Stability of a Functional Equation
Related to Quadratic Mappings

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Abstract
In this paper, we investigate the stability of a functional equation

\[ f(ax + by) + abf(x - y) - (a^2 + ab)f(x) - (b^2 + ab)f(y) = 0, \]

which relates to quadratic mappings.

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1 Introduction
The stability problem of functional equations originated from Ulam’s stability problem [10] of group homomorphisms. The functional equation which is equivalent to the following functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]

is called a quadratic functional equation, and every solution of a quadratic functional equation is said to be a quadratic mapping. F. Skof [9], P. W. Cholewa [1], and S. Czerwik [2] proved the stability of quadratic functional equations,
and the authors have investigated the stability problems of quadratic functional equations, see [3] and [4]. Moreover, Najati and Jung [8] have observed the Hyers-Ulam stability of a generalized quadratic functional equation

\[ f(ax + by) + abf(x - y) = af(x) + bf(y) \]  

(1.1)

where \( a, b \) are non-zero rational numbers with \( a + b = 1 \).

In this paper we consider the stability of the following functional equation

\[ f(ax + by) + abf(x - y) - (a^2 + ab)f(x) - (b^2 + ab)f(y) = 0 \]  

(1.2)

for real numbers \( a, b \) with \( ab(a + b) \neq 0 \), which is a generalization of (1.1). Recall if a mapping \( f \) satisfies the functional equation

\[ f(x + y) - f(x) - f(y) = 0 \]

then we call \( f \) an additive mapping. If a mapping \( f \) is represented by sum of an additive mapping and a quadratic mapping, we call it a quadratic-additive mapping [6]. Now, in this paper, we will show that the solution of the functional equation (1.2) is a quadratic-additive mapping and investigate the stability of it. In [5], H.-K. Kim and J.-R. Lee have investigated the stability of a generalized quadratic functional equation (1.2) for any fixed rational numbers \( a, b \) in fuzzy spaces.

2 Main results

Throughout this paper, let \( X \) be a real normed space and \( Y \) a real Banach space. Let \( V \) and \( W \) be real vector spaces and let \( a \) and \( b \) be real constants. For a given mapping \( f : V \rightarrow W \), we use the following abbreviations

\[ f_o(x) := \frac{f(x) - f(-x)}{2}, \]
\[ f_e(x) := \frac{f(x) + f(-x)}{2}, \]
\[ Af(x, y) := f(x + y) - f(x) - f(y), \]
\[ Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y), \]
\[ D_{a,b}f(x, y) := f(ax + by) + abf(x - y) - (a^2 + ab)f(x) - (b^2 + ab)f(y) \]

for all \( x, y \in V \).

In the following lemma, we will show that every solution of the functional equation (1.2) is a quadratic-additive mapping.
Lemma 2.1 Let $a$ and $b$ be fixed real numbers with $ab(a + b) \neq 0$ and let $f : V \rightarrow W$ be a mapping satisfying the equation $D_{a,b}f(x,y) = 0$ (with $f(0) = 0$ when $a^2 + ab + b^2 = 1$). Then $f$ is a quadratic-additive mapping such that $f(ax) = a^2f(x)$, $f((a + b)x) = (a + b)^2f(x)$, and $f(bx) = -abf(-x) + (b^2 + ab)f(x)$ for all $x \in V$.

Proof. Assume that a mapping $f : V \rightarrow W$ satisfies the functional equation (1.2). Notice that $f_{o}(f(0)) = 0$ if $a^2 + ab + b^2 \neq 1$. For all $x, y \in V$ we get

$$2f_{o}\left(\frac{x + y}{2}\right) - f_{o}(x) - f_{o}(y)$$

$$= \frac{-1}{2a^2b(a + b)} \left( D_{a,b}f_{o}\left( \frac{bx + by}{2}, \frac{ax - ay}{2} \right) - D_{a,b}f_{o}(bx, 0) - a^2D_{a,b}f_{o}(0, x) + D_{a,b}f_{o}\left( \frac{bx + by}{2}, \frac{ay - ax}{2} \right) - D_{a,b}f_{o}(by, 0) - a^2D_{a,b}f_{o}(0, y) \right)$$

$$+ \frac{-1}{ab} D_{a,b}f_{o}(0, \frac{x + y}{2})$$

$$- \frac{1}{2a(a + b)} \left( D_{a,b}f_{o}\left( \frac{x - y}{2}, \frac{x + y}{2} \right) + D_{a,b}f_{o}\left( \frac{y - x}{2}, \frac{x + y}{2} \right) \right)$$

$$= 0.$$

Since $f_{o}(0) = 0$, we easily show that $2f_{o}\left(\frac{x + y}{2}\right) = f_{o}(x + y)$ for all $x, y \in V$. Therefore, we get

