Best Proximity Point Theorems for
Rational Type Proximal Contraction Maps
in Metric Spaces

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Abstract

The notion of rational type proximal contraction maps in metric spaces is introduced, and some best proximity point theorems for this class are established.

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1 Introduction and preliminaries

In the research of many branches of mathematics, mathematical sciences and economics, Banach’s contraction principle is a powerful tool. It is extended and generalized by many authors ([1, 2, 6, 8, 9, 11, 12, 23, 29] and reference therein).

Especially, the author of [5] obtained a fixed point result by introducing the concept of weak contractions.

Recently, the authors of [16] generalized Banach’s contraction principle as follows:
Theorem 1.1. Let \((X,d)\) be a complete metric space. Suppose that a map \(T : X \to X\) satisfies the following condition:

for all \(x, y \in X\),

\[
d(Tx, Ty) \leq \frac{d(x,Ty) + d(y,Tx)}{d(x,Tx) + d(y,Ty) + 1} d(x, y).
\]

Then, \(T\) has a fixed point, and for each \(x_0 \in X\), the Picard iteration \(\{x_n\}\) defined by

\[
x_{n+1} = Tx_n
\]

for all \(n \in \mathbb{N} \cup \{0\}\), converges to some fixed point. Moreover, if \(x_\ast\) and \(y_\ast\) are two distinct fixed points of \(T\), then \(d(x_\ast, y_\ast) \geq \frac{1}{2}\).

The author of [10] introduced the concept of best proximity point, which reduces fixed point when the underlying map is self map, and then best proximity point theorems for certain contractions are proved by many authors (for instance, [3, 7, 10, 14, 15, 17, 18, 22, 24, 25, 26, 27, 28] and reference therein).

A point \(x \in A\) is called best proximity point [10] of a map \(T : A \to B\), where \(A\) and \(B\) is nonempty subsets of a metric space \((X,d)\), if

\[
d(x, Tx) = d(A, B).
\]

Note that the grobal minimum of the real valued map \(x \to d(x, Tx)\) is attained from a best proximity point, because \(d(x, Tx) \geq d(A, B)\) for all \(x \in A\).

In this paper, we introduce the concept of rational type proximal contraction maps and prove the existence of a best proximity point for such maps in complete metric spaces.

Let \((X,d)\) be a metric space.

We denote by \(N(X)\) the family of nonempty subsets of \(X\) and by \(CL(X)\) the family of nonempty closed subsets of \(X\).

Let \(A, B \in N(X)\). We use the following notations:

\[
d(A, B) := \inf \{d(x, y) : x \in A \text{ and } y \in B\},
\]

\[
A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},
\]

\[
B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.
\]

Note that if \(A \cap B \neq \emptyset\), then \(A_0 \neq \emptyset\) and \(B_0 \neq \emptyset\).

\(B\) is called approximatively compact [26] with respect to \(A\) if every sequence \(\{y_n\}\) of points in \(B\) satisfying the condition that \(\lim_{n \to \infty} d(x, y_n) = d(x, B)\) for some \(x \in A\) has a convergent subsequence.

Remark 1.1. It is obvious that

(1) any compact set is approximatively compact with respect to it self;
(2) any subset of a metric spaces is approximatively compact with respect to it self;
(3) if $A$ is compact and $B$ is approximatively compact with respect to $A$, then $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$.

**Lemma 1.1.** [13] Let $(X,d)$ be a metric spaces, and let $A, B \in CL(X)$ with $A_0 \neq \emptyset$. If $B$ is approximatively compact with respect to $A$, then $A_0 \in CL(X)$.

## 2 Best proximity points

Let $(X,d)$ be a metric space, and let $A, B \in N(X)$.

A map $T : A \rightarrow B$ is called rational type proximal contraction of the first kind if there exists $L \geq 0$ such that, for all $x_1, x_2, u_1, u_2 \in A$ with $d(u_1, Tx_1) = d(A,B)$ and $d(u_2, Tx_2) = d(A,B)$,

$$d(u_1, u_2) \leq \frac{d(x_1, u_2) + d(x_2, u_1)}{d(x_1, u_1) + d(x_2, u_2) + 1}d(x_1, x_2) + L \min\{d(x_1, u_2), d(x_2, u_1)\}. \quad (2.1)$$

**Theorem 2.1.** Let $(X,d)$ be a complete metric space and $A, B \in CL(X)$ such that $A_0 \neq \emptyset$, and let $T : A \rightarrow B$ be a map. Suppose that the following conditions hold:

(1) $T$ is a rational type proximal contraction of the first kind;
(2) $T(A_0) \subset B_0$;
(3) either $T$ is continuous or $B$ is approximatively compact with respect to $A$.

