Local Convergence of the Inverse Weierstrass Method for Simultaneous Approximation of Polynomial Zeros

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Abstract

In this work we study the local convergence of the Inverse Weierstrass iterative method for simultaneous approximation of polynomial zeros. We establish new local convergence theorem with error estimates. The main results generalize one of the last known result on local convergence of the Weierstrass’ method provided by Proinov and Petkova in [7].

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1 Introduction

Let \( P(x) \) be a monic polynomial

\[
P(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n,
\]

of degree \( n \geq 2 \), with simple real or complex zeros \( \alpha_1, \alpha_2, \ldots, \alpha_n \).

In this study we consider a simultaneous iterative method defined by

\[
z^{(k+1)} = G (z^{(k)}) = G^{k+1} (z^{(0)}) , \quad k = 0, 1, 2, \ldots,
\]
where \( G : \mathbb{C}^n \to \mathbb{C}^n \) is a vector valued function with components
\[
G_i = G_i(z) = \frac{z_i^2}{z_i + W_i(z)}, \quad i = 1, 2, \ldots, n
\] (3)

where \( z = (z_1, \ldots, z_n) \) and the term
\[
W_i(z) = \frac{P(z_i)}{\prod_{j \neq i}(z_i - z_j)}, \quad (i = 1, \ldots, n)
\] (4)
is the so-called Weierstrass’ correction.

The iteration method (2)-(3) is a modification of the famous Weierstrass’ method for simultaneously finding all the zeros of polynomials defined by
\[
z_i^{(k+1)} = z_i^{(k)} - W_i(z^{(k)}), \quad i = 1, 2, \ldots, n, \quad k \geq 0,
\] (5)

which is known also as the Durand-Kerner or Weierstrass-Dochev method. The modified method (2)-(3) was firstly introduced in [3], and some recent results were obtained in [4, 5].

2 Preliminary Notes

In 2013, Proinov and Petkova [7] established the following local convergence theorem for the Weierstrass method (5) (see also [8, 9]).

**Theorem 2.1** (Theorem 2.3 in [7]) Let \( f \in \mathbb{K}[z] \) be a polynomial of degree \( n \geq 2 \) which has \( n \) simple zeros in \( \mathbb{K} \), \( \xi \) a root-vector of \( f \) and \( 1 \leq p \leq \infty \). Suppose that \( x^0 \in \mathbb{K} \) is an initial guess satisfying
\[
E(x^0) = \left\| x^0 - \xi \right\|_p < R(n, p) = \frac{2^{1/(n-1)} - 1}{2^{1/q(2^{1/(n-1)} - 1)} + (n - 1)^{-1/p}}.
\] (6)

Then the Weierstrass iteration (5) is well defined and converges quadratically to \( \xi \) with error estimates
\[
\|x^{k+1} - \xi\|_p \leq \lambda^{2^k} \|x^k - \xi\| \quad \text{and} \quad \|x^k - \xi\| \leq \lambda^{2^k-1}\|x^0 - \xi\| \quad \text{for all} \quad k \geq 0,
\]
where \( \lambda = E(x^0)/R(n, p) \).

This result is given for polynomials over an arbitrary normed field \( (\mathbb{K}, |.|) \), where \( \mathbb{K}[z] \) denotes the ring of polynomials over \( \mathbb{K} \) and the vector space \( \mathbb{K}^n \) is equipped with the \( p \)-norm
\[
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}
\] (7)
for some $1 \leq p \leq \infty$. The Theorem 2.1 generalizes and improves some of the well known results concerning the local convergence of (5), such as Dochev (1962)[1], Kjurkchiev and Markov (1983)[2] and Yakoubsohn (2002)[11]. Besides, as immediate consequence of Theorem 2.1 is the result of Wang and Zao[10] published in 1991 and rediscovered by Niell (2001)[6].

Our goal in this work is to generalize the Theorem 2.1 in the case of Inverse Weierstrass iterative method (2)-(3). In the next section we present new local convergence theorem for the method (2)-(3).

