Stolarsky Type Inequality for Sugeno Integrals on Fuzzy Convex Functions

Dug Hun Hong

Department of Mathematics, Myongji University
Yongin Kyunggido 449-728, South Korea

Abstract

Recently, Flores-Franulić et al. [A note on fuzzy integral inequality of Stolarsky type, Applied Mathematics and Computation 208 (2008) 55-59] proved the Stolarsky’s inequality for the Sugeno integral on the special class of continuous and strictly monotone functions. This result can be generalized to a general class of fuzzy convex functions in this paper. We also give a fuzzy integral inequality based on addition. Some illustrated examples are given.

Mathematics Subject Classification: 26E50

Keywords: Fuzzy measure; Sugeno integral; Stolarsky type inequality

1 Introduction and preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [16], it has been exhaustively investigated by many authors. Ralescu and Adams [12] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [17] provided an overview of fuzzy measure theory.


The purpose of this paper is to generalize the main results in [6], that is, we prove a Stolarsky type inequality for Sugeno integrals on fuzzy convex functions. We also give a fuzzy integral inequality based on addition. Some examples are provided to illustrate the validity of the proposed inequalities.

Definition 1. Let $\Sigma$ be a $\sigma$-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function. We say that $\mu$ is a fuzzy measure if and only if

(a) $\mu(\emptyset) = 0$.
(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity).
(c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$ (continuity form below).
(d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$ (continuity form above).

If $f$ is a non-negative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in X | f(x) \geq \alpha\} = \{f \geq \alpha\}$ the $\alpha$-level of $f$, for $\alpha > 0$, and $F_0 = \{x \in X | f(x) > 0\} = \text{supp}(f)$ is the support of $f$.

We note that

$\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$

If $\mu$ is a fuzzy measure on $A \subseteq \mathbb{R}$, then we define the following:

$\mathcal{F}^\mu(A) = \{f : A \to [0, \infty)|f$ is $\mu$-measurable}.

Definition 2. Let $\mu$ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathcal{F}^\mu(\mathbb{R})$ and $A \in \Sigma$, then the Sugeno integral (or the fuzzy integral) of $f$ on $A$, with respect to the fuzzy measure $\mu$, is defined as

\[
(S) \int_A f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(A \cap F_\alpha)].
\]
In particular, if $A = X$ then
\[
(S) \int_R f \, d\mu = (S) \int f \, d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(F_\alpha)]
\]

The following properties of the Sugeno integral are well known and can be found in [17]:

**Proposition 1** [17]. If $\mu$ is a fuzzy measure on $\mathbb{R}$ and $f, g \in \mathcal{F}^\mu(\mathbb{R})$, then

(i) $(S) \int_A f \, d\mu \leq \mu(A)$;
(ii) $(S) \int_A K \, d\mu = K \land \mu(A)$ for any constant $K \in [0, \infty)$;
(iii) $(S) \int_A f \, d\mu \leq (S) \int_A g \, d\mu$, if $f \leq g$ on $A$;
(iv) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (S) \int_A f \, d\mu \geq \alpha$;
(v) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (S) \int_A f \, d\mu \leq \alpha$;
(vi) $(S) \int_A f \, d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
(vii) $(S) \int_A f \, d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

## 2 Stolarsky type inequality for fuzzy convex functions

Flores-Franulić et al. [6] presented the following Stolarsky-type inequality for fuzzy integrals under the conditions that $f$ is a continuous and strictly monotone function.

**Theorem 1 (Stolarsky type inequality: monotone case [6])**. Let $a, b > 0$. If $f : [0, 1] \to [0, 1]$ is a continuous and strictly monotone (decreasing or increasing) function and $\mu$ is the Lebesgue measure on $\mathbb{R}$, then the inequality

\[
(S) \int_0^1 f \left( \frac{1}{x^{a+b}} \right) \, dx \geq \left( (S) \int_0^1 f \left( \frac{1}{x^a} \right) \, dx \right) \left( (S) \int_0^1 f \left( \frac{1}{x^b} \right) \, dx \right)
\]

holds.

