Polynomiography of Some Iterative Methods

Shin Min Kang

Department of Mathematics and RINS
Gyeongsang National University
Jinju 52828, Korea

Amir Naseem

Department of Mathematics
Lahore Leads University
Lahore 54810, Pakistan

Waqas Nazeer¹ and Mobeen Munir

Division of Science and Technology
University of Education
Lahore 54000, Pakistan

Chahn Yong Jung²

Department of Business Administration
Gyeongsang National University
Jinju 52828, Korea

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Abstract

The aim of this paper is to present polynomiographs using two step Golbabi and Javidi’s method and Golbabi and Javidi’s method free from second derivative for finding the roots of a given complex polynomial.

¹,² Corresponding authors
Polynomiography is the art and science of visualization in approximation of zeros of complex polynomials. The images thus obtained are called polynomiographs. In this paper, we obtain polynomiographs of different complex polynomials. The obtained polynomiographs have very interesting patterns for complex polynomial equations. We believe that the results of this paper enrich the functionality of the existing polynomiography software.

Mathematics Subject Classification: 65H05, 65D32

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1 Introduction

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. The last interesting contribution to the polynomials root finding history was made by Kalantari [1], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari in 2005 [2, 4]. Polynomiography is defined to be “the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions” [1]. An individual image is called a “polynomiograph”. Polynomiography combines both art and science aspects.

Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iteration methods formulated for the purpose of approximation of the root of polynomials, e.g., Newton’s method, Halley’s method, etc. The word “fractal”, which partially appeared in the definition of polynomiography, was coined by the famous mathematician Benoit Mandelbrot [3]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar has typical structure and independent of scale. On the other hand, polynomiographs are quite different. The “polynomiographer” can control the shape and designed in a more predictable way by using different iteration methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects. Polynomiography has diverse applications in math, science, education, art and design. According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients
Polynomiography of some iterative methods

\{a_n, a_{n-1}, \ldots, a_1, a_0\}:

\[ p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \] \hspace{1cm} (1.1)

or by its zeros (roots) \{r_1, r_2, \ldots, r_{n-1}, r_n\}:

\[ p(z) = (z - r_1)(z - r_2) \cdots (z - r_n) \] \hspace{1cm} (1.2)

of degree \( n \) has \( n \) roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [1, 2, 4].

2 Convergence test

In the numerical algorithms that are based on iterative processes we need a stop criterion for the process, that is, a test that tells us that the process has converged or it is very near to the solution. This type of test is called a convergence test. Usually, in the iterative process that use a feedback, like the root finding methods, the standard convergence test has the following form:

\[ |z_{n+1} - z_n| < \varepsilon, \] \hspace{1cm} (2.1)

where \( z_{n+1} \) and \( z_n \) are two successive points in the iteration process and \( \varepsilon > 0 \) is a given accuracy. In this paper we also use the stop criterion (2.1).

2.1 Iterations

During the last century, various methods have been developed for solving nonlinear equations. Now we define:

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \]

\[ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2(f^3(y_n) - f(y_n)f'(y_n)f''(y_n))}. \]

This is so-called modified Golbabai and Javidi’s method for solving nonlinear equations. Let \( p(z) \) be the complex polynomial, then

\[ y_n = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \ldots, \]

\[ z_{n+1} = y_n - \frac{p(y_n)}{p'(y_n)} - \frac{p^2(y_n)p''(y_n)}{2(p^3(y_n) - p(y_n)p'(y_n)p''(y_n))}. \]
where \( z_0 \in \mathbb{C} \) is a starting point, is so-called modified Golbabai and Javidi’s method for solving nonlinear complex equations.

