Equivalent Multivariate Stochastic Processes

Arnaldo De La Barrera

Departamento de Matemáticas, Universidad de Pamplona
Pamplona, Colombia

Osmin Ferrer

Departamento de Matemáticas, Universidad de Sucre
Carrera 28 No. 5-267, Sincelejo, Colombia

Boris Lora

Coordinación de Matemáticas, Universidad del Atlántico
Barranquilla, Colombia

Abstract


Mathematics Subject Classification: 53C21, 53C42

Keywords: multivariate stochastic processes, positive definite kernels, Toeplitz kernels, Kolmogorov Decomposition
1 Introduction

The weakly stationary stochastic processes are of a great theoretical interest because they can be close related with the Hilbert spaces Riesz bases (see [7] and [9]). In his interesting paper [7], Strandell defined and studied a class of $L^2$ - stochastic processes the so called approximately stationary processes, which contains all weakly stationary processes in $L^2(P)$ and all Riesz bases for $L^2(P)$ as well.

In this paper we define and study a class of stochastic processes called equivalent multivariate stochastic processes. We give necessary and sufficient conditions for two multivariate stochastic processes to be equivalent. Using this and Kolmogorov decomposition theorem ([2, Theorem 3.1]) prove a result on stability of multivariate stochastic processes. This result is similar to that of Paley- Wiener Theorem on stability of bases (see [8, page 38 Theorem 10]).

Finally we take under consideration the class of the vector-valued multivariate approximately weakly stationary stochastic processes, which were introduced in the scalar setting by Strandel in [7].

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ and $P$ is a probability measure on $\mathcal{F}$. A function $x : \Omega \to \mathbb{C}$ which is measurable with respect to the $\sigma$-algebra $\mathcal{F}$ is called a stochastic variable. A stochastic process is a family $\{x_n\}_{n \in \mathbb{Z}}$ of stochastic variables. Let be, as usual, $L^2(P)$ the Hilbert space of all the measurable functions on $\Omega$ which are square integrable with respect to $P$, this is,

$$L^2(P) = \left\{ x : \Omega \to \mathbb{C} : x \text{ is measurable and } \int_{\Omega} |x(\omega)|^2 dP(\omega) < +\infty \right\}$$

equipped with the inner product

$$\langle x, y \rangle_{L^2(P)} = \int_{\Omega} x(\omega)\overline{y(\omega)}dP(\omega).$$

We will consider only stochastic processes with variables on $L^2(P)$.

The geometric settings for the prediction problem may be extended in order to deal with the multivariate case also. To that end, we remark that a variable of a stochastic process can be viewed as an operator from $\mathbb{C}$ to $L^2(P)$ by defining

$$\tilde{x}_n : \mathbb{C} \to L^2(P)$$

to be

$$\tilde{x}_n(\lambda) = \lambda x_n$$

and the elements of the correlation kernel of the process can be calculated by the following rule

$$K(m, n) = (\tilde{x}_m)^{*} \tilde{x}_n.$$
We also note that there are many stochastic processes which may have the same correlation kernel. For this raison it is convenient to adopt the following terminology.

The main object describing a multivariate process will be its correlation kernel, which is assumed to be a positive definite kernel \( K \) such that \( K(m, n) = K_{mn} \) belongs to \( \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n) \) for all \( m, n \in \mathbb{Z} \), where \( \mathcal{H} = \{\mathcal{H}_n\}_{n \in \mathbb{Z}} \) is a family of Hilbert spaces.

A pair \([\mathcal{K}, X]\), where \( \mathcal{K} \) is a Hilbert space and \( X = \{X_n\}_{n \in \mathbb{Z}} \) is a family of operators \( X_n \) in \( \mathcal{L}(\mathcal{H}_n, \mathcal{K}) \), is called a geometrical model of the multivariate process with correlation kernel \( K \), if

\[
K(m, n) = X_m^*X_n.
\]

2 Riesz bases

Let \( X \) and \( \mathcal{L}(X) \) be a Banach space and the algebra of all the bounded linear operators from \( X \) to itself respectively.

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a Schauder basis on the Banach space \( X \) and \( T \in \mathcal{L}(X) \) be a bounded invertible operator with bounded inverse. Let \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence defined by

\[
y_n = Tx_n \quad \text{for } n = 1, 2, \ldots
\]

then \( \{y_n\}_{n \in \mathbb{N}} \) is a Schauder basis in \( X \) too. Two bases \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) are equivalent if there is an operator \( T \in \mathcal{L}(X) \) with bounded inverse such that \( y_n = Tx_n \quad \text{for } n = 1, 2, \ldots \). For more details on bases in Banach spaces see [1, 4].