$$A_{f_{o}}(x, y) = f_{o}(x + y) - f_{o}(x) - f_{o}(y)$$

$$= f_{o}(x + y) - 2f_{o}\left(\frac{x + y}{2}\right)$$

$$= 0$$

for all $x, y \in V$, i.e., $f_{o}$ is an additive mapping. On the other hand, we have

$$Q_{f_{e}}(x, y) = \frac{D_{a,b}f_{e}\left( (1 + \frac{b}{a})x, y \right) - D_{a,b}f_{e}(x, x + y) - \frac{b}{a}D_{f_{e}}(y - x, -x)}{b(a + b)}$$

$$+ \frac{D_{a,b}f_{e}(x, x)}{ab} + \frac{D_{a,b}f_{e}\left( (1 + \frac{b}{a})x - y, 0 \right)}{a(a + b)} - \frac{D_{a,b}f_{e}\left( (1 + \frac{b}{a})x, 0 \right)}{ab}$$

$$= 0$$

for all $x, y \in V$, i.e., $f_{e}$ is a quadratic mapping. Until now, we obtain that $f = f_{e} + f_{o}$ is a quadratic-additive mapping. Moreover, from the equalities

$$f(ax) - a^2f(x) = Df_{a,b}f(x, 0)$$

$$f((a + b)x) - (a + b)^2f(x) = Df_{a,b}f(x, x)$$

$$f(bx) + abf(-x) - (b^2 + ab)f(x) = Df_{a,b}f(0, x)$$
for all \( x \in V \), we get the equalities \( f(ax) = a^2f(x) \), \( f((a+b)x) = (a+b)^2f(x) \), and \( f(bx) = -abf(-x) + (b^2 + ab)f(x) \) for all \( x \in V \).

For the special rational number case, we can show that \( f \) is itself a quadratic mapping.

**Lemma 2.2** Let \( a \) be a fixed rational number with \( ab(a+b)(a-1) \neq 0 \), or let \( a+b \) be a fixed rational number with \( ab(a+b)(a+b-1) \neq 0 \), or let \( b \) be a fixed rational number with \( ab(a+b) \neq 0 \). If a mapping \( f : V \to W \) satisfies the functional equation (1.2), then \( f \) is a quadratic mapping such that \( f(ax) = a^2f(x) \) and \( f(bx) = b^2f(x) \) for all \( x \in V \).

**Proof.** By Lemma 2.1, \( f_0 \) is an additive mapping such that \( f_0(ax) = a^2f_0(x) \), \( f_0((a+b)x) = (a+b)^2f_0(x) \), and \( f_0(bx) = -abf_0(-x) + (b^2 + ab)f_0(x) \) for all \( x \in V \). Observe that, since \( Af_0 \equiv 0 \), \( f_0(rx) = rf_0(x) \) for all rational numbers \( r \) and \( x \in V \). In the first, if \( a \) is a nonzero rational number with \( a \neq 1 \), then the equality \( f_0(ax) = af_0(x) = a^2f_0(x) \) implies that \( f_0 \equiv 0 \). Secondary, if \( a+b \) is a nonzero rational number with \( a+b \neq 1 \), then the equality \( f_0((a+b)x) = (a+b)f_0(x) = (a+b)^2f_0(x) \) implies that \( f_0 \equiv 0 \). Finally, suppose that \( b \) is a nonzero rational number. Then the equality \( f_0(bx) = bf_0(x) = -abf_0(-x) + (b^2 + ab)f_0(x) \) implies the equality \((b+2a-1)bf_0(x) = 0\). In this case, for a nonzero rational number \( b \), we only need to check the case of \( a \notin Q \) or \( a+b \notin Q \) or \( a = 1 \) or \( a+b = 1 \). So \((b+2a-1)b \neq 0\), which means that \( f_0 \equiv 0 \). Therefore, we have shown that \( f \equiv f_0 \) which proved this lemma.

H.-M. Kim and J.-R. Lee [5] showed that for fixed rational numbers \( a \) and \( b \) if \( f : V \to W \) is a solution of the functional equation (1.2), then \( f \) is a quadratic mapping. In the following lemma, we get the converse of it for some conditions.

**Lemma 2.3** Let \( a \) and \( b \) be fixed rational numbers with \( ab(a+b) \neq 0 \). A mapping \( f : V \to W \) satisfies the functional equation (1.2) (with \( f(0) = 0 \) when \( a^2 + ab + b^2 = 1 \)) if and only if \( f \) is a quadratic mapping.

