Some Refinement Operator Inequalities with Matrix Means

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Abstract

In this paper, we refine some operator inequalities with operator means as follows:

Let $A$, $B$ be positive operators on a Hilbert space with $0 < m \leq A \leq m' < M' \leq B \leq M$ and $\sigma$, $\tau$ two arbitrary means between geometric and arithmetic means. Then for every positive unital linear map $\Phi$ and $p \geq 2$,

\[
\Phi^p(A\sigma B) \leq \left( \frac{(M + m)^2}{4^\frac{2}{p} Mm K^\frac{1}{h'}} \right)^p \Phi^p(A\tau B),
\]

and

\[
(\Phi(A)\sigma\Phi(B))^p \leq \left( \frac{(M + m)^2}{4^\frac{2}{p} Mm K^\frac{1}{h'}} \right)^p \Phi^p(A\tau B),
\]

where $K(h') = \frac{(h'+1)^2}{4h'}$ with $h' = \frac{M'}{m}$.

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1. Introduction

Let \( \sigma, \tau \) be two arbitrary means. Taking an axiomatic approach, Kubo and Ando [1] introduced the notions of connection and mean. A binary operation \( \sigma \) defined on the set of positive operators is called a connection if

(i) \( A \leq C, B \leq D \) imply \( A \sigma B \leq C \sigma D \);
(ii) \( C^*(A \sigma B)C \leq (C^* AC) \sigma (C^* BC) \);
(iii) \( A_n \downarrow A \) and \( B_n \downarrow B \) imply \( A_n \sigma B_n \downarrow A \sigma B \).

If \( I \sigma I = I \), then \( \sigma \) is called a mean.

We use the same notation as in [2,3]. Let \( m, m', M, M' \) be scalars and \( I \) be the identity operator. Other capital letters are used to denote the general elements of the \( C^* \)-algebra \( B(\mathcal{H}) \) of all bounded linear operators acting on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). We write \( A \geq 0 \) to mean that the operator \( A \) is positive. If \( A - B \geq 0 (A - B \leq 0) \), then we say that \( A \geq B (A \leq B) \). A linear map \( \Phi \) is positive if \( \Phi(A) \geq 0 \) whenever \( A \geq 0 \). It’s said to be unital if \( \Phi(I) = I \).

For \( A, B > 0 \), the geometric mean \( A \# B \) is defined by

\[
A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}
\]

The harmonic and arithmetic mean are defined \( A!B = (A^{-1} + B^{-1})^{-1} \) and \( A\triangledown B = \frac{A+B}{2} \), respectively. A mean \( \sigma \) is called to be symmetric if \( A \sigma B = B \sigma A \) for any pair of positive definite matrices \( A, B \).

According to the general theory of operator means [1] the arithmetic mean \( \triangledown \) is the biggest and the harmonic mean \( ! \) is the smallest (in the sense of partial order) among symmetric means. Hence, if \( \sigma, \tau \) are two arbitrary means between harmonic and arithmetic means, we know that \( \triangledown \geq \sigma \) and \( \tau \geq ! \).

It is well known that for two positive operator \( A, B \),

\[
A \geq B \Rightarrow A^p \geq B^p \quad \text{for} \quad 0 \leq p \leq 1,
\]

but

\[
A \geq B \nRightarrow A^p \geq B^p \quad \text{for} \quad p > 1.
\]

Let \( 0 < m \leq A \leq M \) with the scalars \( m, M > 0 \) and \( \sigma, \tau \) two arbitrary means between harmonic and arithmetic means. Hoa et al. [4] proved the following inequalities for every positive unital linear map \( \Phi \):

\[
\Phi^2(A \sigma B) \leq K^2(h)\Phi^2(A \tau B),
\]

\[
\Phi^2(A \sigma B) \leq K^2(h)(\Phi(A) \tau \Phi(B))^2,
\]

\[
(\Phi(A) \sigma \Phi(B))^2 \leq K^2(h)\Phi^2(A \tau B),
\]

and

\[
(\Phi(A) \sigma \Phi(B))^2 \leq K^2(h)(\Phi(A) \tau \Phi(B))^2
\]

where \( K(h) = \frac{(h+1)^2}{4h} \) with \( h = \frac{M}{m} \) is called the Kantorovich constant.
When $\sigma = \nabla$ in (1.2) and $\tau = \sharp$ in (1.3), the inequalities (1.2) and (1.3) are reverse AM-GM inequalities in [5]. So, the inequalities (1.2) and (1.3) are generalizations of Lin’s results.