The orthonormal bases are very important in Hilbert space theory. There is another less known but also very useful kind of bases: the Riesz bases. This section will be devoted to them. More about these bases can be found in the Young’s book [8].

**Definition 2.1.** A basis in a Hilbert space is a Riesz basis if it is equivalent to an orthonormal basis.

**Definition 2.2.** Two inner products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \), defined on a vector space are equivalent inner products if they generate equivalent norms.

**Definition 2.3.** Let \( \mathcal{H} \) be a Hilbert space,

(a) A sequence \( \{x_n\}_{n \in \mathbb{N}} \) is complete in \( \mathcal{H} \) if

\[
\text{span}\{x_n\}_{n \in \mathbb{N}} = \mathcal{H}.
\]
(b) Two sequences \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) are said to be bi-orthogonal if
\[
\langle x_n, y_m \rangle = \delta_{nm} \quad \text{for all } n, m \in \mathbb{N}.
\]

**Theorem 2.4.**

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space and let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{H} \). The following statements are equivalent

(i) \( \{x_n\}_{n \in \mathbb{N}} \) is a Riesz basis in \( \mathcal{H} \).

(ii) There exists an inner product \( \langle \cdot, \cdot \rangle_1 \) on the linear space \( \mathcal{H} \), which is equivalent to the inner product of \( \mathcal{H} \) and such that \( \{x_n\}_{n \in \mathbb{N}} \) is an orthonormal basis for \((\mathcal{H}, \langle \cdot, \cdot \rangle_1)\).

(iii) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is complete in \( \mathcal{H} \) and there are constants \( A, B > 0, A \leq B \) such that
\[
A \sum_{n \in \mathbb{N}} |a_n|^2 \leq \left\| \sum_{n \in \mathbb{N}} a_n x_n \right\|^2 \leq B \sum_{n \in \mathbb{N}} |a_n|^2
\]
for every sequence of scalars \( a = \{a_n\}_{n \in \mathbb{N}} \) with finite support.

(iv) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is complete in \( \mathcal{H} \) and its Gram matrix
\[
(\langle x_i, x_j \rangle)_{i,j=1}^\infty
\]
is the matrix associated to an invertible bounded operator in \( l^2(\mathbb{N}) \).

(v) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is complete in \( \mathcal{H} \) and has a bi-orthogonal complete sequence \( \{y_n\}_{n \in \mathbb{N}} \) such that
\[
\sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 < \infty \quad y \quad \sum_{n \in \mathbb{N}} |\langle x, y_n \rangle|^2 < \infty
\]
for each \( x \) in \( \mathcal{H} \).

A proof of this theorem can be found in [8, página 32].

### 3 Paley-Wiener Theorem

The fundamental criterium of stability, and historically the first one, is due to Paley and Wiener [6]. It is based on the known fact that a linear bounded operator \( T \) on a Banach space is invertible if
\[
\|I - T\| < 1.
\]
Theorem 3.1 (Paley-Wiener). Let \( \{x_n\}_{n \in \mathbb{N}} \) be a basis in the Banach space \( X \) and let suppose that \( \{y_n\}_{n \in \mathbb{N}} \) is a sequence of elements of \( X \) such that

\[
\left\| \sum_{n=1}^{N} c_n (x_n - y_n) \right\| \leq \lambda \left\| \sum_{n=1}^{N} c_n x_n \right\|
\]

for all \( N \in \mathbb{N} \), some constant \( \lambda \), with \( 0 \leq \lambda < 1 \) and for any sequence of scalars \( \{c_n\}_{n \in \mathbb{N}} \). Then \( \{y_n\}_{n \in \mathbb{N}} \) is a basis for \( X \) equivalent to \( \{x_n\}_{n \in \mathbb{N}} \).

See [8, Teorema 10] for a proof.

Let \( H \) be a separable Hilbert space and \( \{e_n\}_{n \in \mathbb{N}} \) be an arbitrary but fixed orthonormal basis. The structure of Hilbert spaces allows to reformulate the Paley-Wiener theorem in the following way.

Theorem 3.2. Let \( \{e_n\}_{n \in \mathbb{N}} \) be an orthonormal basis for a Hilbert space \( H \) and let \( \{z_n\}_{n \in \mathbb{N}} \subset H \) be a sequence close to \( \{e_n\}_{n \in \mathbb{N}} \) in the sense that

\[
\left\| \sum_{i} c_i (e_i - z_i) \right\| \leq \lambda \left( \sum_{i} |c_i|^2 \right)^{1/2}
\]

for some constant \( \lambda \), with \( 0 \leq \lambda < 1 \) and for every sequence of scalars \( c_1, c_2, ... c_n \ (n = 1, 2, 3, ...) \). Then \( \{z_n\}_{n \in \mathbb{N}} \) is a Riesz basis for \( H \).