**Proof.** Assume that a mapping \( f : V \to W \) satisfies the functional equation (1.2). By Lemmas 2.1 and 2.2, we conclude that \( f \) is a quadratic mapping. Conversely, assume that \( f \) is a quadratic mapping. Notice that \( f(x) = f(-x) \) and \( f(rx) = r^2f(x) \) for all \( x \in V \) and all rational numbers \( r \). Now, we will prove that \( D_{a,b}f(x,y) = 0 \) for all \( x,y \in V \) and all positive integers \( a \) and \( b \). We apply an induction on \( i \in \{1,2,\ldots,a\} \) and \( j \in \{1,2,\ldots,b\} \) to prove \( D_{a,b}f(x,y) = 0 \) for all \( x,y \in V \). For \( i = 1 \) and \( j = 1 \), we have the following equality

\[ D_{1,1}f(x,y) = Qf(x,y) = 0 \]
for all $x, y \in V$. Assume that $D_{1,j}f(x, y) = 0$ for all $x, y \in V$ and for all $j \leq k$. Then

$$D_{1,k+1}f(x, y) = D_{1,k}f(x + y, y) + (k + 1)Qf(x, y) = 0$$

for all $x, y \in V$. Hence we have

$$D_{1,b}f(x, y) = 0$$

for all $x, y \in V$ and all positive integers $b$. Observe that

$$D_{2,b}f(x, y) = 2D_{1,b}f(x, y) + Qf(x + by, x) - f(by) + b^2f(y) = 0$$

for all $x, y \in V$. Now assume that $D_{i,b}f(x, y) = 0$ for all $x, y \in V$, $b \in \mathbb{Z}^+$, and all $i \leq k$ with $k \geq 2$. Then

$$D_{k+1,b}f(x, y) = 2D_{k,b}f(x, y) - D_{k-1,b}f(x, y) + Qf(kx + by, x) = 0$$

for all $x, y \in V$ and all positive integers $b$. So we have the equality

$$D_{a,b}f(x, y) = 0$$

for all $x, y \in V$ and all positive integers $a, b$. Let $a = \frac{p}{q}$ and $b = \frac{r}{s}$ be positive rational numbers with $p, q, r, s \in \mathbb{Z}^+$. Since $ps, qr$ are nonzero positive integers and $f\left(\frac{x}{qs}\right) = \frac{f(x)}{q^2s^2}$, we have

$$D_{a,b}f(x, y) = D_{ps,qr}f\left(\frac{x}{qs}, \frac{y}{qs}\right) = 0$$

for all $x, y \in V$. From the equality

$$D_{-a,-b}f(x, y) = D_{a,b}f(-x, -y),$$

for all $x, y \in V$, we conclude that

$$D_{a,b}f(x, y) = 0$$

for all $x, y \in V$ and all rational numbers $a, b$ with $ab > 0$. We now consider the case for $ab < 0$ with $a + b \neq 0$. The equality $D_{a,b}f(x, y) = 0$ follows from the equalities $D_{a,b}f(x, y) = D_{a,-a-b}f(x - y, -y)$ when $-a(a + b) > 0$ and $D_{a,b}f(x, y) = D_{-a-b,b}f(-x, y - x)$ when $-b(a + b) > 0$. □

In the following theorem, using Lemma 2.1, we can prove the stability of the functional equation (1.2).
Theorem 2.4 Let $a$ and $b$ be real constants with $ab(a + b) \neq 0$ and let $\varphi : V^2 \to [0, \infty)$ be a function satisfying either

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x, a^i y)}{a^{2i}} < \infty \quad (2.1)$$

or

$$\sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x}{a^i}, \frac{y}{a^i}\right) < \infty \quad (2.2)$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies

$$\|D_{a,b}f(x, y)\| \leq \varphi(x, y) \quad (2.3)$$

for all $x, y \in V$ with $f(0) = 0$, then there exists a unique solution $F : V \to Y$ of the functional equation $(1.2)$ such that the inequality

$$\|f(x) - F(x)\| \leq \left\{ \begin{array}{ll}
\sum_{i=0}^{\infty} \frac{1}{a^{2i+1}} \varphi(a^i x, 0) & \text{if } \varphi \text{ satisfies (2.1),} \\
\sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x}{a^{i+1}}, 0\right) & \text{if } \varphi \text{ satisfies (2.2)}
\end{array} \right. \quad (2.4)$$

holds for all $x \in V$.