Throughout this paper we use the following notations and conventions. Let

$$d = d(\alpha) = \min \{\delta, \gamma\},$$

where $\delta = \min_{j \neq i} |\alpha_i - \alpha_j|$ and $\gamma = \min_{i} |\alpha_i|$ for $i, j = 1, \ldots, n$ and $i \neq j$. Further, for a number $p$ such that $1 \leq p \leq \infty$ we denote by $q$ the conjugate exponent of $p$, i.e $q$ is defined by means of

$$1 \leq q \leq \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We study the local convergence of the Inverse Weierstrass method (2)-(3) with respect to the function of initial conditions $E : \mathbb{C}^n \to \mathbb{R}_+$ defined as follows

$$E(z) = \left\| z - \alpha \right\|_p$$

by analogy of the function of initial conditions in Theorem 2.1.

### 3 Main Results

First, we introduce some auxiliary results. We will state following three known lemmas that we will use without proofs (the proofs may be found, e.g. in [7]).

**Lemma 3.1** Let $u \in \mathbb{C}^n$ and $1 \leq p \leq \infty$. Then

$$|u_i| + |u_j| \leq 2^{\frac{1}{q}}\|u\|_p \quad \text{for any} \quad i, j = 1, 2, \ldots, n.$$  \hfill (9)

**Lemma 3.2** Let $u \in \mathbb{C}^n$ and $1 \leq p \leq \infty$. Then

$$\left( \prod_{i=1}^{n} (1 + |u_i|) - 1 \right) \leq \left( 1 + \frac{\|u\|_p}{n^{1/p}} \right)^n - 1.$$  \hfill (10)

**Lemma 3.3** Let $n \in \mathbb{N}$, $t \geq 0$ and $0 \leq \varphi \leq 1$. Then

$$(1 + \varphi t)^n - 1 \leq \varphi ((1 + t)^n - 1).$$  \hfill (11)
Now, we will prove the following two lemmas.

**Lemma 3.4** Let $0 < t < 1/2^{1/q}$, $h > 1$, $n \geq 2$ and

\[
\phi(t) = \left( 1 + \frac{t}{(n-1)^{1/p}(1-2^{1/q}t)} \right)^{n-1} < h.
\] (12)

Then

\[
\frac{1+t}{1-t} \phi(t) < 2
\] (13)

if and only if

\[
h \leq \frac{3.2^{1/q} - 2 + \sqrt{2^{2/q} - 4.2^{1/q} + 20}}{2(2^{1/q} + 1)}.
\] (14)

**Proof.** It is easy to prove that the following number

\[
\tilde{t} = \frac{h^{\frac{1}{n-1}} - 1}{2^{\frac{1}{q}}(h^{\frac{1}{n-1}} - 1) + (n-1)^{-\frac{1}{q}}}
\] (15)

is the unique root of the equation $\phi(t) = h$ in the interval $(0, 1/2^{1/q})$. Taking into account that the function $\phi$ is continuous and strictly increasing on the interval $I = [0, \tilde{t}]$ and that $\phi(I) = [1, h]$ we deduce that (12) is equivalent to

\[
t < \tilde{t}.
\]

It follows from (15) that

\[
\tilde{t} \leq \frac{h - 1}{2^{\frac{1}{q}}(h - 1) + 1}.
\]

From the last two inequalities, we get

\[
\frac{1+t}{1-t} < \frac{(2^{\frac{1}{q}} + 1)h - 2^{\frac{1}{q}}}{(2^{\frac{1}{q}} - 1)h + 2 - 2^{\frac{1}{q}}},
\]

which implies

\[
\frac{1+t}{1-t} \phi(t) < \frac{((2^{\frac{1}{q}} + 1)h - 2^{\frac{1}{q}})h}{(2^{\frac{1}{q}} - 1)h + 2 - 2^{\frac{1}{q}}}. \tag{16}
\]

Now, we bound the right-hand side of (16) as follows

\[
\frac{((2^{\frac{1}{q}} + 1)h - 2^{\frac{1}{q}})h}{(2^{\frac{1}{q}} - 1)h + 2 - 2^{\frac{1}{q}}} \leq 2
\]

which can be written in the following equivalent form

\[
(2^{\frac{1}{q}} + 1)h^2 - (3.2^{\frac{1}{q}} - 2)h - 2(2 - 2^{\frac{1}{q}}) \leq 0. \tag{17}
\]