4 Kolmogorov and Naimark decomposition theorems

4.1 The Hilbert space associated to a positive definite operator valued kernel

Let \( \{\mathcal{H}_n\}_{n \in \mathbb{Z}} \) be a family of Hilbert spaces. An operator valued kernel on \( \mathbb{Z} \) to \( \{\mathcal{H}_n\}_{n \in \mathbb{Z}} \) is an application \( K : \mathbb{Z} \times \mathbb{Z} \to \bigcup_{m,n \in \mathbb{Z}} \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n) \) such that \( K(n, m) \in \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n) \) for \( n, m \in \mathbb{Z} \).

In this section and the following one, unless it is otherwise stated, all the kernels will be operator valued ones.

A sequence \( \{h_n\} \) in \( \oplus_{n \in \mathbb{Z}} \mathcal{H}_n \) is said to have finite support if \( h_n = 0 \) except for a finite numbers of integers \( n \).

A kernel \( K \) on \( \mathbb{Z} \) to \( \{\mathcal{H}_n\}_{n \in \mathbb{Z}} \) is a positive definite kernel if

\[
\sum_{n,m \in \mathbb{Z}} \langle K(n, m) h_m, h_n \rangle_{\mathcal{H}_n} \geq 0,
\]

for every sequence \( \{h_n\} \) in \( \oplus_{n \in \mathbb{Z}} \mathcal{H}_n \) with finite support.
Let $K$ be a positive definite kernel. Let $\mathcal{F}$ be the linear space of elements $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$ and $\mathcal{F}_o$ the space of elements $\mathcal{F}$ with finite support.

Define $B_K : \mathcal{F}_o \times \mathcal{F}_o \to \mathbb{C}$ with

$$B_K(f, g) = \sum_{m, n \in \mathbb{Z}} \langle K(n, m)f_m, g_n \rangle_{\mathcal{H}_n}, \tag{2}$$

for $f, g \in \mathcal{F}_o$, $f = \{f_n\}$, $g = \{g_n\}$, $f_m, g_n \in \mathcal{H}_n$.

Note that $B_K$ satisfies all the properties of an inner product, except for the fact that the set

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h, h) = 0\}$$

could be non-trivial.

According to the Cauchy-Schwarz inequality

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h, g) = 0, \text{ for all } g \in \mathcal{F}_o\},$$

hence $\mathcal{N}_K$ is a linear subspace of $\mathcal{F}_o$.

The quotient space $\mathcal{F}_o/\mathcal{N}_K$ is also a linear subspace. If $[h]$ stands for the class in $\mathcal{F}_o/\mathcal{N}_K$ of the element $h$ then the application

$$\langle [h], [g] \rangle = B_K(h, g), \quad h, g \in \mathcal{F}_o$$

is well defined. To prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{F}_o/\mathcal{N}_K$ is straightforward.

The completion of $\mathcal{F}_o/\mathcal{N}_K$ with respect to the norm induced by this inner product is a Hilbert space. It is known as the Hilbert space associated to the positive definite kernel $K$ and it is denoted by $\mathcal{H}_K$. The inner product and the norm of $\mathcal{H}_K$ will be represented as $\langle \cdot, \cdot \rangle_{\mathcal{H}_K}$ and $\|\cdot\|_{\mathcal{H}_K}$ respectively. This norm will be named as the norm induced by $K$.

### 4.2 Kolmogorov Decomposition Theorem

The following theorem is a version of the classic result of Kolmogorov (See [3] for a historical review).

**Theorem 4.1** (Kolmogorov). Let $\{\mathcal{H}_n\}_{n \in \mathbb{Z}}$ be a family of Hilbert spaces and let $K : \mathbb{Z} \times \mathbb{Z} \to \bigcup_{m, n \in \mathbb{Z}} L(\mathcal{H}_m, \mathcal{H}_n)$ be a positive definite kernel. Then there exists an application $V$ defined on $\mathbb{Z}$ such that $V(n) \in L(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ and

(a) $K(n, m) = V^*(n)V(m)$ if $n, m \in \mathbb{Z}$.

(b) $\mathcal{H}_K = \bigvee_{n \in \mathbb{Z}} V(n)\mathcal{H}_n$. 