**Proof.** We will prove the theorem in two cases, either $\varphi$ satisfies (2.1) or (2.2).

**Case 1.** Let $\varphi$ satisfy (2.1). It follows from (2.3) that for all $x \in V$

$$\left\| \frac{1}{a^{2n}} f(a^n x) - \frac{1}{a^{2n+2m}} f(a^{n+m} x) \right\| \leq \sum_{i=n}^{n+m-1} \frac{1}{a^{2i}} \left\| D_{a,b}f(a^{i+1} x, 0) \right\| \leq \sum_{i=n}^{n+m-1} \frac{1}{a^{2i}} \varphi(a^{i+1} x, 0) \quad (2.5)$$

So, it is easy to show that the sequence $\{ \frac{f(a^n x)}{a^{2n}} \}$ is a Cauchy sequence for all $x \in V$. Since $Y$ is complete and $f(0) = 0$, the sequence $\{ \frac{f(a^n x)}{a^{2n}} \}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n \to \infty} \frac{f(a^n x)}{a^{2n}}$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (2.5), we obtain the first inequality in (2.4). From the definition of $F$, we get

$$\|D_{a,b}F(x, y)\| = \lim_{n \to \infty} \left\| D_{a,b}f\left(\frac{a^n x}{a^{2n}}, \frac{a^n y}{a^{2n}}\right) \right\| \leq \lim_{n \to \infty} \frac{\varphi(a^n x, a^n y)}{a^{2n}} = 0$$
Stability of a functional equation

i.e., $D_{a,b}F(x, y) = 0$ for all $x, y \in V$. To prove the uniqueness, we assume now that there is another solution $F' : V \to Y$ of the functional equation (1.2) which satisfies the first inequality in (2.4). Since the equality $F'(x) = \frac{F'(a^n x)}{a^{2n}}$ holds for all $n \in \mathbb{N}$ by Lemma 2.1, we obtain

$$\lim_{n \to \infty} \left\| \frac{f(a^n x)}{a^{2n}} - F'(x) \right\| = \lim_{n \to \infty} \left\| \frac{f(a^n x)}{a^{2n}} - \frac{F'(a^n x)}{a^{2n}} \right\| \leq \sum_{i=0}^{\infty} \frac{\varphi(a^{i+n} x, a^{i+n} x)}{a^{2n+2i+2}} \leq \sum_{i=n}^{\infty} \frac{\varphi(a^i x, a^i x)}{a^{2i+2}} = 0$$

for all $x \in V$, i.e., $F'(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^{2n}} = F(x)$ for all $x \in V$.

**Case 2.** Let $\varphi$ satisfy (2.2). It follows from (2.3) that

$$\left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2n+2m} f\left(\frac{x}{a^{n+m}}\right) \right\| \leq \sum_{i=n}^{n+m-1} a^{2i} \left\| D_{a,b} f\left(\frac{x}{a^{i+1}}, 0\right) \right\| \leq \sum_{i=n}^{n+m-1} a^{2i} \varphi\left(\frac{x}{a^{i+1}}, 0\right) \quad (2.6)$$

for all $x \in V$. So, it is easy to show that the sequence $\{a^{2n} f\left(\frac{x}{a^n}\right)\}$ is a Cauchy sequence for all $x \in V$. Since $Y$ is complete and $f(0) = 0$, the sequence $\{a^{2n} f\left(\frac{x}{a^n}\right)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n \to \infty} a^{2n} f\left(\frac{x}{a^n}\right)$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (2.6), we obtain the second inequality in (2.4). From the definition of $F$, we get (2.4) for all $x \in V$ and

$$\|D_{a,b}F(x, y)\| = \lim_{n \to \infty} \left\| a^{2n} D_{a,b} f\left(\frac{x}{a^n}, \frac{y}{a^n}\right) \right\| \leq \lim_{n \to \infty} a^{2n} \varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0$$

for all $x, y \in V$ i.e., $D_{a,b}F(x, y) = 0$ for all $x, y \in V$. To prove the uniqueness, we assume now that there is another solution $F' : V \to W$ of the functional equation (1.2) which satisfies the second inequality in (2.4). Since the equality
\[ F'(x) = a^{2n}F'(\frac{x}{a^n}) \] holds for all \( n \in \mathbb{N} \) by Lemma 2.1, we get
\[
\lim_{n \to \infty} \left\| a^{2n}f\left(\frac{x}{a^n}\right) - F'(x) \right\|
= \lim_{n \to \infty} \left\| a^{2n}f\left(\frac{x}{a^n}\right) - a^{2n}F'(\frac{x}{a^n}) \right\|
\leq \sum_{i=0}^{\infty} a^{2n+2i} \varphi\left(\frac{x}{a^{n+i}}, \frac{x}{a^{n+i}}\right)
\leq \sum_{i=n}^{\infty} a^{2i} \varphi\left(\frac{x}{a^i}, \frac{x}{a^i}\right)
= 0
\]
for all \( x \in V \), i.e, \( F'(x) = \lim_{n \to \infty} a^{2n}f\left(\frac{x}{a^n}\right) = F(x) \) for all \( x \in V \). \( \square \)

From the Lemma 2.2 and the previous theorem, we obtained the following corollary clearly.

