Equivalent Conditions of Hermite-Hadamard Type Inequality for Sugeno Integrals

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Abstract

The following is the classical Hermite-Hadamard inequality [4, 5]:

\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, d\mu \leq \frac{f(a) + f(b)}{2}. \]

which provides estimates of the mean value of a convex function \( f \) on \([a, b]\) where \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). This inequality in general, is not valid in the fuzzy context. In this paper, we find necessary and sufficient conditions of Hermite-Hadamard type inequality for Sugeno integrals.

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1 Introduction and preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [16], it has been exhaustively investigated by many authors. Ralescu and Adams [12] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [17] provided an overview of fuzzy measure theory.

**Definition 1.** Let $\Sigma$ be a $\sigma$-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function. We say that $\mu$ is a fuzzy measure if and only if

(a) $\mu(\emptyset) = 0$.

(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity).

(c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^\infty E_p\right)$ (continuity form below).

(d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^\infty E_p\right)$ (continuity form above).

If $f$ is a non-negative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in X | f(x) \geq \alpha\} = \{f \geq \alpha\}$ the $\alpha$-level of $f$, for $\alpha > 0$, and $F_0 = \{x \in X | f(x) > 0\} = \text{supp}(f)$ is the support of $f$.

We note that

$$\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$$

If $\mu$ is a fuzzy measure on $A \subset \mathbb{R}$, then we define the following:

$$\mathfrak{F}^\mu(A) = \{f : A \to [0, \infty) | f \text{ is } \mu\text{-measurable}\}.$$

**Definition 2.** Let $\mu$ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathfrak{F}^\mu(\mathbb{R})$ and $A \in \Sigma$, then the Sugeno integral (or the fuzzy integral) of $f$ on $A$, with respect to the
fuzzy measure \( \mu \), is defined as
\[
(S) \int_A f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(A \cap F_\alpha)].
\]
In particular, if \( A = X \) then
\[
(S) \int_\mathbb{R} f d\mu = (S) \int f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(F_\alpha)].
\]

The following properties of the Sugeno integral are well known and can be found in [17]:

**Proposition 1** [17]. If \( \mu \) is a fuzzy measure on \( \mathbb{R} \) and \( f, g \in \mathcal{F}(\mathbb{R}) \), then
(i) \( (S) \int_A f d\mu \leq \mu(A) \);
(ii) \( (S) \int_A K d\mu = K \land \mu(A) \) for any constant \( K \in [0, \infty) \);
(iii) \( (S) \int_A f d\mu \leq (S) \int_A g d\mu \), if \( f \leq g \) on \( A \);
(iv) \( \mu(A \cap \{ f \geq \alpha \}) \geq \alpha \Rightarrow (S) \int_A f d\mu \geq \alpha \);
(v) \( \mu(A \cap \{ f \geq \alpha \}) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha \);
(vi) \( (S) \int_A f d\mu < \alpha \Leftrightarrow \) there exists \( \gamma \leq \alpha \) such that \( \mu(A \cap \{ f \geq \gamma \}) < \alpha \);
(vii) \( (S) \int_A f d\mu > \alpha \Leftrightarrow \) there exists \( \gamma \geq \alpha \) such that \( \mu(A \cap \{ f \geq \gamma \}) > \alpha \).

**Note 1.** Let \( F(\alpha) = \mu(A \cap \{ f \geq \alpha \}) \), then by Proposition 1, (v), (vi),
\[
F(\alpha) = \alpha \Rightarrow (S) \int_0^1 f(x) d\mu = \alpha.
\]

**Theorem 1** [10]. Let \( f : [0, \infty) \rightarrow [0, \infty) \) be continuous and non-increasing or non-decreasing functions and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Let \( (S) \int_0^a f(x) d\mu = p \). If \( 0 < p < a \), then \( f(p) = p \), \( f(a - p) = p \), respectively.

### 2 Equivalent conditions of Hermite-Hadamard type inequalities

The classical Hermite-Hadamard inequality provides estimates of the mean value of a convex function \( f \) on \([a, b]\)
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} (S) \int_a^b f(x) d\mu \leq \frac{f(a) + f(b)}{2}.
\]

Recently, Caballero and Sadarangani [2] showed two examples which proves the left part and right part of Hermite-Hadamard type inequality are not valid in the fuzzy context.
The aim of this section is to find some sufficient or necessary conditions so that the left or right inequality (1) is valid for Sugeno integral.