### 2.1.1 Polynomiograph for \( z^2 - 1 = 0 \)

The polynomiograph of the complex polynomial equation \( z^2 - 1 = 0 \), having two roots 1, \(-1\), can be visualized in the following Figure 1:

![Figure 1. Polynomiograph for \( z^2 - 1 = 0 \)](image)

### 2.1.2 Polynomiograph for \( z^3 - 1 = 0 \)

The polynomiograph of complex polynomial equation \( z^3 - 1 = 0 \), having three roots: 1, \(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\), \(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\), can be visualized in the following Figure 2:

![Figure 2. Polynomiograph for \( z^3 - 1 = 0 \)](image)

### 2.1.3 Polynomiograph for \( z^3 - z^2 + z - 1 = 0 \)

The polynomiograph of complex polynomial equation \( z^3 - z^2 + z - 1 = 0 \), having three roots: 1, \(i\), \(-i\), can be visualized in the following Figure 3:
2.1.4 Polynomiograph for $z^4 - 1 = 0$

The polynomiograph of complex polynomial equation $z^4 - 1 = 0$, having four roots: -1, $-I$, $I$, 1, can be visualized in the following Figure 4:

![Figure 4. Polynomiograph for $z^4 - 1 = 0$](image)

2.1.5 Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$

The polynomiograph of the complex polynomial equation $z^4 - z^3 + z^2 - z + 1 = 0$, having four roots: $-0.30902 - 0.95106I$, $-0.30902 + 0.95106I$, $0.80902 - 0.58779I$, $0.80902 + 0.58779I$, can be visualized in the following Figure 5:

![Figure 5. Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$](image)

2.1.6 Polynomiograph for $z(z^2 + 1)(z^2 + 4) = 0$

The polynomiograph of the complex polynomial equation $z(z^2 + 1)(z^2 + 4) = 0$, having five roots: 0, $0 - 1I$, $0 + 1I$, $0 - 2I$, $0 + 2I$, can be visualized in the
following Figure 6:

![Figure 6](image)

Figure 6. Polynomiograph for $z(z^2 + 1)(z^2 + 4) = 0$

### 2.1.7 Polynomiograph for $z^5 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^5 - 1 = 0$, having five roots: $1, -\frac{1}{4} + \frac{\sqrt{5}}{4}, -\frac{1}{4} - \frac{\sqrt{5}}{4}, -\frac{1}{4} - \frac{\sqrt{5}}{4} + \frac{i\sqrt{2\sqrt{5}+\sqrt{5}}}{4}$, can be visualized in the following Figure 7:

![Figure 7](image)

Figure 7. Polynomiograph for $z^5 - 1 = 0$

### 2.1.8 Polynomiograph for $z^6 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^6 - 1 = 0$, having six roots: $1, -1, 0.5 + 0.86603I, -0.5 - 0.86603I, -0.5 + 0.86603I, 0.5 - 0.86603I$, can be visualized in the following Figure 8:

![Figure 8](image)

Figure 8. Polynomiograph for $z^6 - 1 = 0$
2.1.9 Polynomiograph for $z^7 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^7 - 1 = 0$, having seven roots: 1, 0.62349 − 0.78183$I$, 0.62349 + 0.78183$I$, −0.90097 − 0.43388$I$, −0.90097 + 0.43388$I$, −0.22252 − 0.97493$I$, −0.22252 + 0.97493$I$, can be visualized in the following Figure 9:

![Figure 9. Polynomiograph for $z^7 - 1 = 0$](image)

2.1.10 Polynomiograph for $z^7 - z^6 + z^5 - z^4 + z^3 - z^2 + z - 1 = 0$

The polynomiograph of the complex polynomial equation $z^7 - z^6 + z^5 - z^4 + z^3 - z^2 + z - 1 = 0$, having seven roots: 1, $I$, $-I$, $-0.70711 - 0.70711I$, $-0.70711 + 0.70711I$, $0.70711 + 0.70711I$, $0.70711 - 0.70711I$, can be visualized in the following Figure 10:

![Figure 10. Polynomiograph for $z^7 - z^6 + z^5 - z^4 + z^3 - z^2 + z - 1 = 0$](image)

2.1.11 Polynomiograph for $z^8 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^8 - 1 = 0$, having eight roots: 1, $-1$, $I$, $-I$, $-0.70711 - 0.70711I$, $-0.70711 + 0.70711I$, $0.70711 - 0.70711I$, $0.70711 + 0.70711I$, can be visualized in the following Figure 11:
2.1.12 Polynomiograph for \( z^9 - 1 = 0 \)

