $\{a, c\}$, $f^{-1}(\{a, c\}) = \{b, d\}$ is not $gw$-open, and so $f$ is not $gw^*$-continuous.

We recall that: Let $(X, w_\tau)$ be an associated $w$-space with a topology $\tau$ and $A \subseteq X$. Then $A$ is called a generalized $w_\tau$-closed set (simply, $gw_\tau$-closed set) [16] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $w$-open.

Let $f : X \rightarrow Y$ be a function in two associated $w$-spaces $w$-spaces. Then $f$ is said to be

1. $gw_\tau$-continuous [17] if for $x \in X$ and for each open set $V$ containing $f(x)$, there is a $gw_\tau$-open set $U$ containing $x$ such that $f(U) \subseteq V$;

2. $gw^*_\tau$-continuous [17] if for every open set $V$ in $Y$, $f^{-1}(V)$ is a $gw_\tau$-open set in $X$.

Obviously, the following things are obtained:

Theorem 3.4. (1) Every $gw_\tau$-continuous function is $gw$-continuous.

(2) Every $gw^*_\tau$-continuous function is $gw^*$-continuous.

Proof. Since every $gw_\tau$-open set is $gw$-open, the things are obvious. \qed

The following example supports that the converses of the above theorem are not true in general.

Example 3.5. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $w_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, X\}$ be a $w$-structure in $X$. Note that:
Continuity induced by generalized $w$-open sets

$WC(X) = \{\emptyset, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{c\}, X\};$

$GW_xC(X) = \{\emptyset, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\};$

$GW_\tau(X) = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, X\};$

$GWC(X) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, X\};$

$GW(X) = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, d\}, X\}.$

Consider a function $f : X \to X$ defined as: $f(a) = f(c) = a; f(b) = b; f(d) = d$. Then $f$ is $gw$-continuous and $gw^*$-continuous. But for an open set $\{a\}$, $f^{-1}(\{a\}) = \{a, c\}$ is not $gw_\tau$-open. So $f$ is not neither $gw_\tau$-continuous nor $gw^*_\tau$-continuous.

Let $f : X \to Y$ be a function in two associated $w$-spaces. Then $f$ is said to be

1. **WO-continuous** [13] if for $x \in X$ and for each open set $V$ containing $f(x)$, there is a $w$-open set $U$ containing $x$ such that $f(U) \subseteq V$;
2. **WK-continuous** [13] if for every open set $V$ in $Y$, $f^{-1}(V)$ is a $w$-open set in $X$.

Obviously, the following things are obtained:

**Theorem 3.6.**
1. Every WO-continuous function is $gw$-continuous.
2. Every WK-continuous function is $gw$-continuous.

**Proof.** Since every $w$-open set is $gw$-open, they are obtained.

The following example supports that the converses of the above theorem are not true in general.

**Example 3.7.** Consider the function $f$ defined in Example 3.5. Then $f$ is $gw$-continuous and $gw^*$-continuous but neither WO-continuous nor WK-continuous.

Let $f : X \to Y$ be a function on $w$-spaces. Then $f$ is said to be

1. **gW-continuous** [18] if for $x \in X$ and for each $w$-open set $V$ containing $f(x)$, there is a $gw$-open set $U$ containing $x$ such that $f(U) \subseteq V$;
2. **gW*-continuous** [18] if for every $w$-open set $V$ in $Y$, $f^{-1}(V)$ is a $gw$-open set in $X$.

Obviously, the following things are obtained:

**Theorem 3.8.**
1. Every gW-continuous function is $gw$-continuous.
2. Every gW*-continuous function is $gw^*$-continuous.
Proof. Since every open set is \( w \)-open, the things are obtained.

The following example supports that the converses of the above theorem are not true in general.

**Example 3.9.** (1) The function \( f \) defined in Example 3.5 is obviously \( gw^* \)-continuous but not \( gW^* \)-continuous.

(2) In Example 3.5, consider a function \( g : X \to X \) defined by \( g(a) = b ; g(b) = a ; g(c) = c ; g(d) = d \). Then \( g \) is \( gw \)-continuous. For a \( w \)-open set \( V = \{a, c\} \) and for \( g(c) = c \in U \), there is no any \( gw \)-open set \( U \) containing \( c \) such that \( g(U) \subseteq V \). So, \( g \) is not \( gW^* \)-continuous.

Let \( f : (X, w_\tau) \to (Y, w_\mu) \) be a function on two associated \( w \)-spaces with \( \tau \) and \( \mu \). Then \( f \) is said to be

(1) \( gW \)-irresolute [18] if for \( x \in X \) and for each \( gw \)-open set \( V \) containing \( f(x) \), there is \( gw \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \);

(2) \( gW^* \)-irresolute [18] if for every \( gw \)-open set \( V \) in \( Y \), \( f^{-1}(V) \) is \( gw \)-open in \( X \).