It is easily seen that (17) implies (14). The lemma is proved.
Lemma 3.5 Let \( P \in \mathbb{C}[z] \) be a monic polynomial of degree \( n \geq 2 \), where \( \alpha = \{ \alpha \in \mathbb{C}^n : \alpha_i \neq 0 \text{ and } \alpha_i \neq \alpha_j \text{ for } i, j = 1, \ldots, n \} \) is the root-vector of \( P \), \( 1 \leq p \leq \infty \), and let \( d = d(\alpha) = \min(\delta, \gamma) \), where \( \delta = \min_{j \neq i}|\alpha_i - \alpha_j| \) and \( \gamma = \min_{i} |\alpha_i| \) for \( i, j = 1, \ldots, n \) (\( i \neq j \)). Let for any \( k \geq 0 \)

\[
E_k = E(z^{(k)}) = \left\| \frac{z^{(k)} - \alpha}{d(\alpha)} \right\|_p < \frac{1}{2^k},
\]

Then the iteration \( z^{(k)} \) is well defined and it has distinct components. Besides,

\[
\|z^{(k+1)} - \alpha\|_p \leq \sigma_k \|z^{(k)} - \alpha\|_p
\]

and

\[
E(z^{(k+1)}) \leq \sigma_k E(z^{(k)}),
\]

where

\[
\sigma_k = \sigma(E_k) = \frac{1 - E_k}{1 - E_k} \left( 1 + \frac{E_k}{(n-1)^{1/p}(1-2^k E_k)} \right)^{n-1} - 1.
\]

**Proof.** Using the triangle inequality, Lemma 3.1 and (18), we get for \( i \neq j \)

\[
|z_i^{(k)} - z_j^{(k)}| \geq |\alpha_i - \alpha_j| - |z_i^{(k)} - \alpha_i| - |z_j^{(k)} - \alpha_j| = \left(1 - \frac{|z_i^{(k)} - \alpha_i|}{|\alpha_i - \alpha_j|} \right) |\alpha_i - \alpha_j| \geq \left( 1 - 2^{\frac{1}{p}} \left\| \frac{z^{(k)} - \alpha}{d(\alpha)} \right\| \right) d(\alpha) = (1 - 2^{\frac{1}{p}} E_k) d > 0,
\]

which means that \( z^{(k)} \) has distinct components, i.e. the iteration is well defined. Now we will prove the following estimates

\[
|z_i^{(k+1)} - \alpha_i| \leq \sigma_k |z_i^{(k)} - \alpha_i|, \quad (i = 1, 2, \ldots, n).
\]

For easy of later comparisons, we will use the following equivalent form of (3)

\[
z_i^{(k+1)} = z_i^{(k)} - \frac{W_i(z^{(k)})}{1 + W_i(z^{(k)})}, \quad i = 1, 2, \ldots, n,
\]

which implies

\[
z_i^{(k+1)} - \alpha_i = z_i^{(k)} - \alpha_i - \frac{W_i(z^{(k)})}{1 + W_i(z^{(k)})} = (z_i^{(k)} - \alpha_i) \left[ 1 - \frac{1}{1 + W_i(z^{(k)})} \right] \left[ 1 - \prod_{j \neq i}^{n} \frac{z_j^{(k)} - \alpha_j}{z_i^{(k)} - z_i^{(k)}} \right]
\]

and consequently

\[
(z_i^{(k+1)} - \alpha_i) = (z_i^{(k)} - \alpha_i) \left[ 1 - \prod_{j \neq i}^{n} \frac{z_j^{(k)} - \alpha_j}{z_i^{(k)} - z_i^{(k)}} + \frac{W_i(z^{(k)})}{1 + W_i(z^{(k)})} \right].
\]
Therefore
\[ |z_i^{(k+1)} - \alpha_i| = A_i^{(k)} |z_i^{(k)} - \alpha_i| , \]
where
\[ A_i^{(k)} := \left| 1 - \frac{\prod_{j \neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + W_i(z^{(k)})}{1 + W_i(z^{(k)})} \right| . \]

Now, we can bound the amplification factor \( A_i^{(k)} \) as follows
\[ A_i^{(k)} \leq \left| \frac{\prod_{j \neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} - 1 + \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)} - z_j^{(k)}} \left| \prod_{j \neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|}{1 - \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)} - z_j^{(k)}} \left| \prod_{j \neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|} \right| . \] (26)