Combining (1.1) and (1.2)-(1.5), we know that the following inequalities hold for $0 \leq p \leq 2$

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(A\tau B),$$

(1.6)

$$\Phi^p(A\sigma B) \leq K^p(h)(\Phi(A)\tau\Phi(B))^p,$$

(1.7)

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(A\tau B),$$

(1.8)

and

$$(\Phi(A)\sigma\Phi(B))^p \leq K^p(h)(\Phi(A)\tau\Phi(B))^p$$

(1.9)

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is so called the Kantorovich constant.

Are the inequalities (1.6)-(1.9) true for $p > 2$? Fu in [6, Theorem 4] show the following inequalities for $\sigma, \tau$ are two arbitrary means between harmonic and arithmetic means and $p \geq 2$

$$\Phi^p(A\sigma B) \leq \left(\frac{(M+m)^2}{4^p Mm}\right)^p\Phi^p(A\tau B),$$

(1.10)

$$\Phi^p(A\sigma B) \leq \left(\frac{(M+m)^2}{4^p Mm}\right)^p(\Phi(A)\tau\Phi(B))^p,$$

(1.11)

$$\Phi^p(A\sigma B) \leq \left(\frac{(M+m)^2}{4^p Mm}\right)^p\Phi^p(A\tau B),$$

(1.12)

and

$$(\Phi(A)\sigma\Phi(B))^p \leq \left(\frac{(M+m)^2}{4^p Mm}\right)^p(\Phi(A)\tau\Phi(B))^p$$

(1.13)

Marshall and Olkin [7] proved the following operator Kantorovich inequality:

$$\Phi(A^{-1}) \leq K(h)\Phi(A)^{-1}$$

(1.14)

Lin [3, Theorem 2.8] showed that the operator inequality (1.14) can be squared:

$$\Phi^2(A^{-1}) \leq K^2(h)\Phi(A)^{-2}$$

(1.15)

By (1.1) and (1.15) we know that

$$\Phi^p(A^{-1}) \leq K^p(h)\Phi(A)^{-p}$$

(1.16)

holds when $0 \leq p \leq 2$.

Fu [2, Theorem 3] also showed that

$$\Phi^p(A^{-1}) \leq \left(\frac{(M+m)^2}{4^p Mm}\right)^p\Phi(A)^{-p}$$

(1.17)

holds when $p \geq 2$.

The main purpose of this paper is to present some refinement results of the above inequalities for arbitrary matrix means $\sigma, \tau$ between geometric and arithmetic means.

2. Main results

We need some Lemmas to prove the main theorems of this paper:

**Lemma 2.1.** [8] Let $A, B > 0$. Then the following norm inequality holds:
\[ \|AB\| \leq \frac{1}{4}\|A + B\|^2. \]  

**Lemma 2.2.**[9] Let \( A \) and \( B \) be positive operators. Then for \( 1 \leq r < \infty \),
\[ \|A^r + B^r\| \leq \|(A + B)^r\|. \]  

**Lemma 2.3.**[10] Let \( \Phi \) be an unital positive linear map on \( M_n \). Then the positive definite matrix \( A \in M_n \)
\[ \Phi(A^{-1}) \geq \Phi^{-1}(A). \]  

**Lemma 2.4.**[11] Suppose that two operators \( A, B \) and positive real numbers \( m, m', M, M' \) satisfy either of the following conditions:
(i) \( 0 < m \leq A \leq m' < M' \leq B \leq M \)
(ii) \( 0 < m \leq B \leq m' < M' \leq A \leq M \)
Then
\[ A \nabla_t B \geq K(h') A_{2t} B, \]  
for all \( t \in [0, 1] \), where \( \mu = \min\{t, 1 - t\} \) and \( h' = \frac{M'}{m} \).

Now, we prove the first main result in the following theorem.

**Theorem 2.5.** Let \( A, B \) be positive operators on a Hilbert space with \( 0 < m \leq A \leq m' < M' \leq B \leq M \) and \( \sigma, \tau \) two arbitrary means between geometric and arithmetic means. Then for every positive unital linear map \( \Phi \) and \( p \geq 2 \),
\[ \Phi^p(A \sigma B) \leq \left( \frac{(M+m)^2}{4\pi MmK\frac{1}{2}(h')} \right)^p \Phi^p(A \tau B), \]  
\[ \Phi^p(A \sigma B) \leq \left( \frac{(M+m)^2}{4\pi MmK\frac{1}{2}(h')} \right)^p (\Phi(A) \tau \Phi(B))^p, \]  
\[ (\Phi(A) \sigma \Phi(B))^p \leq \left( \frac{(M+m)^2}{4\pi MmK\frac{1}{2}(h')} \right)^p \Phi^p(A \tau B), \]  
and
\[ (\Phi(A) \sigma \Phi(B))^p \leq \left( \frac{(M+m)^2}{4\pi MmK\frac{1}{2}(h')} \right)^p (\Phi(A) \tau \Phi(B))^p, \]  
where \( K(h') = (\frac{h'+1}{2h'})^2 \) with \( h' = \frac{M'}{m} \).