Then, $T$ has a best proximity point. Moreover, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some best proximity point $x_*$, and the following estimates hold:

there exists a constant $k \in (0,1)$ such that

(i) for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_n, x_*) \leq \frac{k^n}{1-k}d(x_0, x_1); \quad (2.2)$$
(ii) for \( n \in \mathbb{N} \),

\[
d(x_n, x_*) \leq \frac{k}{1-k} d(x_{n-1}, x_n).
\]  

(2.3)

Furthermore, if \( x_* \) and \( y_* \) are two distinct best proximity points of \( T \), then

\[
d(x_*, y_*) \geq \frac{1}{2}(1 - L).
\]

Proof. Let \( x_0 \in A_0 \) be a point.

Then, from (2) there exists \( x_1 \in A_0 \) such that

\[
d(x_1, Tx_0) = d(A, B).
\]

Further, since \( Tx_1 \in T(A_0) \subset B_0 \), there exists \( x_2 \in A_0 \) such that

\[
d(x_2, Tx_1) = d(A, B).
\]

By continuing this process, we can find a sequence \( \{x_n\} \) in \( A_0 \) such that, for all \( n \in \mathbb{N} \),

\[
d(x_n, Tx_{n-1}) = d(A, B).
\]  

(2.4)

If there exists \( n_0 \in \mathbb{N} \) such that \( x_{n_0-1} = x_{n_0} \), then \( d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0-1}, Tx_{n_0-1}) \). Hence, the proof is finished.

Now, we assume that \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N} \).

Since \( T \) is rational type proximal contraction of the first kind, from (2.1) with

\[
u_1 = x_n, u_2 = x_{n+1}, x_1 = x_{n-1} \text{ and } x_2 = x_n,
\]

we have

\[
\begin{align*}
&d(x_n, x_{n+1}) \\
&\leq \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}d(x_{n-1}, x_n) + L \min \{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\
&\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}d(x_{n-1}, x_n) \\
&= \alpha_n d(x_{n-1}, x_n)
\end{align*}
\]  

(2.5)

for all \( n \in \mathbb{N} \), where

\[
\alpha_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}.
\]

Note that \( 0 < \alpha_{n+1} < \alpha_n < 1 \) for all \( n \in \mathbb{N} \). Let \( k \in (0, 1) \) be such that \( \alpha_1 \leq k \).

From (2.5) we have

\[
d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)
\]  

(2.6)
for all $n \in \mathbb{N}$.

Then, we have

\[
\begin{align*}
d(x_n, x_{n+1}) & \leq kd(x_{n-1}, x_n) \\
& \leq k^2d(x_{n-2}, x_{n-1}) \\
& \quad \cdots \cdots \\
& \leq k^nd(x_0, x_1).
\end{align*}
\]

For $m > n$, we obtain

\[
\begin{align*}
d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
& \leq (k^n + k^{n+1} + \cdots k^{m-1})d(x_0, x_1) \\
& \leq \frac{k^n}{1-k}d(x_0, x_1)
\end{align*}
\]

which implies \( \{x_n\} \) is a Cauchy sequence in \( A \).

Since \( A \in CL(X) \) and \((X, d)\) is complete, there exists \( x_* \in A \) such that

\[
\lim_{n \to \infty} x_n = x_*.
\]

If \( T \) is continuous, then from (2.4) we obtain

\[
d(x_*, Tx_*) = \lim_{n \to \infty} d(x_n, Tx_{n-1}) = d(A, B).
\]

Assume that \( B \) is approximatively compact with respect to \( A \).