(c) The decomposition is unique in the following sense: if $\mathcal{H}'$ is another Hilbert space and $V'$ defined on $\mathbb{Z}$ is an application such that $V'(n) \in L(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ that satisfy (a) and (b), then there exists an unitary operator $\Phi : \mathcal{H}_K \to \mathcal{H}'$ such that $\Phi V(n) = V'(n)$ for all $n \in \mathbb{Z}$.

A proof of this theorem can be found in [2, Teorema 3.1].

An application $V$ that satisfies the property (a) in the former theorem is called The Kolmogorov Decomposition of the Kernel $K$ or simply, a Decomposition of the kernel $K$ (see [2]). The property (b) is known as minimality condition of Kolmogorov Decomposition. The meaning of property (c) is that, given the minimality condition (b) the Kolmogorov Decomposition is essentially unique.

### 4.3 Naimark Theorem

A particular case is that when the family $\{\mathcal{H}_n\}_{n \in \mathbb{Z}}$ is one single operator, that is, when $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$. An operator valued kernel is an application $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$. An important instance is the so-called Toeplitz kernels.

A kernel $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ is an operator valued Toeplitz kernel if

$$K(n, m) = W(m - n) \quad \text{for all} \quad n, m \in \mathbb{Z}$$

for some application $W : \mathbb{Z} \to \mathcal{L}(\mathcal{H})$.

**Theorem 4.2** (Naimark). Let $\mathcal{H}$ be a Hilbert space and let $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be a positive definite Toeplitz kernel. There exists an unitary operator $S$ in $\mathcal{L}(\mathcal{H}_K)$ and an operator $Q$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ such that

(a) $K(n, m) = Q^* S^{m-n} Q$, $n, m \in \mathbb{Z}$.

(b) $\mathcal{H}_K = \bigvee_{n \in \mathbb{Z}} S^n Q \mathcal{H}$.

(c) The decomposition is unique in the following sense: if $\mathcal{H}'$ is a Hilbert space such that there exists an unitary operator $S'$ in $\mathcal{L}(\mathcal{H}')$ and an operator $Q'$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ satisfying (a) and (b) (changing $\mathcal{H}, S$ and $Q$ for $\mathcal{H}', S'$ and $Q'$ respectively), then there is an unitary operator $\Phi : \mathcal{H}_K \to \mathcal{H}'$ such that $\Phi Q h = Q' h$ for all $h \in \mathcal{H}$ and $S' \Phi = \Phi S$.

A proof of this Theorem can be found in [2, Teorema 3.2].

The operator $S$ is usually known as Naimark Dilation of the considered positive definite Toeplitz kernel or shift in $\mathcal{H}_K$. 
5 Multivariate Stochastic Processes

In this section it will be used the decomposition of the covariance Kernels of the stochastic processes (see [2] section 1 Chapter 6).

Let \((\Omega, F, P)\) be a probability space, where \(F\) is a \(\sigma\)-algebra of subsets of \(\Omega\) and \(P\) is a probability measure on \(F\). A stochastic variable is a measurable function \(x: \Omega \rightarrow \mathbb{C}\).

A stochastic process is a family \(\{x_n\}_{n \in \mathbb{Z}}\) of stochastic variables. Let \(L^2(P)\) be the Hilbert space of the measurable functions from \(F\) to \(\Omega\) with integrable square, this is,

\[
L^2(P) = \left\{ x: \Omega \rightarrow \mathbb{C} : x \text{ is a measurable function } \int_{\Omega} |x(\omega)|^2 dP(\omega) < +\infty \right\}
\]
equipped with the inner product

\[
\langle x, y \rangle_{L^2(P)} = \int_{\Omega} x(\omega)\overline{y(\omega)}dP(\omega).
\]

From here on, only stochastic processes with variables in \(L^2(P)\) will be considered.

The mean-value variable is defined by

\[
m_n = E(x_n) = \int_{\Omega} x_n(\omega)dP(\omega)
\]
and it is convenient to assume that \(m_n = 0\) for all \(n \in \mathbb{Z}\). The correlation of the stochastic process \(\{x_n\}_{n \in \mathbb{Z}}\) is given by

\[
K(m, n) = K_{mn} = \int_{\Omega} x_n(\omega)x_m(\omega)dP(\omega) = \langle x_n, x_m \rangle_{L^2(P)}
\]
for all \(m, n \in \mathbb{Z}\).