The following proposition is immediate.

**Proposition 2.** Let $f : [a, b] \rightarrow [0, \infty)$ be real-valued nonnegative convex function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. Let $f^*(x) = f(a + (b-a) x), \ x \in [0,1]$. Then we have

$$f^*(0) = f(a), \ f^*(1) = f(b), \ f^* \left( \frac{1}{2} \right) = f \left( \frac{a+b}{2} \right)$$

and

$$\mu \{ f^* \geq \alpha \} = \frac{1}{b-a} \mu \{ f \geq \alpha \}$$

which implies

$$\frac{1}{b-a} (S) \int_a^b f(x) d\mu = (S) \int_0^1 f^*(x) d\mu.$$ 

**Theorem 2.** Let $f : [a, b] \rightarrow [0, \infty)$ be real-valued convex function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. If $f((a+b)/2) \leq (a+b)/2$, then the inequality

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} (S) \int_a^b f(x) d\mu.$$

holds.

**Proof.** From Proposition 2, we may assume that $a = 0, b = 1$. We first assume that $f$ is non-increasing or non-decreasing. We notice that since $f(1/2) \leq 1/2$,

$$\mu(F_{f(1/2)}) \geq \frac{1}{2} \geq f \left( \frac{1}{2} \right).$$

Then we have

$$(S) \int_0^1 f(x) d\mu = \sup_{\alpha \in [0,1]} [\alpha \wedge \mu(F_{\alpha})]$$

$$\geq f \left( \frac{1}{2} \right) \wedge \mu(F_{f(1/2)}) \geq f \left( \frac{1}{2} \right).$$

We now assume that $f(0) > f \left( \frac{1}{2} \right) < f(1)$. Since $f$ is continuous on $[0, 1]$, there exists $c \in (0, 1)$ such that $\inf \{ f(x) | x \in [0, 1] \} = f(c)$. Suppose that $\frac{1}{2} \leq c$. Define a function $f^*$ as

$$f^*(x) = \begin{cases} f(x), & x \in [0, c], \\ f(c), & x \in [c, 1]. \end{cases}$$
Then $f^*$ is a non-increasing convex function such that $f^*(x) \leq f(x)$. By Proposition 1 and (2), we have
\[ f\left(\frac{1}{2}\right) = f^*(\frac{1}{2}) \leq (S) \int_{0}^{1} f^*(x) d\mu \leq (S) \int_{0}^{1} f(x) d\mu. \]
The case for $\frac{1}{2} \geq c$ can be proved in a similar manner and we complete the proof.

**Remark 1.** In Theorem 2, the condition $f(1/2) \geq 1/2$ is essential and the converse of Theorem 2 is not true in general. Counterexamples can be easily found.

We need additional condition so that the converse of Theorem 2 is true.

**Theorem 3.** Let $f : [a, b] \to [0, \infty)$ be real-valued convex monotone function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. If the inequality
\[ f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} (S) \int_{a}^{b} f(x) d\mu. \] (3)
holds if and only if $f((a + b)/2) \leq (a + b)/2$.

**Proof.** From Proposition 2, without loss of generality, we may assume that $a = 0$, $b = 1$, and $f$ is non-increasing. Let $(S) \int_{0}^{1} f(x) d\mu = p$. Suppose that $f(1/2) \leq (S) \int_{0}^{1} f(x) d\mu$ then, by Theorem 1, we have
\[ f\left(\frac{1}{2}\right) \leq (S) \int_{0}^{1} f(x) d\mu = p = f(p). \]
Since $f$ is convex and non-increasing, $p \leq \frac{1}{2}$, and hence
\[ f\left(\frac{1}{2}\right) \leq \frac{1}{2}. \]
The converse is from Theorem 2 and the proof is completed.

**Theorem 4.** Let $f : [a, b] \to [0, \infty)$ be real-valued convex function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. If $f((a + b)/2) \geq (a + b)/2$, then the inequality
\[ \frac{1}{b - a} (S) \int_{a}^{b} f(x) d\mu \leq \frac{f(a) + f(b)}{2}. \] (4)
holds.