This result can be generalized to a general class of fuzzy convex functions in this paper. We need the following lemma.

**Lemma 1.** Let $a, b > 0$. If $f : [0, 1] \to \mathbb{R}$ is a fuzzy convex function, and $\mu$ is the Lebesgue measure on $\mathbb{R}$, then the inequalities

\[
\mu \left\{ f \left( \frac{1}{x^{a+b}} \right) \geq \alpha \right\} \geq \mu \left\{ f \left( \frac{1}{x^a} \right) \geq \alpha \right\} \mu \left\{ f \left( \frac{1}{x^b} \right) \geq \alpha \right\}
\]

and

\[
\mu \left\{ f \left( \frac{1}{x^{a+b}} \right) \geq \alpha \right\} \leq \mu \left\{ f \left( \frac{1}{x^a} \right) \geq \alpha \right\} + \mu \left\{ f \left( \frac{1}{x^b} \right) \geq \alpha \right\}
\]
hold.

Proof. Suppose that for $0 < \alpha$,

\[
\left\{ f \left( x^{\frac{1}{\alpha + \delta}} \right) \geq \alpha \right\} = [\alpha_1, \alpha_2].
\]

Then we have

\[
f \left( \alpha_1^{\frac{1}{\alpha + \delta}} \right) = f \left( \alpha_2^{\frac{1}{\alpha + \delta}} \right) = \alpha,
\]

and hence

\[
\left\{ f(x) \geq \alpha \right\} = [\alpha_1^{\frac{1}{\alpha + \delta}}, \alpha_2^{\frac{1}{\alpha + \delta}}].
\]

Similarly, suppose that

\[
\left\{ f \left( x^{\frac{1}{\beta}} \right) \geq \alpha \right\} = [\beta_1, \beta_2], \quad \left\{ f \left( x^{\frac{1}{\gamma}} \right) \geq \alpha \right\} = [\gamma_1, \gamma_2],
\]

then we have

\[
\left\{ f(x) \geq \alpha \right\} = [\beta_1^{\frac{1}{\beta}}, \beta_2^{\frac{1}{\beta}}] = [\gamma_1^{\frac{1}{\gamma}}, \gamma_2^{\frac{1}{\gamma}}].
\]

Let \( \{ f(x) \geq \alpha \} = [h_1, h_2] \). Then we have

\[
[\alpha_1, \alpha_2] = [h_1^{a+b}, h_2^{a+b}], \quad [\beta_1, \beta_2] = [h_1^a, h_2^a] \text{ and } [\gamma_1, \gamma_2] = [h_1^b, h_2^b].
\]

We now consider that

\[
\mu \left\{ f \left( x^{\frac{1}{\alpha + \delta}} \right) \geq \alpha \right\} - \mu \left\{ f \left( x^{\frac{1}{\beta}} \right) \geq \alpha \right\} \mu \left\{ f \left( x^{\frac{1}{\gamma}} \right) \geq \alpha \right\}
\]

\[
= (\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)(\gamma_2 - \gamma_1)
\]

\[
= (h_2^{a+b} - h_1^{a+b}) - (h_2^a - h_1^a)(h_2^b - h_1^b)
\]

\[
= h_1^a h_2^b + h_1^b h_2^a - 2h_1^{a+b}
\]

\[
\geq 0
\]

and

\[
\mu \left\{ f \left( x^{\frac{1}{\alpha + \delta}} \right) \geq \alpha \right\}
\]

\[
= \alpha_2 - \alpha_1 = \beta_2 \gamma_2 - \beta_1 \gamma_1
\]

\[
= \beta_2 \gamma_2 - \beta_2 \gamma_1 + \beta_2 \gamma_1 - \beta_1 \gamma_1
\]

\[
= \beta_2 (\gamma_2 - \gamma_1) + \gamma_1 (\beta_2 - \beta_1)
\]

\[
\leq (\gamma_2 - \gamma_1) + (\beta_2 - \beta_1)
\]

\[
= \mu \left\{ f \left( x^{\frac{1}{\beta}} \right) \geq \alpha \right\} + \mu \left\{ f \left( x^{\frac{1}{\gamma}} \right) \geq \alpha \right\}.
\]

which completes the proof.