The polynomiograph of the complex polynomial equation \( z^9 - 1 = 0 \), having nine roots: 1, \(-0.5 - 0.86603I, 0.76604 - 0.64279I, 0.76604 + 0.64279I, -0.93969 - 0.34202I, -0.93969 + 0.34202I, 0.17365 - 0.98481I, 0.17365 + 0.98481I, -0.5 + 0.86603I \), can be visualized in the following Figure 12:

![Figure 12. Polynomiograph for \( z^9 - 1 = 0 \)](image)

2.1.13 Polynomiograph for \( z^{10} - 1 = 0 \)

The polynomiograph of the complex polynomial equation \( z^{10} - 1 = 0 \), having ten roots: 1, \(-1, -0.80902-0.58779I, -0.80902+0.58779I, 0.80902-0.58779I, 0.80902 + 0.58779I, -0.30902 - 0.95106I, -0.30902 + 0.95106I, 0.30902 - 0.95106I, 0.30902 + 0.95106I \), can be visualized in the following Figure 13:

![Figure 13. Polynomiograph for \( z^{10} - 1 = 0 \)](image)
2.1.14 Polynomiograph for \( z^{20} - 1 = 0 \)

The polynomiograph of the complex polynomial equation \( z^{20} - 1 = 0 \), having twenty roots, can be visualized in the following Figure 14:

![Polynomiograph for \( z^{20} - 1 = 0 \)](image_url)

Figure 14. Polynomiograph for \( z^{20} - 1 = 0 \)

2.2 Polynomiography via Golbabai And Javidi’s method free from 2nd derivative

The Golbabai and Javidi’s method free from second derivative is given as follows:

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...
\]

\[
x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f'(x_n)(f'(x_n) - f'(y_n))}{2f(x_n)f''(y_n) - 2f(y_n)f'(x_n)f''(y_n)(f'(x_n) - f'(y_n))}.
\]

Let \( p(z) \) be the complex polynomial, then

\[
y_n = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, ...
\]

\[
z_{n+1} = y_n - \frac{p(y_n)}{p'(y_n)} - \frac{p^2(y_n)p'(z_n)(p'(z_n) - p'(y_n))}{2p(z_n)p''(y_n) - 2p(y_n)p'(z_n)p'(y_n)(p'(z_n) - p'(y_n))},
\]

where \( z_o \in \mathbb{C} \) is a starting point, is so-called Golbabai and Javidi’s method (GJM) free from second derivative for solving nonlinear complex equations. The sequence \( \{z_n\}_{n=0}^{\infty} \) is called the orbit of the point \( z_o \) converges to a root \( z^* \) of \( f \) then, we say that \( z_o \) is attracted to \( z^* \). A set of all such starting points for which \( \{z_n\}_{n=0}^{\infty} \) converges to root \( z^* \) is called the basin of attraction of \( z^* \).

2.2.1 Polynomiograph for \( z^{2} - 1 = 0 \)

Complex polynomial equation \( z^{2} - 1 = 0 \) has two roots: 1, -1. The polynomiograph via new GJM is presented in the following Figure 15:
Complex polynomial equation $z^3 - 1 = 0$ has three roots: $1$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}I$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}I$. The polynomiograph via new GJM is presented in the following Figure 16:

![Figure 16. Polynomiograph for $z^3 - 1 = 0$](image1)

Complex polynomial equation $(z + 1)(z^2 + 2) = 0$ has three roots: $-1$, $\sqrt{2}I$, $-\sqrt{2}I$. The polynomiograph via new GJM is presented in the following Figure 17:

![Figure 17. Polynomiograph for $(z + 1)(z^2 + 2) = 0$](image2)

Complex polynomial equation $z^4 - 1 = 0$ has four roots: $-1$, $-I$, $I$, $1$. The polynomiograph via new GJM is presented in the following Figure 18:
2.2.5 Polynomiograph for $z^4 + 4 = 0$