**Corollary 2.5** Let \( a \) be a fixed rational number with \( ab(a+b)(a-1) \neq 0 \), or let \( a+b \) be a fixed rational number with \( ab(a+b)(a+b-1) \neq 0 \), or let \( b \) be a fixed rational number with \( ab(a+b) \neq 0 \). And let \( \varphi : V^2 \to [0, \infty) \) be a function satisfying either the condition (2.1) or the condition (2.2) for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies \( f(0) = 0 \) and (2.3) for all \( x, y \in V \), then there exists a unique quadratic mapping \( F : V \to Y \) satisfying the equation (1.2) and the equality (2.4) for all \( x \in V \).

Now, we obtain another stability result of the functional equation (1.2) by following.

**Theorem 2.6** Let \( a \) and \( b \) be real constants with \( ab(a+b)^2 > 0 \) and and let \( \varphi : V^2 \to [0, \infty) \) be a function satisfying either
\[
\sum_{i=0}^{\infty} \frac{\varphi(b^ix, b^iy)}{b^{2i}} < \infty
\]  
(2.7)
or
\[
\sum_{i=0}^{\infty} (b^2 + 2ab)^i \varphi\left(\frac{x}{b^i}, \frac{y}{b^i}\right) < \infty
\]  
(2.8)
for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies (2.3) for all \( x, y \in V \) with \( f(0) = 0 \), then there exists a unique solution \( F : V \to Y \) of the functional equation (1.2) such that
\[
\|f(x) - F(x)\| \leq \begin{cases}
\sum_{i=0}^{\infty} \left( \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{2(b^i + 1)} + \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{2(b^i + 2ab)^i + 1} \right) \\
\text{if } \varphi \text{ satisfies (2.7)}, \\
\sum_{i=0}^{\infty} \frac{b^{2i} + (b^2 + 2ab)^i}{2} \left( \varphi(0, \frac{x}{b^i}) + \varphi(0, \frac{x}{b^{i+1}}) \right) \\
\text{if } \varphi \text{ satisfies (2.8)}
\end{cases}
\]  
(2.9)
for all $x \in V$.

**Proof.** We will prove the theorem in two cases, either $\varphi$ satisfies (2.7) or (2.8).

**Case 1.** Let $\varphi$ satisfy (2.7). It follows from (2.3) that

$$
\left\| \frac{f(b^n x) + f(-b^n x)}{2 \cdot b^{2n}} - \frac{f(b^{n+m} x) + f(-b^{n+m} x)}{2 \cdot b^{2n+2m}} \right\| \\
\leq \sum_{i=n}^{n+m-1} \frac{1}{2} \cdot \frac{1}{b^{2i+2}} \left\| - D_{a,b} f(0, b^i x) - D_{a,b} f(0, -b^i x) \right\| \\
\leq \sum_{i=n}^{n+m-1} \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{2 \cdot b^{2i+2}},
$$

(2.10)

$$
\left\| \frac{f(b^n x) - f(-b^n x)}{2 \cdot (b^2 + 2ab)^n} - \frac{f(b^{n+m} x) - f(-b^{n+m} x)}{2 \cdot (b^2 + 2ab)^{n+m}} \right\| \\
\leq \sum_{i=n}^{n+m-1} \frac{1}{2} \cdot \frac{1}{(b^2 + 2ab)^{i+1}} \left\| - D_{a,b} f(0, b^i x) + D_{a,b} f(0, -b^i x) \right\| \\
\leq \sum_{i=n}^{n+m-1} \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{2 \cdot (b^2 + 2ab)^{i+1}}.
$$

(2.12)

for all $x \in V$. So, it is easy to show that the sequences $\{\frac{f(b^n x) + f(-b^n x)}{2 \cdot b^{2n}}\}$ and $\{\frac{f(b^n x) - f(-b^n x)}{2 \cdot (b^2 + 2ab)^n}\}$ are Cauchy sequences for all $x \in V$. Since $Y$ is complete and $f(0) = 0$, the sequences $\{\frac{f(b^n x) + f(-b^n x)}{2 \cdot b^{2n}}\}$ and $\{\frac{f(b^n x) - f(-b^n x)}{2 \cdot (b^2 + 2ab)^n}\}$ converge for all $x \in V$. Hence, we can define mappings $F', F'' : V \to Y$ by

$$
F'(x) := \lim_{n \to \infty} \frac{f(b^n x) + f(-b^n x)}{2 \cdot b^{2n}},
$$

$$
F''(x) := \lim_{n \to \infty} \frac{f(b^n x) - f(-b^n x)}{2 \cdot (b^2 + 2ab)^n}
$$