By using Lemma 1, we obtain the following main result. We show that the condition of continuity and strictly monotonicity of \( f \) can be released. Fuzzy
convexity of $f$ is sufficient for validity of a Stolarsky type inequality for fuzzy integrals.

**Theorem 2.** Let $a, b > 0$. If $f : [0, 1] \to \mathbb{R}$ is a fuzzy convex function, and $\mu$ is the Lebesgue measure on $\mathbb{R}$, then the inequality

$$
(S) \int_0^1 f \left( x^{\frac{1}{a+b}} \right) d\mu \geq \left( (S) \int_0^1 f \left( x^{\frac{1}{a}} \right) d\mu \right) \left( (S) \int_0^1 f \left( x^{\frac{1}{b}} \right) d\mu \right)
$$

holds.

**Proof.** Let $(S) \int_0^1 f \left( x^{\frac{1}{a}} \right) d\mu = p$ and $(S) \int_0^1 f \left( x^{\frac{1}{b}} \right) d\mu = q$. And let $\varepsilon > 0$ such that $\min\{p - \varepsilon, q - \varepsilon\} > 0$.

Then we have

$$
(S) \int_0^1 f \left( x^{\frac{1}{a}} \right) d\mu > p - \varepsilon \quad \text{and} \quad (S) \int_0^1 f \left( x^{\frac{1}{b}} \right) d\mu > q - \varepsilon
$$

holds. By Proposition 1, there exist $\alpha$ and $\beta$ such that $1 \geq \alpha > p - \varepsilon$ and $1 \geq \beta > q - \varepsilon$,

$$
\mu \{ f \left( x^{\frac{1}{a}} \right) \geq \alpha \} > p - \varepsilon \quad \text{and} \quad \mu \{ f \left( x^{\frac{1}{b}} \right) \geq \beta \} > q - \varepsilon
$$

hold. By Lemma 1, we have

$$
\mu \{ f \left( x^{\frac{1}{a+b}} \right) \geq \alpha \beta \} \geq \mu \{ f \left( x^{\frac{1}{a}} \right) \geq \alpha \} \mu \{ f \left( x^{\frac{1}{b}} \right) \geq \beta \} > (p - \varepsilon)(q - \varepsilon).
$$

Since $\alpha \beta > (p - \varepsilon)(q - \varepsilon)$, by Proposition 1,

$$
(S) \int_0^1 f \left( x^{\frac{1}{a+b}} \right) d\mu > (p - \varepsilon)(q - \varepsilon).
$$

Since $\varepsilon$ is arbitrary,

$$
(S) \int_0^1 f \left( x^{\frac{1}{a+b}} \right) d\mu \geq pq = \left( (S) \int_0^1 f d\mu \right) \left( (S) \int_0^1 g d\mu \right).
$$

In a similar manner, we can prove the following result using Lemma 1.

**Theorem 3.** Let $a, b > 0$. If $f : [0, 1] \to \mathbb{R}$ is a fuzzy convex function, and $\mu$ is the Lebesgue measure on $\mathbb{R}$, then the inequality

$$
(S) \int_0^1 f \left( x^{\frac{1}{a+b}} \right) d\mu \leq \left( (S) \int_0^1 f \left( x^{\frac{1}{a}} \right) d\mu \right) + \left( (S) \int_0^1 f \left( x^{\frac{1}{b}} \right) d\mu \right)
$$

holds.
holds.