We next establish the following inequalities
\[ |z_i^{(k)}| \geq |\alpha_i| - |z_i^{(k)} - \alpha_i| \geq d - |z^{(k)} - \alpha| \geq (1 - E_k)d > 0 . \] (27)

It follows from (22) and the definition of \( E(z^{(k)}) \) that
\[ \left| \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right| = 1 + \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \leq 1 + \frac{|z_i^{(k)} - \alpha_j|}{(1 - 2^{-\frac{1}{p}}E_k)d(\alpha)} . \] (28)

From (26), the last two inequalities (28) and (27), and Lemma 3.2, we obtain
\[ A_i^{(k)} \leq \left( 1 + \frac{E_k}{(n-1)^\frac{1}{p}(1-2^{-\frac{1}{p}}E_k)} \right)^{n-1} - 1 + \frac{E_k}{1-E_k} \left( 1 + \frac{E_k}{(n-1)^\frac{1}{p}(1-2^{-\frac{1}{p}}E_k)} \right)^{n-1} \left( 1 - \frac{E_k}{1-E_k} \left( 1 + \frac{E_k}{(n-1)^\frac{1}{p}(1-2^{-\frac{1}{p}}E_k)} \right) \right)^{n-1} \] (29)

and consequently
\[ A_i^{(k)} \leq \frac{1}{1-E_k} \left( 1 + \frac{E_k}{(n-1)^\frac{1}{p}(1-2^{-\frac{1}{p}}E_k)} \right)^{n-1} - 1 \].

Finally, from the last expression and (25) we obtain (23). Taking the \( p \)-norm in (23), we deduce the first inequality in (20). We get the second inequality in (20) by dividing both sides of inequality (23) by \( d(\alpha) \) and taking the \( p \)-norm.

Now we are ready to state the main result of this paper which generalizes the above mentioned Theorem 2.1 introduced by Proinov and Petkova in [7].
Theorem 3.6 Let \( P \in \mathbb{C}[z] \) be a monic polynomial of degree \( n \geq 2 \), where \( \alpha = \{ \alpha \in \mathbb{C}^n : \alpha_i \neq 0 \text{ and } \alpha_i \neq \alpha_j \text{ for } i, j = 1, \ldots, n \} \) is the root vector of \( P \), and let \( 0 \leq p \leq \infty, d = d(\alpha) = \min\{\delta, \gamma\}, \) where \( \delta = \min_{j \neq i} |\alpha_i - \alpha_j| \) and \( \gamma = \min_i |\alpha_i| \) for \( i, j = 1, \ldots, n \) \((i \neq j)\). Suppose \( z^{(0)} \in \mathbb{C}^n \) is an initial guess satisfying
\[
E(z^{(0)}) = \left\| \frac{z^{(0)} - \alpha}{d(\alpha)} \right\|_p < R(n, p) = \frac{\theta^\frac{1}{q} - 1}{2^\frac{1}{q} (\theta^\frac{1}{q} - 1) + (n - 1) - \frac{1}{p}},
\]
where
\[
\theta = \frac{3b - 2 + \sqrt{b^2 - 4b + 20}}{2(b + 1)} \text{ and } b = 2^\frac{1}{q}.
\]
Then the following statements hold true.

(i) **Convergence.** The Inverse Weierstrass iteration (2)-(3) is well defined and converges quadratically to the root-vector \( \alpha \) of \( P \).

(ii) **A posteriori error estimate.** For all \( k \geq 0 \) we have the estimate
\[
\|z^{(k+1)} - \alpha\|_p \leq \lambda^{2^{k+1}} (z^{(k)} - \alpha),
\]

(iii) **A priori error estimate.** For all \( k \geq 1 \) we have the estimate
\[
\|z^{(k)} - \alpha\|_p \leq \lambda^{2^{k-1}} (z^{(0)} - \alpha),
\]
where \( \lambda = E(z^{(0)})/R(n, p) \).