**Proof.** From Proposition 2, we may assume that $a = 0$, $b = 1$. We first assume that $f$ is non-increasing. Let $(S) \int_{0}^{1} f(x) d\mu = p$. Then we have
\[ f(p) \leq f\left(\frac{1}{2}\right). \]
If not, then \( f(p) > f(\frac{1}{2}) \). Since \( f \) is non-increasing and convex, \( p < \frac{1}{2} \). Then

\[
f(p) > f(\frac{1}{2}) \geq \frac{1}{2} > p,
\]

which is contradict that \( f(p) = p \). Hence we have \( f(p) \leq f(\frac{1}{2}) \) and since \( f \) is non-increasing, \( \frac{1}{2} \leq p \). Therefore we have that

\[
f(p) \leq f(\frac{1}{2}) \leq \frac{f(0) + f(1)}{2}.
\]

Next, we assume that \( f \) is non-decreasing. We consider the function \( f^*(x) = f(1 - x) \). Then \( f^* \) is a non-increasing convex function such that \( f^*(1/2) = f(1/2) \geq 1/2 \) and

\[
(S) \int_0^1 f(x) d\mu = (S) \int_0^1 f^*(x) d\mu.
\]

Therefore we can use the result for non-increasing case

\[
(S) \int_0^1 f(x) d\mu = (S) \int_0^1 f^*(x) d\mu \leq \frac{f^*(0) + f^*(1)}{2} = \frac{f(0) + f(1)}{2}.
\]

Now, we consider general case. Let \( g \) be the piecewise linear continuous and convex function connecting points \((0, f(0)), (1/2, f(1/2))\) and \((1, f(1))\). Since \( f \) is convex, \( f(x) \leq g(x), \ x \in [0, 1] \). If \( g \) is monotone i.e., non-increasing or non-decreasing then by Proposition 1 and using previous cases

\[
(S) \int_0^1 f(x) d\mu \leq (S) \int_0^1 g(x) d\mu \leq \frac{g(0) + g(1)}{2} = \frac{f(0) + f(1)}{2}.
\]

Assume that \( g \) is not monotone. Without loss of generality, we assume that \( g(1/2) < g(1) \leq g(0) \). Then there exists \( c \in [0, 1/2) \) such that \( g(c) = g(1) \). Define a function \( g^* \) as

\[
g^*(x) = \begin{cases} g(x), & x \in [0, c], \\ g(1), & x \in [c, 1]. \end{cases}
\]

Then \( g^* \) is non-increasing convex function such that \( g(x) \leq g^*(x), \ x \in [0, 1] \) and \( g^*(0) = g(0) = f(0), \ g^*(1) = g(1) = f(1) \). Hence we have

\[
(S) \int_0^1 f(x) d\mu \leq (S) \int_0^1 g^*(x) d\mu \leq \frac{g(0) + g(1)}{2} = \frac{f(0) + f(1)}{2}.
\]

We complete the proof.
Remark 2. In Theorem 4, the condition $f(1/2) \geq 1/2$ is essential and the converse of Theorem 4 is not true in general. Counterexamples can be easily found.

We need additional condition so that the converse of Theorem 4 is true.

Theorem 5. Let $f : [a, b] \to [0, \infty)$ be real-valued linear function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. If the inequality

$$\frac{1}{b-a} \left( \mathcal{S} \right) \int_a^b f(x) d\mu \leq \frac{f(a) + f(b)}{2},$$

holds if and only if $f((a + b)/2) \geq (a + b)/2$.

Proof. From Proposition 2, without loss of generality, we may assume that $a = 0, b = 1$ and $f$ is non-increasing. First suppose that $\frac{f(0) + f(1)}{2} \geq \mathcal{S} \int_0^1 f(x) d\mu = p$. Then, since $f$ is linear and $f(p) = p$,

$$\frac{f(0) + f(1)}{2} = f \left( \frac{1}{2} \right) \geq f(p).$$

Since $f$ is non-increasing, $p \geq 1/2$ and hence

$$f \left( \frac{1}{2} \right) \geq f(p) = p \geq \frac{1}{2}.$$

The converse is from Theorem 4 and we complete the proof.

Combining Theorem 3 and 5, we have the following result.

Theorem 6. Let $f : [a, b] \to [0, \infty)$ be real-valued linear function and $\mu$ be the Lebesgue measure on $\mathbb{R}$. If the inequality

$$\frac{1}{b-a} \left( \mathcal{S} \right) \int_a^b f(x) d\mu = f \left( \frac{a+b}{2} \right),$$

holds if and only if $f((a + b)/2) = (a + b)/2$.

References


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