By Lemma 1.1, \( A_0 \in CL(X) \). Since \( \{x_n\} \) is Cauchy sequence in \( A_0 \),

\[
\lim_{n \to \infty} x_n = \bar{x} \in A_0 \text{ exists}.
\]

Since \( T\bar{x} \in T(A_0) \subset B_0 \), there exists \( u \in A \) such that

\[
d(u, Tx) = d(A, B).
\]

From (2.1) we have

\[
\begin{align*}
d(x_{n+1}, u) & \leq \frac{d(x_n, u) + d(\bar{x}, x_{n+1})}{d(x_n, x_{n+1}) + d(\bar{x}, u) + 1}d(x_n, \bar{x}) + L \min\{d(x_n, u), d(\bar{x}, x_{n+1})\}.
\end{align*}
\]
Letting $n \to \infty$ in the above inequality, we have
\[ d(\overline{x}, u) \leq \frac{d(\overline{x}, u)}{d(\overline{x}, u) + 1} \cdot 0 + L \cdot 0 = 0 \]
which implies
\[ u = \overline{x}. \]

So from (2.8)
\[ d(\overline{x}, T\overline{x}) = d(A, B). \]

We now show that (2.2) and (2.3) are satisfied.

From (2.6) we have
\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \]
\[ \leq (k + k^2 + \cdots + k^p)d(x_{n-1}, x_n) \]
\[ \leq \frac{k(1 - k^p)}{1 - k}d(x_{n-1}, x_n). \quad (2.9) \]

Letting $p \to \infty$ in (2.10), we have
\[ d(x_n, x_*) \leq \frac{k}{1 - k}d(x_{n-1}, x_n) \]
for all $n \in \mathbb{N}$. Thus, (2.3) is satisfied.

From (2.7) and (2.9) we have
\[ d(x_n, x_{n+p}) \leq \frac{k(1 - k^p)}{1 - k}d(x_{n-1}, x_n) \]
\[ \leq \frac{k^n(1 - k^p)}{1 - k}d(x_0, x_1). \]

Letting $p \to \infty$ in above inequality, we have
\[ d(x_n, x_*) \leq \frac{k^n}{1 - k}d(x_0, x_1) \]
for all $n \in \mathbb{N} \cup \{0\}$. Hence, (2.2) is satisfied.

Suppose that $y_*$ is another best proximity point of $T$ such that $x_* \neq y_*$. From (2.1) we have
\[ d(x_*, y_*) \leq \frac{d(x_*, y_*) + d(y_*, x_*)}{d(x_*, x_*) + d(y_*, y_*) + 1}d(x_*, y_*) + L \min\{d(x_*, y_*), d(y_*, x_*)\} \]
\[ = 2\{d(x_*, y_*)\}^2 + Ld(x_*, y_*), \]
which implies $d(x_*, y_*) \geq \frac{1}{2}(1 - L)$. \(\square\)
Corollary 2.2. Let \((X, d)\) be a complete metric space and \(A, B \in CL(X)\) such that \(A_0 \neq \emptyset\), and let \(T : A \to B\) be a map. Suppose that the following conditions hold:

1. For all \(x_1, x_2, u_1, u_2 \in A\) with \(d(u_1, Tx_1) = d(A, B)\) and \(d(u_2, Tx_2) = d(A, B)\),
   \[
   d(u_1, u_2) \leq \frac{d(x_1, u_2) + d(x_2, u_1)}{d(x_1, u_1) + d(x_2, u_2) + 1} d(x_1, x_2);
   \]

2. \(T(A_0) \subset B_0\);

3. Either \(T\) is continuous or \(B\) is approximatively compact with respect to \(A\).

Then, \(T\) has a best proximity point. Further, if \(x_*\) and \(y_*\) are two distinct best proximity points of \(T\), then
   \[
   d(x_*, y_*) \geq \frac{1}{2}.
   \]

Corollary 2.3. Let \((X, d)\) be a complete metric space and \(A, B \in CL(X)\) such that \(A_0 \neq \emptyset\), and let \(T : A \to B\) be a map. Suppose that the following conditions hold:

1. For all \(x_1, x_2, u_1, u_2 \in A\) with \(d(u_1, Tx_1) = d(A, B)\) and \(d(u_2, Tx_2) = d(A, B)\),
   \[
   d(u_1, Tx_1) + d(Tx_1, Tx_2) + d(Tx_2, u_2) \\
   \leq \frac{d(x_1, u_2) + d(x_2, u_1)}{d(x_1, u_1) + d(x_2, u_2) + 1} d(x_1, x_2) + L \min\{d(x_1, u_2), d(x_2, u_1)\}
   \]
   where \(L \geq 0\);

2. \(T(A_0) \subset B_0\);