It is straightforward that the correlation kernel of this process is a positive definite kernel. In fact

\[
\sum_{i,j=m}^{n} K_{ij} \lambda_j \lambda_i = \sum_{i,j=m}^{n} \langle x_j, x_i \rangle_{L^2(P)} \lambda_j \lambda_i = \sum_{i,j=m}^{n} \langle \lambda_j x_j, \lambda_i x_i \rangle_{L^2(P)} = \left\| \sum_{j=m}^{n} \lambda_j x_j \right\|_{L^2(P)}^2 \geq 0
\]
for all \(m, n \in \mathbb{Z}, m \leq n\), and \(\lambda_k \in \mathbb{C} \quad (k = m, m+1, \ldots, n)\).

A stochastic process \(\{x_n\}_{n \in \mathbb{Z}}\) is said to be stationary (in a wide sense) if its correlation kernel is a Toeplitz kernel, that is

\[
K(m, n) = K_{n-m} \quad \text{for all} \quad m, n \in \mathbb{Z}
\]
In this case it can be used the Naimark Decomposition Theorem in order to associate the stationary stochastic process \( \{x_n\}_{n \in \mathbb{Z}} \) with the Hilbert space \( \mathcal{H}_K \), the unitary operator \( S \in L(\mathcal{H}_K) \) and the operator \( Q \in L(\mathcal{C}, \mathcal{H}_K) \) such that

\[
K_n = Q^* S^n Q, \quad n \in \mathbb{Z}.
\]

The geometric settings for the prediction problem can be extended in order to deal with the multivariate case too. Let notice that a random variable \( x_n : \Omega \to \mathbb{C} \), of a stochastic process \( \{x_n\}_{n \in \mathbb{Z}} \subset L^2(\mathcal{P}) \), can be interpreted as an operator from \( \mathcal{C} \) to \( L^2(\mathcal{P}) \) defining \( \bar{x}_n : \mathcal{C} \to L^2(\mathcal{P}) \) as

\[
\bar{x}_n(\lambda) = \lambda x_n
\]

and the elements of the correlation kernel of the process can be calculated according to the rule

\[
K(m, n) = (\bar{x}_m)^* \bar{x}_n.
\]

Also it must be noticed that many stochastic processes have the same correlation kernel. Having this in mind it is convenient to adopt the following terminology. The main object used to describe a multivariate process will be its correlation kernel \( K \) which is supposed to be positive definite and \( K(m, n) \in L(\mathcal{H}_n, \mathcal{H}_m) \) for all \( m, n \in \mathbb{Z} \), where \( \mathcal{H} = \{\mathcal{H}_n\}_{n \in \mathbb{Z}} \) is a family of Hilbert spaces.

**Definition 5.1.** A pair \([\mathcal{K}, \mathcal{X}]\), where \( \mathcal{K} \) is a Hilbert space and \( \mathcal{X} = \{X_n\}_{n \in \mathbb{Z}} \) is a family of operators \( X_n \) in \( L(\mathcal{H}_n, \mathcal{K}) \), is called a geometric model of the multivariate process with correlation kernel \( K \), if

\[
K(m, n) = X^*_m X_n.
\]

The Kolmogorov Decomposition Theorem shows that given a positive definite kernel \( K \), there exists a geometric model of the multivariate process with correlation kernel \( K \). If \([\mathcal{K}, \mathcal{X}]\) is the geometric model of the multivariate process with covariance kernel \( K \) then \( \mathcal{H}_X \) will be the subspace of \( \mathcal{K} \) generated for this model, that is,

\[
\mathcal{H}_X = \bigvee_{n \in \mathbb{Z}} X_n \mathcal{H}_n. \quad (3)
\]

If \([\mathcal{K}', \mathcal{X}']\) is another geometric model of the same process, then the Kolmogorov Decomposition Theorem guarantees the existence of an unitary operator \( \Phi : \mathcal{H}_X \to \mathcal{H}_X' \) such that \( \Phi X_n = X'_n \) for all \( n \in \mathbb{Z} \). This means that the geometry of the process is essentially determined by the choice of a geometric model such that

\[
\mathcal{K} = \bigvee_{n \in \mathbb{Z}} X_n \mathcal{H}_n. \quad (4)
\]
5.1 Equivalent Multivariate Stochastic Processes

From here on, \( H_n = H \) for all \( n \in \mathbb{Z} \) and the covariance kernels of the processes will be positive definite.

**Definition 5.2.** Two geometric models of multivariate processes \([K, X]\) and \([L, Y]\) are said to be equivalent, if \( \dim(H_X) = \dim(H_Y) \) and there are two constants \( A, B \) with \( 0 < A \leq B \) such that

\[
A \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{H_X}^2 \leq \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{H_Y}^2 \leq B \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{H_X}^2,
\]

where \( \{h_n\}_{n \in \mathbb{Z}} \) is a sequence in \( H \) with finite support.