for all $x \in V$. From the definitions of $F'$ and $F''$, we obtain that $F'(bx) = b^2 F'(x)$, $F''(bx) = (b^2 + 2ab) F''(x)$, $DF'(x, y) = 0$, and $DF''(x, y) = 0$ for all $x, y \in V$. Therefore, $F'$ and $F''$ are quadratic-additive mapping by Lemma 2.1. Moreover, if we put $n = 0$ and let $m \to \infty$ in (2.10) and (2.12), we obtain the inequalities

$$
\left\| \frac{f(x) + f(-x)}{2} - F'(x) \right\| \leq \sum_{i=0}^{\infty} \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{2 \cdot b^{2i+2}},
$$
\[ \left\| \frac{f(x) - f(-x)}{2} - F''(x) \right\| \leq \sum_{i=0}^{\infty} \varphi(0, b^i x) + \varphi(0, -b^i x) \]

hold for all \( x \in V \). Put \( F(x) := F'(x) + F''(x) \) for all \( x \in V \), then \( F'(x) = F_e(x) \) and \( F''(x) = F_o(x) \) for all \( x \in V \). Since

\[ f(x) - F(x) = \frac{f(x) + f(-x)}{2} - F'(x) + \frac{f(x) - f(-x)}{2} - F''(x) \]

for all \( x \in V \), we obtain the first inequality in (2.9). To prove the uniqueness, we assume now that there is another solution \( \overline{F} : V \to W \) of the functional equation (1.2) which satisfies the first inequality in (2.9). Since the equalities \( \overline{F}_e(bx) = b^2 \overline{F}_e(x) \) and \( \overline{F}_o(bx) = (b^2 + 2ab) \overline{F}_o(x) \) hold by Lemma 2.1, we obtain for all \( n \in \mathbb{N} \)

\[ \left\| \frac{f_e(b^n x)}{b^{2n}} + \frac{f_o(b^n x)}{b^{2n} + 2ab} - \overline{F}(x) \right\| \]

\[ = \left\| \frac{f_e(b^n x)}{b^{2n}} - \overline{F}_e(x) + \frac{f_o(b^n x)}{b^{2n} + 2ab} - \overline{F}_o(x) \right\| \]

\[ = \left\| \frac{f_e(b^n x)}{b^{2n}} - \overline{F}_e(b^n x) \right\| + \left\| \frac{f_o(b^n x)}{b^{2n} + 2ab} - \overline{F}_o(b^n x) \right\| \]

\[ \leq \sum_{i=0}^{\infty} \left( \frac{\varphi(0, b^{i+n} x) + \varphi(0, -b^{i+n} x)}{2 \cdot b^{2i+2+2n}} + \frac{\varphi(0, b^{i+n} x) + \varphi(0, -b^{i+n} x)}{2 \cdot (b^2 + 2ab)^{i+1} b^{2n}} \right) \]

\[ \leq \sum_{i=0}^{\infty} 2 \left\| \frac{\varphi(0, b^{i+n} x) + \varphi(0, -b^{i+n} x)}{b^{2i+2+2n}} \right\| \]

\[ = \sum_{i=n}^{\infty} 2 \left( \frac{\varphi(0, b^i x) + \varphi(0, -b^i x)}{b^{2i+2}} \right) \]

i.e., \( \overline{F}(x) = \lim_{n \to \infty} \left( \frac{f(b^n x) + f(-b^n x)}{2 b^{2n}} + \frac{f(b^n x) - f(-b^n x)}{2 (b^2 + 2ab)^n} \right) = F(x) \) for all \( x \in V \).

**Case 2.** Let \( \varphi \) satisfy (2.8). Together with (2.3), it follows that

\[ \left\| \frac{b^{2n}}{2} \left( f \left( \frac{x}{b^n} \right) + f \left( -\frac{x}{b^n} \right) \right) + \frac{b^{2n+2m}}{2} \left( f \left( \frac{x}{b^{n+m}} \right) + f \left( -\frac{x}{b^{n+m}} \right) \right) \right\| \]

\[ \leq \sum_{i=n}^{n+m-1} \frac{b^{2i}}{2} \left( D_{a,b} f \left( 0, \frac{x}{b^{i+1}} \right) + D_{a,b} f \left( 0, -\frac{x}{b^{i+1}} \right) \right) \]