3. Either \(T\) is continuous or \(B\) is approximatively compact with respect to \(A\).

Then, \(T\) has a best proximity point. Further, if \(x_*\) and \(y_*\) are two distinct best proximity points of \(T\), then
   \[
   d(x_*, y_*) \geq \frac{1}{2}.
   \]

By taking \(A = B = X\) in Theorem 2.1, and using Remark 1.1 (2), we have the following fixed point result.
Corollary 2.4. Let \((X,d)\) be a complete metric space. Suppose that a map \(T : X \to X\) satisfies the following condition:

there exists \(L \geq 0\) such that, for all \(x,y \in X\),

\[
d(Tx,Ty) \leq \frac{d(x,Ty) + d(y,Tx)}{d(x,Tx) + d(y,Ty) + 1}d(x,y) + L \min\{d(x,Ty),d(y,Tx)\}.
\]

Then, \(T\) has a fixed point in \(X\). Moreover, the Picard iteration \(\{x_n\}\) defined by \(x_{n+1} = Tx_n\) for all \(n \in \mathbb{N} \cup \{0\}\), converges to some fixed point \(x_*\). Furthermore, if \(x_*\) and \(y_*\) are two fixed points of \(T\), then

\[
d(x_*,y_*) \geq \frac{1}{2}(1 - L).
\]

Corollary 2.5. \([16]\) Let \((X,d)\) be a complete metric space. Suppose that a map \(T : X \to X\) satisfies the following condition:

for all \(x,y \in X\),

\[
d(Tx,Ty) \leq \frac{d(x,Ty) + d(y,Tx)}{d(x,Tx) + d(y,Ty) + 1}d(x,y).
\]

Then, \(T\) has a fixed point in \(X\). Further, if \(x_*\) and \(y_*\) are two fixed points of \(T\), then \(d(x_*,y_*) \geq \frac{1}{2}\).

Remark 2.1. Consider the following condition:

for all \(x_1,x_2,u_1,u_2 \in A\) with \(d(u_1,Tx_1) = d(A,B)\) and \(d(u_2,Tx_2) = d(A,B)\), there exist \(k \in (0,1)\) and \(L \geq 0\) such that

\[
d(u_1,u_2) \leq kd(x_1,x_2) + L \min\{d(x_1,u_2),d(x_2,u_1)\}, \quad (2.10)
\]

where \((X,d)\) is metric space, \(A,B \in N(X)\) and \(T : A \to B\) is a map.

Note that if condition (1) in Theorem 2.1 replace by (2.10), then one can prove the existence of best proximity points of \(T\) similar to proof of Theorem 2.1.

Let \((X,d)\) be a metric space, and let \(A,B \in N(X)\).

A map \(T : A \to B\) is called rational type proximal contraction of the second kind if there exists \(L \geq 0\) such that, for all \(x_1,x_2,u_1,u_2 \in A\) with \(d(u_1,Tx_1) = d(A,B)\) and \(d(u_2,Tx_2) = d(A,B)\),

\[
d(Tu_1,Tu_2) \\ \leq \frac{d(Tx_1,Tu_2) + d(Tx_2,Tu_1)}{d(Tx_1,Tu_1) + d(Tx_2,Tu_2) + 1}d(Tx_1,Tx_2) \\ + L \min\{d(Tx_1,Tu_2),d(Tx_2,Tu_1)\}. \quad (2.11)
\]
**Theorem 2.6.** Let \((X, d)\) be a complete metric space, and let \(A, B \in CL(X)\) such that \(A_0 \neq \emptyset\). Suppose that a map \(T : A \to B\) satisfies the following conditions:

1. \(T\) is a rational type proximal contraction of the second kind;
2. \(T(A_0) \subset B_0\);
3. \(T\) is continuous.

Assume that \(A\) is approximatively compact with respect to \(B\).

Then, \(T\) has a best proximity point. Moreover, there exists a subsequence \(\{x_n(k)\}\) of the sequence \(\{x_n\}\) defined by \(d(x_{n+1}, Tx_n) = d(A, B)\) for all \(n \in \mathbb{N} \cup \{0\}\), converges to some best proximity point \(x_\ast\). Furthermore, if \(x_\ast\) and \(y_\ast\) are two distinct best proximity points of \(T\), then

\[d(Tx_\ast, Ty_\ast) \geq \frac{1}{2}(1 - L).\]

**Proof.** As in the proof of Theorem 2.1, we can find a sequence \(\{x_n\}\) of points in \(A_0\) such that \(d(x_n, Tx_{n-1}) = d(A, B)\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\).