The following theorem is the first of our results. It is similar to a theorem about equivalent basic sequences in Banach spaces (see [1, 4]) and generalizes a result of Strandell in [7] for approximately weakly stationary scalar valued stochastic processes.

**Theorem 5.3.** Let \([K, X]\) and \([L, Y]\) two geometric models of multivariate processes. The following conditions are equivalent:

(i) The models of the multivariate processes \([K, X]\) and \([L, Y]\) are equivalent.

(ii) There is a bijective bounded linear application with bounded inverse

\[\psi : H_X \rightarrow H_Y\]

such that

\[\psi X_n = Y_n \quad \text{for all } n \in \mathbb{Z}.
\]

(iii) There exists two contants \( A, B \) with \( 0 < A \leq B \) such that

\[
A \sum_{n, m \in \mathbb{Z}} \langle X_n^* X_m h_m, h_n \rangle_H \leq \sum_{n, m \in \mathbb{Z}} \langle Y_n^* Y_m h_m, h_n \rangle_H \leq B \sum_{n, m \in \mathbb{Z}} \langle X_n^* X_m h_m, h_n \rangle_H,
\]

for each sequence with finite support \( \{h_n\}_{n \in \mathbb{Z}} \subset H \).

**Proof 5.4.** The equivalence of (i) and (iii) follows immediately from definitions. Suppose (ii) holds. Since \( \Phi \) is a bijective bounded linear application with bounded inverse, there are two constants \( a_o, b_o \) with \( 0 < a_o \leq b_o \) such that

\[a_o \|f\|_{H_X} \leq \|\Phi(f)\|_{H_Y} \leq b_o \|f\|_{H_X},
\]
for every \( f \in \mathcal{H}_X \).

Let \( f \in \mathcal{H}_X \) be given by

\[
f = \sum_{n \in \mathbb{Z}} X_n h_n,\]

where \( \{h_n\}_{n \in \mathbb{Z}} \) is a sequence in \( \mathcal{H} \) with finite support.

Then

\[
a_o^2 \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_X}^2 \leq \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2 \leq b_o^2 \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_X}^2.
\]

In the other hand

\[
\left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_X}^2 = \sum_{n,m \in \mathbb{Z}} \langle X^*_n X_m h_m, h_n \rangle_{\mathcal{H}},
\]

similarly

\[
\left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2 = \sum_{n,m \in \mathbb{Z}} \langle Y^*_n Y_m h_m, h_n \rangle_{\mathcal{H}}.
\]

So, taking \( A = a_o^2 \) and \( B = b_o^2 \) we have the existence of the two constants \( A, B \) with \( 0 < A \leq B \) such that

\[
A \sum_{n,m \in \mathbb{Z}} \langle X^*_n X_m h_m, h_n \rangle_{\mathcal{H}} \leq \sum_{n,m \in \mathbb{Z}} \langle Y^*_n Y_m h_m, h_n \rangle_{\mathcal{H}} \leq B \sum_{n,m \in \mathbb{Z}} \langle X^*_n X_m h_m, h_n \rangle_{\mathcal{H}},
\]

for every sequence \( \{h_n\}_{n \in \mathbb{Z}} \subset \mathcal{H} \) with finite support.

Now, suppose condition (iii) holds.

Define the application \( \Phi_o : \mathcal{F}_{o,X} \rightarrow \mathcal{F}_{o,Y} \) with

\[
\Phi_o \left( \sum_{n \in \mathbb{Z}} X_n h_n \right) = \sum_{n \in \mathbb{Z}} Y_n h_n,
\]

where \( \{h_n\}_{n \in \mathbb{Z}} \) is a sequence in \( \mathcal{H} \) with finite support. It is easy to prove that \( \Phi_o \) is a linear operator.

We prove now that \( \Phi_o \) is a bounded operator and also that it is bounded from below.

\[
\sum_{n,m \in \mathbb{Z}} \langle Y^*_n Y_m h_m, h_n \rangle_{\mathcal{H}} = \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2.
\]
taking this and the definition of $\Phi_o$ in account we have

$$
\sum_{m,n \in \mathbb{Z}} \langle Y_n^* Y_m h_m, h_n \rangle_H = \left( \sum_{m \in \mathbb{Z}} Y_m h_m, \sum_{n \in \mathbb{Z}} Y_n h_n \right)_{\mathcal{H}_Y} = \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2
$$

$$
= \left\| \Phi_o \left( \sum_{n \in \mathbb{Z}} X_n h_n \right) \right\|_{\mathcal{H}_Y}^2.
$$