**Proof.** (i) From the Lemma 3.4 it follows that \( R = R(n, p) \) is the unique solution of the equation \( \phi(t) = \theta \) in the interval \((0, 1/2^{1/q})\), where \( \phi(t) \) is defined by (12). First, we will prove that for any \( k \geq 0 \) the iteration \( z^{(k)} \) in (2)-(3) is well defined and
\[
E(z^{(k)}) \leq R \lambda^{2^k}.
\]
We shall use mathematical induction to prove the statement. First, we confirm that the base case \( k = 0 \) is true due to definition of \( \lambda \). It can be shown that \( \theta > 1 \) for any \( q \geq 1 \), which implies that \( R < 1/2^{1/q} \). From the initial assumption \( E(z^{(0)}) < R \) it follows that \( E(z^{(0)}) < 1/2^{1/q} \). From this and Lemma 3.5 we deduce that the iteration \( z^{(0)} \) is well defined.

Now, we will prove that
\[
\sigma(E_0) \leq \lambda.
\]
From (21) it follows that \( \sigma(E_k) \) can be written in the following equivalent form
\[
\sigma(E_k) = \frac{\left( 1 + \frac{E_k}{(n-1)^{1/p}(1-2^{1/q} E_k)} \right)^{n-1} - 1 + E_k}{1 - E_k - E_k \left( 1 + \frac{E_k}{(n-1)^{1/p}(1-2^{1/q} E_k)} \right)^{n-1}}.
\]
which implies
\[ \sigma(E_0) \leq \frac{\left( 1 + \frac{\lambda R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1} - 1 + \lambda R}{1 - R - R \left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1}}. \]

From Lemma 3.3, where \( \varphi = \lambda \) and \( t = \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \) we deduce
\[ \sigma(E_0) \leq \frac{\lambda \left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1} - 1 + \lambda R}{1 - R - R \left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1}} \leq \lambda \sigma(R), \quad (35) \]

where
\[ \sigma(R) = \frac{\left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1} - 1 + R}{1 - R - R \left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1}}. \]

It is easy to show that the assumption \( \sigma(R) < 1 \) is equivalent to the assumption defined by (13), where \( t = R \). Therefore, from (35) using Lemma 3.3 and the definition of \( R \) we deduce that \( \sigma(E_0) \leq \lambda \).

Suppose that for any \( k \geq 0 \) is fulfilled
\[ E(z^{(k)}) \leq R \lambda^{2^k} \tag{36} \]
and we will prove that
\[ E(z^{(k+1)}) \leq R \lambda^{2^{k+1}}. \]

From (34) and the assumption by induction we obtain
\[ \sigma(E_k) \leq \frac{\left( 1 + \frac{\lambda^{2^k} R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1} - 1 + \lambda^{2^k} R}{1 - R - R \left( 1 + \frac{R}{(n-1)^{1/p}(1-2^{1/q} R)} \right)^{n-1}} \leq \lambda^{2^k} \sigma(R) < \lambda^{2^k}. \tag{37} \]

Therefore, from (37), the assumption (36) and the estimate (20) it follows that
\[ E(z^{(k+1)}) \leq \sigma_k E(z^{(k)}) \leq \lambda^{2^k} R \lambda^{2^k} = \lambda^{2^{k+1}} R. \]

Using the inequality \( R < 1/2^{1/q} \) it follows that \( E(z^{(k+1)}) < 1/2^{1/q} \), which implies by Lemma 3.5 that the iteration \( z^{(k+1)} \) is well defined.

(ii) From (37) and Lemma 3.5 estimate (19) it is trivial to prove that
\[ \| z^{(k+1)} - \alpha \|_p \leq \lambda^{2^k} \| z^{(k)} - \alpha \|_p. \]

(iii) From the assertion (ii) and the sum of geometric progression, we obtain
\[ \| z^{(k)} - \alpha \|_p \leq \lambda^{2^{k-1}} \| z^{(k-1)} - \alpha \|_p \leq \lambda^{2^{k-1}} \lambda^{2^{k-2}} \| z^{(k-2)} - \alpha \|_p \leq \ldots \leq \lambda^{2^k-1} \lambda^{2^{k-2}} \ldots \lambda^{2^2} \| z^{(0)} - \alpha \|_p = \lambda^{2^k-1} \| z^{(0)} - \alpha \|_p, \]
which implies (33). The theorem is proved.
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References


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