\[ \leq \sum_{i=n}^{n+m-1} \frac{b^{2i}}{2} \left( \varphi \left( 0, \frac{x}{b^{i+1}} \right) + \varphi \left( 0, -\frac{x}{b^{i+1}} \right) \right) , \]
\[ \left\| \frac{(b^2 + 2ab)^n}{2} \left( f\left( \frac{x}{b^n} \right) + f\left( \frac{-x}{b^n} \right) \right) + \frac{(b^2 + 2ab)^{n+m}}{2} \left( f\left( \frac{x}{b^{n+m}} \right) + f\left( \frac{-x}{b^{n+m}} \right) \right) \right\| \leq \sum_{i=n}^{n+m-1} \frac{(b^2 + 2ab)^i}{2} \left( D_{a,b} f\left( 0, \frac{x}{b^{i+1}} \right) - D_{a,b} f\left( 0, -\frac{x}{b^{i+1}} \right) \right) \]
\[ \leq \sum_{i=n}^{n+m-1} \frac{(b^2 + 2ab)^i}{2} \left( \varphi\left( 0, \frac{x}{b^{i+1}} \right) + \varphi\left( 0, -\frac{x}{b^{i+1}} \right) \right) \]

for all \( x \in V \). From the above inequalities, we obtain that there exists a unique solution \( F : V \to Y \) of the functional equation (2.4) satisfying the second inequality in (2.9) by using the similar method used in the case 1. □

From the Lemma 2.2 and the previous theorem, we obtained the following corollary clearly.

**Corollary 2.7** Let \( a \) be a fixed rational number with \( ab(a+b)(a-1) \neq 0 \), or let \( a+b \) be a fixed rational number with \( ab(a+b)(a+b-1) \neq 0 \), or let \( b \) be a fixed rational number with \( ab(ab) \neq 0 \). Suppose that \( ab > 0 \) and \( \varphi : V^2 \to \mathbb{R} \) is a function satisfying either the condition (2.7) or the condition (2.8) for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies (2.3) for all \( x, y \in V \), then there exists a unique quadratic mapping \( F : V \to Y \) satisfying the inequality (1.2) and the inequality (2.9) for all \( x \in V \).

We can easily following stability theorem by using the similar method used in the previous theorems.

**Theorem 2.8** Let \( a \) or \( b \) be a real constants with \( ab(a+b)^2(2a+b)^2 < 0 \) and let \( \varphi : V^2 \to \mathbb{R} \) be a function satisfying either
\[ \sum_{i=0}^{\infty} \frac{\varphi(b^i x, b^i y)}{(b^2 + 2ab)^{2i}} < \infty \]  
(2.13)

or

\[ \sum_{i=0}^{\infty} b^{2i} \varphi\left( \frac{x}{b^i}, \frac{y}{b^i} \right) < \infty \]  
(2.14)

for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies (2.3) for all \( x, y \in V \) with \( f(0) = 0 \), then there exists a unique solution \( F : V \to Y \) of the functional equation (1.2) satisfying the inequality (2.9) for all \( x \in V \).

**Corollary 2.9** Let \( a \) be a fixed rational number with \( ab(a+b)(a-1)(2a+b) \neq 0 \), or let \( a+b \) be a fixed rational number with \( ab(a+b)(a+b-1)(2a+b) \neq 0 \), or let \( b \) be a fixed rational number with \( ab(a+b)(2a+b) \neq 0 \). Suppose that
ab < 0 and \( \varphi : V^2 \to [0, \infty) \) is a function satisfying either the condition (2.13) or the condition (2.14) for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies \( f(0) = 0 \) and (2.3) for all \( x, y \in V \), then there exists a unique quadratic mapping \( F : V \to Y \) satisfying the functional equation (1.2) and the inequality (2.9) for all \( x \in V \).

Recall the following theorem shows the stability of the functional equation (1.2) on the restricted domain \( V \setminus \{0\} \).

**Theorem 2.10** (Theorem 4.3 and Theorem 4.4 in [7]) Let \( a \) and \( b \) be real constants with \( ab(a + b) \neq 0 \) and let \( \varphi : (V \setminus \{0\})^2 \to [0, \infty) \) be a function satisfying either

\[
\sum_{i=0}^{\infty} \frac{\varphi((a + b)^i x, (a + b)^i y)}{(a + b)^{2i}} < \infty, \tag{2.15}
\]

or

\[
\sum_{i=0}^{\infty} (a + b)^{2i} \varphi \left( \frac{x}{(a + b)^i}, \frac{y}{(a + b)^i} \right) < \infty \tag{2.16}
\]

for all \( x, y \in V \setminus \{0\} \). If a mapping \( f : V \to Y \) satisfies \( f(0) = 0 \) and (2.3) for all \( x, y \in V \setminus \{0\} \), then there exists a unique mapping \( F : V \to Y \) satisfying the equation (1.2) for all \( x \in V \setminus \{0\} \) and the inequality

\[
\|f(x) - F(x)\| \leq \begin{cases} 
\sum_{i=0}^{\infty} \frac{\varphi((a + b)^i x, (a + b)^i y)}{(a + b)^{2i+2}} & \text{if } \varphi \text{ satisfies (2.15),} \\
\sum_{i=0}^{\infty} (a + b)^{2i} \varphi \left( \frac{x}{(a + b)^{i+1}}, \frac{x}{(a + b)^{i+1}} \right) & \text{if } \varphi \text{ satisfies (2.16)}
\end{cases} \tag{2.17}
\]

for all \( x \in V \setminus \{0\} \).