Similarly it follows that

$$
\sum_{n,m \in \mathbb{Z}} \langle X_n^* X_m h_m, h_n \rangle_H = \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_X}^2.
$$

Replacing these expressions in the inequelities (iii)

$$
A \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_X}^2 \leq \left\| \Phi_o \left( \sum_{n \in \mathbb{Z}} X_n h_n \right) \right\|_{\mathcal{H}_Y}^2 \leq B \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_{K_1}}^2. \quad (5)
$$

This shows that $\Phi_o$ is a bounded operator bounded from below. Furthermore the dominium and the range of $\Phi_o$ are dense in $\mathcal{H}_X$ and $\mathcal{H}_Y$ respectively. Then this operator can be extended to a bounded one with bounded inverse $\Phi : \mathcal{H}_X \to \mathcal{H}_Y$.

By construction

$$
\Phi X_n = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.
$$

In the multivariate stochastic processes setting it is posible to obtain a result similar to that of the theorem on stability (see Theorem 3.1).

The following is our result about stability of multivariate stochastic processes.

**Theorem 5.5.** Let $[\mathcal{W}, Y]$ be a geometrical model of a multivariate stochastic process, $\mathcal{H}_Y$ the subspace generated by the process, and suppose $X_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}_Y)$ for all $n \in \mathbb{Z}$ such that

$$
\left\| \sum_{n \in \mathbb{Z}} (Y_n - X_n) h_n \right\|_{\mathcal{H}_Y} \leq \delta \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}, \quad (6)
$$

for some constant $\delta$, $0 < \delta < 1$, and any sequence $\{h_n\}_{n \in \mathbb{Z}}$ in $\mathcal{H}$ with fi-
nite support. Then the geometric model of the multivariate process $[\mathcal{K}, X]$ is equivalent to $[\mathcal{W}, Y]$. 

\textbf{Proof 5.6.} Define the mapping \( T : \mathcal{H}_Y \to \mathcal{H}_Y \) by setting

\[
T \left( \sum_{n \in \mathbb{Z}} Y_n h_n \right) = \sum_{n \in \mathbb{Z}} (Y_n - X_n) h_n,
\]

with \( \{h_n\}_{n \in \mathbb{Z}} \) a sequence in \( \mathcal{H} \) with finite support. According to the hypothesis \( T \) is well defined and it is a linear operator.

From the definition of \( T \) and the hypothesis it follows that

\[
\left\| T \left( \sum_{n \in \mathbb{Z}} Y_n h_n \right) \right\|_{\mathcal{H}_Y}^2 \leq \delta^2 \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2.
\]

Then, \( T \) is bounded and

\[
\|T\| \leq |\delta| < 1.
\]

Consider the operator \( I - T : \mathcal{H}_Y \to \mathcal{H}_Y \), where as usual, \( I : \mathcal{H}_Y \to \mathcal{H}_Y \) is the identity operator. Since \( \|T\| < 1 \), \( I - T \) is invertible and

\[
(I - T) \left( \sum_{n \in \mathbb{Z}} Y_n h_n \right) = \sum_{n \in \mathbb{Z}} Y_n h_n - T \left( \sum_{n \in \mathbb{Z}} Y_n h_n \right)
= \sum_{n \in \mathbb{Z}} Y_n h_n - \left( \sum_{n \in \mathbb{Z}} (Y_n - X_n) h_n \right) = \sum_{n \in \mathbb{Z}} X_n h_n.
\]

From here it follows that there are two positive constants \( m \) and \( M \), with \( m \leq M \) such that

\[
m \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y} \leq \left\| (I - T) \left( \sum_{n \in \mathbb{Z}} Y_n h_n \right) \right\|_{\mathcal{H}_Y} = \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_Y}
\leq M \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}.
\]

On the other side

\[
\left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}^2 = \sum_{n,m \in \mathbb{Z}} \langle Y_n^* Y_m h_n, h_n \rangle_{\mathcal{H}}.
\]

And by hypothesis \( X_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}_Y) \) for all \( n \in \mathbb{Z} \), so \( X_n h_n \in \mathcal{H}_Y \). Then

\[
\sum_{n,m \in \mathbb{Z}} \langle X_n^* X_m h_m, h_n \rangle_{\mathcal{H}} = \sum_{m,n \in \mathbb{Z}} \langle X_m h_m, X_n h_n \rangle_{\mathcal{H}_Y} = \left\| \sum_{n \in \mathbb{Z}} X_n h_n \right\|_{\mathcal{H}_Y}^2.
\]
Replacing these expressions in the inequalities we obtain the existence of positive constants $A$ and $B$ with $A \leq B$ such that

$$A \sum_{m,n \in \mathbb{Z}} \langle Y^*_m h_m h_n, h_n \rangle_{\mathcal{H}} \leq \sum_{m,n \in \mathbb{Z}} \langle X^*_n X_m h_m h_n, h_n \rangle_{\mathcal{H}} \leq B \sum_{m,n \in \mathbb{Z}} \langle Y^*_m h_m h_n, h_n \rangle_{\mathcal{H}},$$

for every sequence $\{h_n\}_{n \in \mathbb{Z}}$ in $\mathcal{H}$ with finite support.