**Proof.** The inequality (2.5) is equivalent to
\[ \|\Phi^\frac{1}{2}(A \sigma B)\Phi^{-\frac{1}{2}}(A \tau B)\| \leq \left( \frac{(M+m)^2}{4\pi MmK\frac{1}{2}(h')} \right)^\frac{1}{2}, \]  
By \( \nabla \geq \sigma \) and \( \tau \geq \sharp \), we have
\[ \Phi(A \sigma B) + MmK\frac{1}{2}(h')\Phi^{-1}(A \tau B) \]
\[ \leq \Phi(A \nabla B) + MmK\frac{1}{2}(h')\Phi^{-1}(A_{2} B) \]
\[ \leq \Phi(A \nabla B) + MmK\frac{1}{2}(h')\Phi((A_{2} B)^{-1}) \quad (by (2.3)) \]
\[ \leq \Phi(A \nabla B) + Mm\Phi(A^{-1} \nabla B^{-1}) \quad (by (2.4)) \]
\[ \leq M + m \]
So, by (2.1), (2.2) and (2.10), we can compute
\[
\| \Phi^\frac{p}{2} (A\sigma B) M^\frac{p}{2} m^\frac{p}{2} K^\frac{p}{2} (h') \Phi^{-\frac{p}{2}} (A\tau B) \|
\leq \frac{1}{4} \| \Phi^\frac{p}{2} (A\sigma B) + M^\frac{p}{2} m^\frac{p}{2} K^\frac{p}{2} (h') \Phi^{-\frac{p}{2}} (A\tau B) \|^2
\leq \frac{1}{4} \| \Phi (A\sigma B) + M m K^\frac{1}{2} (h') \Phi^{-1} (A\tau B) \|^p
\leq \frac{1}{4} (M + m)^p
\]
which is equivalent to
\[
\| \Phi^\frac{p}{2} (A\sigma B) \Phi^{-\frac{p}{2}} (A\tau B) \| \leq (\frac{(M + m)^2}{4^\frac{p}{2} M m K^\frac{1}{2} (h')})^\frac{p}{2}
\]
Thus, (2.9) holds.

Also,
\[
\| (\Phi (A) \sigma \Phi (B))^\frac{p}{2} M^\frac{p}{2} m^\frac{p}{2} K^\frac{p}{2} (h') \Phi^{-\frac{p}{2}} (A\tau B) \|
\leq \frac{1}{4} \| (\Phi (A) \sigma \Phi (B))^\frac{p}{2} + M^\frac{p}{2} m^\frac{p}{2} K^\frac{p}{2} (h') \Phi^{-\frac{p}{2}} (A\tau B) \|^2
\leq \frac{1}{4} \| \Phi (A) \sigma \Phi (B) + M m K^\frac{1}{2} (h') \Phi^{-1} (A\tau B) \|^p
\leq \frac{1}{4} \| \Phi (A \triangledown B) + M m K^\frac{1}{2} (h') \Phi^{-1} (A \sharp B) \|^p
\]
\[
= \frac{1}{4} \| \Phi (A \triangledown B) + M m K^\frac{1}{2} (h') \Phi^{-1} (A \sharp B) \|^p
\leq \frac{1}{4} (M + m)^p
\]
which is equivalent to
\[
\| (\Phi (A) \sigma \Phi (B))^\frac{p}{2} \Phi^{-\frac{p}{2}} (A\tau B) \| \leq (\frac{(M + m)^2}{4^\frac{p}{2} M m K^\frac{1}{2} (h')})^\frac{p}{2}
\]
Thus, (2.7) holds.

By a similar argument, we can obtain the inequalities (2.6) and (2.8).

This completes the proof.

**Remark 2.6.** Since \( K(h') > 1 \), when \( \sigma, \tau \) are two arbitrary means between geometric and arithmetic means, (2.5)-(2.8) are refinements of (1.10)-(1.13), respectively. Moreover, if \( \sigma = \triangledown \) in (2.5) and \( \tau = \sharp \) in (2.6), the inequalities (2.5) and (2.6) are the refinement results in [2, Theorem 4].