Complex polynomial equation $z^4 + 4 = 0$ has four roots: $1 + I$, $1 - I$, $-1 + I$, $-1 - I$. The polynomiograph via new GJM is presented in the following Figure 19:

![Figure 19. Polynomiograph for $z^4 + 4 = 0$](image)

2.2.6 Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$

Complex polynomial equation $z^4 - z^3 + z^2 - z + 1 = 0$ has four roots: $-0.30902 - 0.95106I$, $-0.30902 + 0.95106I$, $0.80902 - 0.58779I$, $0.80902 + 0.58779I$. The polynomiograph via new GJM is presented in the following Figure 20:

![Figure 20. Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$](image)
2.2.7 Polynomiograph for \((z^2 + 1)(z^2 + 2) = 0\)

Complex polynomial equation \((z^2 + 1)(z^2 + 2) = 0\) has four roots: \(I, -I, \sqrt{2}I, -\sqrt{2}I\). The polynomiograph via new GJM is presented in the following Figure 21:

![Figure 21. Polynomiograph for \((z^2 + 1)(z^2 + 2) = 0\)](image)

2.2.8 Polynomiograph for \(z^5 - 1 = 0\)

Complex polynomial equation \(z^5 - 1 = 0\) has five roots: 1, \(\frac{-1}{4} + \frac{\sqrt{5}}{4} + \frac{i\sqrt{2\sqrt{5} + \sqrt{5}}}{4}\), \(\frac{-1}{4} - \frac{\sqrt{5}}{4} + \frac{i\sqrt{2\sqrt{5} - \sqrt{5}}}{4}\), \(\frac{-1}{4} + \frac{\sqrt{5}}{4} - \frac{i\sqrt{2\sqrt{5} + \sqrt{5}}}{4}\). The polynomiograph via new GJM is presented in the following Figure 22:

![Figure 22. Polynomiograph for \(z^5 - 1 = 0\)](image)

2.2.9 Polynomiograph for \(16z^5 - 20z^3 + 3 = 0\)

Complex polynomial equation \(16z^5 - 20z^3 + 3 = 0\) has five roots: \(1.04107, -1.16931, 0.59324, -0.23250 + 0.45341I, -0.23250 - 0.45341I\). The polynomiograph via new GJM is presented in the following Figure 23:
2.2.10 Polynomiograph for $z(z^2 + 1)(z^2 + 4) = 0$

Complex polynomial equation $z(z^2 + 1)(z^2 + 4) = 0$ has five roots: 0, 0 − 1I, 0 + 1I, 0 − 2I, 0 + 2I. The polynomiograph via new GJM is presented in the following Figure 24:

2.2.11 Polynomiograph for $z^6 − 1 = 0$

The complex polynomial equation $z^6 − 1 = 0$ has six roots: 1, −1, 0.5 + 0.86603I, −0.5 − 0.86603I, −0.5 + 0.86603I, 0.5 − 0.86603I. The polynomiograph via new GJM is presented in the following Figure 25:
2.2.12 Polynomiograph for \( z^6 - \frac{1}{2}z^5 + \frac{11(1+I)}{4}z^4 - \frac{19+3I}{4}z^3 + \frac{11+5I}{4}z^2 - \frac{11+I}{4}z + \frac{3}{2} - 3I = 0 \)

The complex polynomial equation \( z^6 - \frac{1}{2}z^5 + \frac{11(1+I)}{4}z^4 - \frac{19+3I}{4}z^3 + \frac{11+5I}{4}z^2 - \frac{11+I}{4}z + \frac{3}{2} - 3I = 0 \) has six roots: \(-1 + 2I, -0.5 - 0.5I, 0 + 1I, 0 - 1.5I, 1 - 1I, 1\). The polynomiograph via new GJM is presented in the following Figure 26:

Figure 26. Polynomiography for \( z^6 - \frac{1}{2}z^5 + \frac{11(1+I)}{4}z^4 - \frac{19+3I}{4}z^3 + \frac{11+5I}{4}z^2 - \frac{11+I}{4}z + \frac{3}{2} - 3I = 0 \)