In the following, we present an example to illustrate the validity of Theorem 2. and 3.

**Example 1.** Let $f(x) = 4(x - x^2)$, $x \in [0,1]$ and $a = 2, b = 1/6$. Then, a straightforward calculus with the aid of computer work shows that

\[
i(S) \int_0^1 f(x^\frac{1}{7}) \, d\mu = (S) \int_0^1 (x^\frac{1}{7} - x) \, d\mu = \bigvee_{\alpha \in [0,\infty)} [\alpha \wedge \mu(4(x^\frac{1}{7} - x) \geq \alpha)] = 0.618,
\]

\[
ii(S) \int_0^1 f(x^\frac{1}{6}) \, d\mu = (S) \int_0^1 (x^6 - x^{12}) \, d\mu = \bigvee_{\alpha \in [0,\infty)} [\alpha \wedge \mu(4(x^6 - x^{12}) \geq \alpha)] = 0.319,
\]

\[
iii(S) \int_0^1 f(x^\frac{1}{a+b}) \, d\mu = (S) \int_0^1 (x^{\frac{6}{a+b}} - x^{\frac{12}{a+b}}) \, d\mu = \bigvee_{\alpha \in [0,\infty)} [\alpha \wedge \mu(4(x^{\frac{6}{a+b}} - x^{\frac{12}{a+b}}) \geq \alpha)] = 0.610.
\]

Therefore

\[
0.610 = (S) \int_0^1 f(x^\frac{1}{a+b}) \, d\mu \geq \left( (S) \int_0^1 f(x^\frac{1}{7}) \, d\mu \right) \left( (S) \int_0^1 f(x^\frac{1}{6}) \, d\mu \right) = 0.197.
\]

**Example 2.** Let

\[
f(x) = \begin{cases} 
x & \text{if } x \in [0,1/4) \\
1/2 & \text{if } x \in [1/4,3/8) \\
4x - 1 & \text{if } x \in [3/8,1/2) \\
2(1-x) & \text{if } x \in [1/2,1],
\end{cases}
\]

and $a = 1, b = 2$. Then $f$ is not continuous and not monotone but is fuzzy convex. A straightforward calculus shows that

\[
\mu\{f(x^\frac{1}{7}) \geq 1/2\} = 1/2 \quad \text{and} \quad \mu\{f(x^\frac{1}{7}) \geq 1/2\} = 1/2,
\]

and hence by Proposition 1, we have

\[
(S) \int_0^1 f(x^\frac{1}{7}) \, d\mu = 1/2 \quad \text{and} \quad (S) \int_0^1 f(x^\frac{1}{6}) \, d\mu = 1/2.
\]

Now, a straightforward calculus shows that

\[
\mu\left\{f(x^\frac{1}{a+b}) \geq \alpha\right\} = \begin{cases} 
(1 - \alpha/2)^2 - \alpha^2 & \text{if } x \in [0,1/4], \\
(1 - \alpha/2)^2 - 1/16 & \text{if } x \in [1/4,1/2], \\
(1 - \alpha/2)^2 - (1/4 + \alpha/4)^2 & \text{if } x \in (1/2,1].
\end{cases}
\]
By proving the equation \((1 - \alpha/2)^2 - 1/16 = \alpha\), we see that
\[
\mu\{ f \left( \frac{x}{\alpha + x} \right) \geq 0.4499 \} = 0.4499
\]
and hence
\[
(S) \int_0^1 f \left( \frac{1}{\alpha + x} \right) d\mu = 0.4499.
\]
Therefore
\[
0.4499 = (S) \int_0^1 f \left( \frac{1}{\alpha + x} \right) d\mu \geq \left( (S) \int_0^1 f \left( \frac{1}{x} \right) d\mu \right) \left( (S) \int_0^1 f \left( \frac{1}{x} \right) d\mu \right) = 0.25.
\]

References


**Received**: December 6, 2016; **Published**: January 11, 2017