**Remark 2.7.** As is mentioned above, we proved our main result for arbitrary means between geometric and arithmetic means. If the means do not lie between geometric and arithmetic means, the following example shows that the main theorem fails even for commutative case:

For example,
\[
f(t) = \frac{2t}{1 + t} (t \neq -1) \quad \text{and} \quad f(1) = 1.
\]
It is clear that
\[ f(t) = \frac{2t}{1+t} \leq \sqrt{t} \leq \frac{1+t}{2} \quad t \in [0, 1] \]
And the mean \( \sigma_f \) corresponding to the function \( f \) is harmonic mean, which does not lie between geometric and arithmetic means.

Let \( 0 < \frac{1}{6} I \leq A \leq \frac{141 - 36\sqrt{11}}{100} I < \frac{3}{4} I \leq B \leq \frac{5}{6} I \), such that \( K(h) = \frac{9}{8} \) and \( K(h') = \frac{30}{23} \). As mentioned above, we can show that
\[
(A \nabla B)^p \leq (4^{1-h} \frac{K(h)}{K'(h')})^p (A!B)^p \tag{2.11}
\]
is not valid even in the scalar case. In fact, since its scalar version is equivalent to
\[
\frac{1+t}{2} \leq 4^{1-\frac{2}{h'}} \frac{3}{2} \frac{2t}{1+t} \leq \frac{12t}{1+t}, \quad \left( \frac{K(h)}{K'(h')} = \frac{3}{2} \right) \tag{2.12}
\]
which does not hold for any \( t \in [0, 1] \). Indeed, (2.12) is equivalent to the following inequality
\[
t^2 - 22t + 1 \leq 0, \quad t \in [0, 1].
\]
The last inequality fails for any \( t \in [0, 11 - 2\sqrt{30}] \).

**Theorem 2.8** Assume the conditions as in Theorem 2.5. Then
\[
\Phi^p((A \tau B)^{-1}) \leq \left( \frac{(M+m)^2}{4\tilde{p} M m K^{1\frac{1}{h'}}(h')} \right)^p \Phi(\sigma A B)^{-p} \tag{2.13}
\]

**Proof.** The inequality (2.13) is equivalent to
\[
\left\| \Phi^{1\frac{1}{h'}}((A \tau B)^{-1}) \Phi(\sigma A B)^{1\frac{1}{h'}} \right\| \leq \left( \frac{(M+m)^2}{4\tilde{p} M m K^{1\frac{1}{h'}}(h')} \right)^p \tag{2.14}
\]
By \( \nabla \geq \sigma \) and \( \tau \geq \sharp \), we can compute
\[
\left\| \Phi^{1\frac{1}{h'}}((A \tau B)^{-1}) M^{1\frac{1}{h'}} m^{1\frac{1}{h'}} K^{1\frac{1}{h'}}(h') \Phi(\sigma A B)^{1\frac{1}{h'}} \right\|
\leq \frac{1}{4} \left\| \Phi^{1\frac{1}{h'}}(\sigma A B) + M^{1\frac{1}{h'}} m^{1\frac{1}{h'}} K^{1\frac{1}{h'}}(h') \Phi((A \tau B)^{-1})^{1\frac{1}{h'}} \right\|^2 \quad \text{(by (2.1))}
\leq \frac{1}{4} \left\| \Phi(\sigma A B) + M m K^{1\frac{1}{h'}}(h') \Phi((A \tau B)^{-1}) \right\|^p \quad \text{(by (2.2))}
\leq \frac{1}{4} \left\| \Phi(A \nabla B) + M m K^{1\frac{1}{h'}}(h') \Phi((A \nabla B)^{-1}) \right\|^p
\leq \frac{1}{4} \left\| \Phi(A \nabla B) + M m \Phi(A^{-1} \nabla B^{-1}) \right\|^p
\leq \frac{1}{4} (M + m)^p
\]
which is equivalent to
\[
\left\| \Phi^{1\frac{1}{h'}}((A \tau B)^{-1}) \Phi(\sigma A B)^{1\frac{1}{h'}} \right\| \leq \left( \frac{(M+m)^2}{4\tilde{p} M m K^{1\frac{1}{h'}}(h')} \right)^\frac{1}{h'}.
\]
Thus, (2.14) holds.
This completes the proof.
Remark 2.9. Let $B \downarrow A$ in Theorem 2.8, then $K(h') \downarrow 1$. By (2.13) we obtain the operator inequality (1.17). Specially, put $p = 2$, we obtain (1.15). Moreover, $p \geq 2$, by (1.1) we can obtain (1.14).

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References


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