In [18], we showed that every \( gW \)-irresolute is \( gW \)-continuous and very \( gW^* \)-irresolute function is \( gW^* \)-continuous. From Theorem 3.4, the following theorem is directly obtained:

**Theorem 3.10.**

1. Every \( gW \)-irresolute is \( gw \)-continuous.
2. Every \( gW^* \)-irresolute function is \( gw^* \)-continuous.

**Remark 3.11.** For a function from an associated \( w \)-space to an associated \( w \)-space, we have the following diagram:

\[
\begin{array}{ccc}
\text{Continuity} & \rightarrow & WK\text{-conti.} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
gw^*\text{-conti.} & \rightarrow & gw\text{-conti.} \\
\downarrow & & \downarrow \\
gw^*_r\text{-conti.} & \rightarrow & gw_r\text{-conti.} \\
\uparrow & & \uparrow \\
\uparrow & & \uparrow \\
gw_rW^*\text{-conti.} & \rightarrow & gw_rW\text{-conti.} \\
\uparrow & & \uparrow \\
gW^*\text{-conti.} & \rightarrow & gW\text{-conti.} \\
\uparrow & & \uparrow \\
W^*\text{-conti.} & \rightarrow & W\text{-conti.}
\end{array}
\]

Continuity \( \rightarrow gw^*_r\)-continuity \( \rightarrow gw_r\)-continuity
Theorem 3.12. Let \( f : X \rightarrow Y \) be a function on \( w \)-spaces. Then \( f \) is \( gw^* \)-continuous if and only if for every closed set \( F \) in \( Y \), \( f^{-1}(F) \) is \( gw \)-closed in \( X \).

Proof. It is obvious. \(\square\)

Theorem 3.13. Let \( f : X \rightarrow Y \) be a function on \( w \)-spaces. Then the following statements are equivalent:

1. \( f \) is \( gw \)-continuous.
2. \( f(gwC(A)) \subseteq cl(f(A)) \) for \( A \subseteq X \).
3. \( gwC(f^{-1}(V)) \subseteq f^{-1}(cl(V)) \) for \( V \subseteq Y \).
4. \( f^{-1}(int(V)) \subseteq gwI(f^{-1}(V)) \) for \( V \subseteq Y \).

Proof. Obvious. \(\square\)

Corollary 3.14. Let \( f : X \rightarrow Y \) be a function on \( w \)-spaces. Then the following statements are equivalent:

1. \( f \) is \( gw \)-continuous.
2. \( f^{-1}(V) = gwI(f^{-1}(V)) \) for every open set \( V \in Y \).
3. \( f^{-1}(B) = gwC(f^{-1}(B)) \) for every closed set \( B \subseteq Y \).

Proof. From Theorem 2.2 and Theorem 3.13, it is obvious. \(\square\)

Let \((X, w)\) be a \( w \)-space. Let \( gw(x) \) (resp., \( w(x) \)) denote the set of all \( gw \)-open (resp., \( w \)-open) set containing \( x \) in \( X \). A collection \( \mathcal{H} \) of subsets of \( X \) is called an \( m \)-family \([10]\) on \( X \) if \( \cap \mathcal{H} \neq \emptyset \). Let \( \mathcal{H} \) be an \( m \)-family on \( X \). Then we say that an \( m \)-family \( \mathcal{H} \) \( gw \)-converges (resp., \( gw \)-converges) to \( x \in X \) if \( \mathcal{H} \) is finer than \( gw(x) \) (resp., \( O(x) \)) i.e., \( gw(x) \subseteq \mathcal{H} \) (resp., \( O(x) \subseteq \mathcal{H} \)). Let \( f : X \rightarrow Y \) be a function; then it is obvious \( f(\mathcal{H}) = \{ f(F) : F \in \mathcal{H} \} \) is an \( m \)-family on \( Y \).

Theorem 3.15. Let \( f : X \rightarrow Y \) be a function on \( w \)-spaces. If \( f \) is \( gw \)-continuous, then for an \( m \)-family \( \mathcal{H} \) \( gw \)-converging to \( x \in X \), an \( m \)-family \( < f(\mathcal{H}) > = \{ F : H \subseteq F \) for some \( H \in f(\mathcal{H}) \} \) converges to \( f(x) \).

Proof. Let \( f \) be \( gw \)-continuous and let \( \mathcal{H} \) be an \( m \)-family \( gw \)-converging to \( x \in X \). By \( gw \)-continuity, for an open set \( V \) containing \( f(x) \), there exists a \( gw \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \). Since \( f(gw(x)) \subseteq f(\mathcal{H}) \), \( V \in < f(\mathcal{H}) > \), and so \( O(f(x)) \subseteq < f(\mathcal{H}) > \). Hence the \( m \)-family \( < f(\mathcal{H}) > \) converges to \( f(x) \). \(\square\)
Theorem 3.16. Let $f : X \rightarrow Y$ be a bijective function on $w$-spaces. Then $f$ is $gw^*$-continuous iff for an $m$-family $\mathcal{H}$ $gw$-converging to $x \in X$, the $m$-family $f(\mathcal{H})$ converges to $f(x)$.

Proof. Suppose $f$ is $gw^*$-continuous and $\mathcal{H}$ is an $m$-family $gw$-converging to $x \in X$. By hypothesis and surjectivity, $O(f(x)) \subseteq f(gW(x)) \subseteq f(\mathcal{H})$, and so the $m$-family $f(\mathcal{H})$ converges to $f(x)$.

For the converse, let $U \in O(f(x))$ for $U \subseteq Y$. Since the family $gW(x)$ clearly $gw$-converges to $x$, by hypothesis, we get $O(f(x)) \subseteq f(gW(x))$ for $x \in X$. Since $f$ is injectivity, $f^{-1}(U) \in gW(x)$.

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References


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