From condition (2) \(\{Tx_n\}\) is a sequence in \(B_0\).

From (2.11) with \(u_1 = x_n, u_2 = x_{n+1}, x_1 = x_{n-1}\) and \(x_2 = x_n\), we have

\[
d(Tx_n, Tx_{n+1}) \leq \frac{d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)}{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + 1}d(Tx_{n-1}, Tx_n) + L \min\{d(Tx_{n-1}, Tx_{n+1}), d(Tx_n, Tx_n)\}
\]

\[
= \beta_n d(Tx_{n-1}, Tx_n)
\]

for all \(n \in \mathbb{N}\), where

\[
\beta_n = \frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + 1}.
\]

Note that \(0 < \beta_{n+1} < \beta_n < 1\) for all \(n \in \mathbb{N}\). Let \(k \in (0, 1)\) be such that \(\beta_1 \leq k\).

From (2.12) we have

\[
d(Tx_n, Tx_{n+1}) \leq kd(Tx_{n-1}, Tx_n)
\]
for all \(n \in \mathbb{N}\), which implies \(\{Tx_n\}\) is a Cauchy sequence in \(B_0\).

Since \(A_0 \neq \emptyset\) and \(T(A_0) \subset B_0, B_0 \neq \emptyset\). By Lemma 1.1, \(B_0 \in \text{CL}(X)\).

It follows from the completeness that there exists \(y_* \in B_0\) such that

\[
\lim_{k \to \infty} Tx_n = y_*.
\]

We deduce

\[
d(y_*, A) \leq d(y_*, x_{n+1})
\leq d(y_*, Tx_n) + d(x_{n+1}, Tx_n)
\leq d(y_*, Tx_n) + d(A, B)
\leq d(y_*, Tx_n) + d(y_*, A),
\]

which implies

\[
\lim_{n \to \infty} d(y_*, x_n) = d(y_*, A).
\]

Since \(A\) is approximatively compact with respect to \(B\), there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that

\[
\lim_{k \to \infty} x_{n(k)} = x_*,
\]

where \(x_* \in A\).

Since \(T\) is continuous, we have

\[
d(A, B) = \lim_{k \to \infty} d(x_{n(k)+1}, Tx_{n(k)}) = d(x_*, Tx_*).
\]

The remainder part of proof is similar to proof of Theorem 2.1. \(\square\)

By taking \(A = B = X\) in Theorem 2.6, we have the following fixed point results.

**Corollary 2.7.** Let \((X, d)\) be a complete metric space. Suppose that a map \(T : X \to X\) satisfies the following conditions:

1. there exists \(L \geq 0\) such that, for all \(x, y \in X\),

\[
d(TT x, TT y) \leq \frac{d(Tx, TT y) + d(Ty, TT x)}{d(Tx, TT x) + d(Ty, TT y) + 1} \cdot d(Tx, Ty) + L \min\{d(Tx, TT y), d(Ty, TT x)\};
\]

2. \(T\) is continuous.

Then, \(T\) has a fixed point. Further, if \(x_*\) and \(y_*\) are two fixed points of \(T\), then

\[
d(x_*, y_*) \geq \frac{1}{2}(1 - L).
\]
Theorem 2.8. Let \((X, d)\) be a complete metric space, and let \(A, B \in CL(X)\) such that \(A_0 \neq \emptyset\). Suppose that a map \(T : A \to B\) satisfies the following conditions:

1. \(T\) is rational type proximal contraction of the first and second kinds;
2. \(T(A_0) \subseteq B_0\);

Then, \(T\) has a best proximity point. Moreover, the sequence \(\{x_n\}\) defined by 
\[d(x_{n+1},Tx_n) = d(A,B)\]
for all \(n \in \mathbb{N} \cup \{0\}\), converges to some best proximity point. Furthermore, if \(x_*\) and \(y_*\) are two distinct best proximity points of \(T\), then 
\[d(x_*, y_*) \geq \frac{1}{2}(1 - L)\]
and 
\[d(Tx_*, Ty_*) \geq \frac{1}{2}(1 - L)\].