2.2.13 Polynomiograph for \((z^3 - 1)(z^3 - 2) = 0\)

The complex polynomial equation \((z^3 - 1)(z^3 - 2) = 0\) has six roots: \(1, 1.25992, -0.62996 - 1.09112I, -0.62996 + 1.09112I, -0.5 - 0.86603I, -0.5 + 0.86603I\). The polynomiograph via new GJM is presented in the following Figure 27:

Figure 27. Polynomiograph for \((z^3 - 1)(z^3 - 2) = 0\)

2.2.14 Polynomiograph for \( z^7 - 1 = 0 \)

The complex polynomial equation \( z^7 - 1 = 0 \) has seven roots: \(1, 0.62349 - 0.78183I, 0.62349 + 0.78183I, -0.90097 - 0.43388I, -0.90097 + 0.43388I, -0.22252 - 0.97493I, -0.22252 + 0.97493I\). The polynomiograph via new GJM is presented in the following Figure 28:
Figure 28. Polynomiograph for \( z^7 - 1 = 0 \)

### 2.2.15 Polynomiograph for \( z^8 - 1 = 0 \)

The complex polynomial equation \( z^8 - 1 = 0 \) has eight roots: 1, \(-1\), \(i\), \(-i\), \(-0.70711 - 0.70711i\), \(-0.70711 + 0.70711i\), \(0.70711 - 0.70711i\), \(0.70711 + 0.70711i\). The polynomiograph via new GJM is presented in the following Figure 29:

Figure 29. Polynomiograph for \( z^8 - 1 = 0 \)

### 2.2.16 Polynomiograph for \( z^9 - 1 = 0 \)

The complex polynomial equation \( z^9 - 1 = 0 \) has nine roots: 1, \(-0.5 - 0.86603i\), \(0.76604 - 0.64279i\), \(0.76604 + 0.64279i\), \(-0.93969 - 0.34202i\), \(-0.93969 + 0.34202i\), \(0.17365 - 0.98481i\), \(0.17365 + 0.98481i\), \(-0.5 + 0.86603i\). The polynomiograph via new GJM is presented in the following Figure 30:

Figure 30. Polynomiograph for \( z^9 - 1 = 0 \)
2.2.17 Polynomiograph for $z^{10} - 1 = 0$

The complex polynomial equation $z^{10} - 1 = 0$ has ten roots: 1, $-1$, $-0.80902 - 0.58779I$, $-0.80902 + 0.58779I$, $0.80902 - 0.58779I$, $0.80902 + 0.58779I$, $-0.30902 - 0.95106I$, $-0.30902 + 0.95106I$, $0.30902 - 0.95106I$, $0.30902 + 0.95106I$. The polynomiograph via new GJM is presented in the following Figure 31:

![Figure 31. Polynomiograph for $z^{10} - 1 = 0$](image)

2.2.18 Polynomiograph for $z^{20} - 1 = 0$

Complex polynomial equation $z^{20} - 1 = 0$ has twenty roots. The polynomiograph via new GJM is presented in the following Figure 32:

![Figure 32. Polynomiograph for $z^{20} - 1 = 0$](image)

2.2.19 Polynomiograph for $z^{21} + 5z^{14} - \frac{22}{15}z^7 - \frac{11}{675} = 0$

Complex polynomial equation $z^{21} + 5z^{14} - \frac{22}{15}z^7 - \frac{11}{675} = 0$ has twenty one roots. The polynomiograph via new GJM is presented in the following Figure 33:
Figure 33. Polynomiograph for $z^{21} + 5z^{14} - \frac{22}{15}z^7 - \frac{11}{675} = 0$

3 Conclusions

In this paper, we presented some new polynomiographs of different complex polynomials using two-step Golbabai and Javidi’s method and Golbabai and Javidi’s method free from second derivative for solving nonlinear complex polynomials. The obtained images are nice looking, interesting, quite new and different from images obtained using the Newton’s method, Halley’s method and Householder’s method free from second derivatives and interesting from the aesthetic point of view.

References


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