Using the previous theorem and Lemma 2.2, we get the following corollary.

**Corollary 2.11** Let \( a \) be a fixed rational number with \( ab(a + b)(a - 1) \neq 0 \), or let \( a + b \) be a fixed rational number with \( ab(a + b)(a + b - 1) \neq 0 \), or let \( b \) be a fixed rational number with \( ab(a + b) \neq 0 \). Let \( \varphi : V^2 \to [0, \infty) \) be a function satisfying either the condition (2.15) or the condition (2.16) for all \( x, y \in V \). If a mapping \( f : V \to Y \) satisfies \( f(0) = 0 \) and (2.3) for all \( x, y \in V \setminus \{0\} \), then there exists a unique quadratic mapping \( F : V \to Y \) satisfying the inequality (1.2) and the inequality (2.17) for all \( x \in V \setminus \{0\} \).

The following corollary follows from Theorem 2.4 and Theorem 2.10 by taking \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^p) \) defined on a real normed space \( X \).
Corollary 2.12 Let $a$, $b$, $p$, and $\theta$ be real constants such that $ab(a + b) \neq 0$, $a^2 \neq 1$, $(a + b)^2 \neq 1$, $p \neq 2$ and $p, \theta > 0$. If a mapping $f : X \to Y$ satisfies the inequality
\[ \|D_{ab}f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \] (2.18)
for all $x, y \in X$, then there exists a unique mapping $F : X \to Y$ satisfying the equation (1.2) and the inequality
\[ \|f(x) - F(x)\| \leq \min \left\{ \frac{2\theta\|x\|^p}{|(a + b)^2 - |a + b|^p|}, \frac{\theta\|x\|^p}{|a^2 - |a|^p|} \right\} \] (2.19)
for all $x \in X \setminus \{0\}$.

The following corollary follows from Lemma 2.2, Theorem 2.4, and Theorem 2.10.

Corollary 2.13 Let $a$ be a fixed rational number with $ab(a + b)(a - 1) \neq 0$, or let $a + b$ be a fixed rational number with $ab(a + b)(a + b - 1) \neq 0$, or let $b$ be a fixed rational number with $ab(a + b) \neq 0$. Let $p$ and $\theta$ be positive real numbers such that $p \neq 2$. If a mapping $f : X \to Y$ satisfies the inequality (2.18) for all $x, y \in X$, then there exists a unique quadratic mapping $F : X \to Y$ satisfying the inequality (2.19) for all $x \in X \setminus \{0\}$.

The following corollary follows from Theorem 2.10 with the negative $p$.

Corollary 2.14 Let $a$, $b$, $p$, and $\theta$ be real constants such that $ab(a + b) \neq 0$, $p < 0$ and $\theta > 0$. If a mapping $f : X \to Y$ satisfies the inequality (2.18) for all $x, y \in X \setminus \{0\}$ with $f(0) = 0$, then the mapping $f : X \to Y$ satisfies the equation (1.2) for all $x, y \in X \setminus \{0\}$.

Proof. According to Theorem 2.10, there exists a unique mapping $F : V \to Y$ satisfying the equation (1.2) for all $x, y \in X \setminus \{0\}$ and the inequality
\[ \|f(x) - F(x)\| \leq \frac{2\theta\|x\|^p}{|(a + b)^2 - |a + b|^p|} \] (2.20)
for all $x \in X \setminus \{0\}$. Since the equality $D_{ab}F(x, y) = 0$ holds for all $x, y \in X \setminus \{0\}$ and the inequality (2.20) holds for all $x \in X \setminus \{0\}$, we obtain the inequality
\[
|ab||f(x) - F(x)| \\
\leq \|(D_{ab}f - D_{ab}F)((k + 1)x, kx)\| + \|(F - f)(((a + b)k + 1)x)\| \\
+ |a(a + b)||f - F|(k + 1)x)\| + |b(a + b)||f - F|((k + 1)x)\| \\
\leq \left( \frac{((a + b)k + 1|^p + |a(a + b)|(k + 1)^p + |b(a + b)|k^p)}{|(a + b)^2 - |a + b|^p|} + (k + 1)^p + k^p \right) \theta\|x\|^p \\
\to 0, \text{ as } k \to \infty,
\]
for all $x \in X \setminus \{0\}$.

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References


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