Proof. As in the proof of Theorem 2.1, we can find a Cauchy sequence \(\{x_n\}\) in \(A\) such that 
\[d(x_n, Tx_{n-1}) = d(A, B)\]
for all \(n \in \mathbb{N}\).

It follows from the closedness of \(A\) and completeness of \(X\) that there exists \(x_* \in A\) such that 
\[\lim_{n \to \infty} x_n = x_*\].

Further, as in the proof of Theorem 2.6, \(\{Tx_n\}\) is a Cauchy sequence in \(B\). Thus, 
\[\lim_{n \to \infty} Tx_n = y_* \in B\] exists.

Hence, we have 
\[d(x_*, y_*) = \lim_{n \to \infty} d(x_n, Tx_{n-1}) = d(A, B),\]
and hence, \(x_* \in A_0\). From condition (2) \(Tx_* \in B_0\), and so there exists \(\overline{x} \in A\) such that 
\[d(\overline{x}, Tx_*) = d(A, B)\].

From (2.1) we have
\[
d(x_n, \overline{x}) \leq \frac{d(x_*, x_n) + d(x_{n-1}, \overline{x})}{d(x_*, \overline{x}) + d(x_{n-1}, x_n) + 1} d(x_*, x_{n-1}) + L \min\{d(x_*, x_n), d(x_{n-1}, \overline{x})\}. \tag{2.13}
\]

From (2.13) we obtain 
\[\lim_{n \to \infty} x_n = \overline{x}\].

Hence, \(x_* = \overline{x}\), and hence \(d(x_*, Tx_*) = d(A, B)\).
We now give an example to illustrate Theorem 2.1.

**Example 2.1.** Let \( X = \mathbb{R}^2 \) with Euclidean metric \( d \). Let \( A = \{(x, 0) : 0 \leq x \leq 1\} \) and \( B = \{(x, 1) : 0 \leq x \leq 1\} \), and let \( L = 1 \).

Then, \((X, d)\) is a complete metric space, \( A, B \in \text{CL}(X) \), \( A_0 = A, B_0 = B \) and \( d(A, B) = 1 \).

Let \( T : A \to B \) be a map defined by \( T((x, 0)) = (x, 1) \).

Then, \( T \) is continuous and \( T(A_0) \subset B_0 \).

Let \( u_1 = (a_1, 0), u_2 = (a_2, 0), x_1 = (b_1, 0), x_2 = (b_2, 0) \in A \) be any points such that
\[
d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2).
\]

Thus, \( a_1 = b_1 \) and \( a_2 = b_2 \). Hence, \( u_1 = x_1 \) and \( u_2 = x_2 \).

Hence, we have
\[
\begin{align*}
d(x_1, u_2) + d(x_2, u_1) & \leq \frac{d(x_1, u_1) + d(x_2, u_2)}{d(x_1, u_1) + d(x_2, u_2) + 1} d(x_1, x_2) + L \min\{d(x_1, u_2), d(x_2, u_1)\} \\
& = \frac{d(x_1, x_2) + d(x_2, x_1)}{d(x_1, x_2) + d(x_2, x_1) + 1} d(x_1, x_2) + d(x_2, x_1) \\
& \geq d(x_1, x_2) \\
& = d(u_1, u_2),
\end{align*}
\]

and so \( T \) is a rational type proximal contraction of the first kind.

Thus, all condition of Theorem 2.1 are satisfied. By Theorem 2.1, there exists \((x_*, 0) \in A \) such that \( d((x_*, 0), T((x_*, 0))) = 1 = d(A, B) \). More precisely, the point \((0, 0) \in A \) is the best proximity point of \( T \).

Note that condition (2.10) in Remark 2.1 is not satisfied. In fact, assume that there exist \( k \in (0, 1) \) and \( L \geq 0 \) such that
\[
d(u_1, u_2) \leq kd(x_1, x_2) + L \min\{d(x_1, u_2), d(x_2, u_1)\}
\]
whenever \( d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2) \).

Then we have, for \( u_1 = x_1 = (0, 0), u_2 = x_2 = (1, 0) \) and \( L = 0 \),
\[
d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2).
\]

It follows from (2.10) that
\[
1 = d(u_1, u_2) \leq kd(x_1, x_2) = k.
\]

However, it is not possible. Thus, condition (2.10) in Remark 2.1 is not satisfied.
Best proximity points

References


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