Applying the theorem 5.3 it follows that the geometric model of the multivariate process $[K, X]$ is equivalent to $[W, Y]$.

### 5.2 Approximately Stationary Multivariate Stochastic Processes

The following are original results that generalize the theorem of Strandell in [7].

**Definition 5.7.** A geometric model of a multivariate process $[K, X]$ is said to be approximately stationary if there exist a geometric model of a multivariate stationary process $[L, Y]$ such that $[K, X]$ and $[L, Y]$ are equivalent.

**Corollary 5.8.** Let $[K, X]$ be a geometric model of a multivariate process. Then the following conditions are equivalent:

(i) The geometric model of the multivariate process $[K, X]$ is approximately stationary.

(ii) There exist a geometric model of a multivariate stationary process $[L, Y]$ and a bijective bounded linear application with bounded inverse, $\psi : \mathcal{H}_Y \to \mathcal{H}_X$ such that $\psi Y_n = X_n$ for all $n \in \mathbb{Z}$.

(iii) There exist a definite positive sequence $W : \mathbb{Z} \to L(\mathcal{H})$ and two constants $A, B$ con $0 < A \leq B$ such that

$$A \sum_{n,m \in \mathbb{Z}} \langle W(m - n) h_m, h_n \rangle_{\mathcal{H}} \leq \sum_{n,m \in \mathbb{Z}} \langle X^*_n X_m h_m, h_n \rangle_{\mathcal{H}} \leq B \sum_{n,m \in \mathbb{Z}} \langle W(m - n) h_m, h_n \rangle_{\mathcal{H}}.$$

where $\{h_n\}_{n \in \mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.

**Proof 5.9.** The result follows from Theorem 5.3.
Let $[K, X]$ be a geometric model of a multivariate process. If $[K, X]$ is close -in some sense- to a geometric model of a multivariate stationary process $[L, Y]$. Under which conditions is $[K, X]$ approximately stationary?

The answer to the former question is the following result and it is an extension of the Paley- Wiener Stability Theorem.

**Corollary 5.10.** Let $[W, Y]$ be a geometrical model of a multivariate stationary process with correlation kernel $G$ defined for $G(n, m) = T(m - n)$ for all $m, n \in \mathbb{Z}$ and for a certain map $T$ from $\mathbb{Z}$ to $L(H)$. If $X_n \in L(H, H_Y)$ for all $n \in \mathbb{Z}$ such that

$$\left\| \sum_{n \in \mathbb{Z}} (Y_n - X_n) h_n \right\|_{H_Y}^2 \leq \delta^2 \sum_{m, n \in \mathbb{Z}} \langle T(m - n) h_m, h_n \rangle_{H_Y}, \quad (7)$$

for some constant $\delta$, $0 < \delta < 1$, and all sequences $\{h_n\}_{n \in \mathbb{Z}}$ in $H$ with finite support. Then $[K, X]$ is approximately stationary.

**Proof 5.11.** By the definition of $G$ follows that

$$\sum_{m, n \in \mathbb{Z}} \langle T(m - n) h_m, h_n \rangle_{H_Y} = \delta^2 \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{H_Y}^2.$$

Hence

$$\left\| \sum_{n \in \mathbb{Z}} (Y_n - X_n) h_n \right\|_{H_Y} \leq \delta \left\| \sum_{n \in \mathbb{Z}} Y_n h_n \right\|_{H_Y},$$

for some constant $\delta$, $0 < \delta < 1$, and all sequences $\{h_n\}_{n \in \mathbb{Z}}$ in $H$ with finite support.

Then by the theorem 5.5 follows that $[K, X], [W, Y]$ are equivalent. And since $[W, Y]$ is stationary, $[K, X]$ must be approximately stationary.

Strandell gave a similar result for scalar valued approximately stationary stochastic processes (ver[7], Theorem 2 page 19).

**Acknowledgements.** Thanks to the Departaments of Mathematics and vice-rectorías de investigacion of the Universidad de Pamplona, Unverisad de Sucre and Universidad del Atlántico.
References


Received: October 12, 2016; Published: January 16, 2017