Some Generalizations of Du-Lakzian’s Convergence Theorem and Best Proximity Point Theorem

Yi-Chou Chen

Department of General Education
National Army Academy, Taiwan

Abstract
In this paper, we establish some new existence and convergence theorems of iterates of best proximity points for new nonlinear mappings.

Mathematics Subject Classification: 41A17, 47H09

Keywords: $\mathcal{M}\mathcal{T}$-function ($\mathcal{R}$-function); cyclic mapping; best proximity point; convergence theorem

1. Introduction and preliminaries

It is well-known that fixed point theory plays as a powerful tool in the study of nonlinear analysis. However, as we know, the equation $Tx = x$ ($T$ is a selfmapping or non-selfmapping) is not necessarily to have a solution. In other words, a mapping $T$ does not necessarily possess a fixed point. So, in such situations, we often turn to find the best approximation of the existence of solutions. Many authors have investigated to generalize the best proximity point theory by using new nonlinear conditions; see, e.g., [1-3, 10, 14, 15, 17-22] and references therein.

Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. A mapping $T : A \cup B \mapsto A \cup B$ is called a cyclic mapping if

$$T(A) \subset B \quad \text{and} \quad T(B) \subset A.$$
For any nonempty subsets $A$ and $B$ of $X$, denote 
\[ \text{dist}(A, B) = \inf \{d(x, y) : x \in A, y \in B\}. \]

A point $x \in A \cup B$ is called to be a best proximity point for a mapping $T : A \cup B \to A \cup B$ if 
\[ d(x, Tx) = \text{dist}(A, B). \]

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let $f$ be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that 
\[ \limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0 < x - c < \varepsilon} f(x). \]

**Definition 1.1.** [4-13, 16, 18, 19] A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-function) if it satisfies Mizoguchi-Takahashi’s condition (i.e. $\limsup_{s \to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$).

It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then $\varphi$ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

In 2012, Du [8] proved some characterizations of $\mathcal{MT}$-functions.

**Theorem 1.2.** [8, Theorem 2.1] Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.

(a) $\varphi$ is an $\mathcal{MT}$-function.

(b) For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.

(c) For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.

(d) For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)})$.

(e) For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)}]$.

(f) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$. 


(g) $\varphi$ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

**Definition 1.3.** [14] Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. A map $T : A \cup B \to A \cup B$ is called a cyclic contraction if the following conditions hold:

1. $T(A) \subset B$ and $T(B) \subset A$;
2. There exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (1 - k)\text{dist}(A, B)$ for all $x \in A, y \in B$.

In [14], Eldred and Veeramani first proved the following interesting best proximity point theorem.

**Theorem 1.4 (see [14, Proposition 3.2]).** Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$. Let $T : A \cup B \to A \cup B$ be a cyclic contraction mapping, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in $A$. Then there exists $x \in A$ such that $d(x, Tx) = \text{dist}(A, B)$.

Du and Lakzian introduced the concept of $\mathcal{MT}$-cyclic contractions [10] to generalize the concept of cyclic contractions as follows.

**Definition 1.5 (see [10, Definition 1.3]).** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. If a mapping $T : A \cup B \to A \cup B$ satisfies

1. $T(A) \subset B$ and $T(B) \subset A$;
2. There exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that $d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))\text{dist}(A, B)$ for any $x \in A$ and $y \in B$,

then $T$ is called an $\mathcal{MT}$-cyclic contraction with respect to $\varphi$ on $A \cup B$.

In [10], Du and Lakzian gave the following example to show that there exists an $\mathcal{MT}$-cyclic contraction but not a cyclic contraction.

**Example 1.6 (see [10, Example A]).** Let $X = \{v_1, v_2, v_3, ...\}$ be a countable set and $\{\tau_n\}$ be a strictly increasing convergent sequence of positive real numbers. Denote by $\tau_\infty := \lim_{n \to \infty} \tau_n$. Then $\tau_2 < \tau_\infty$. Let $d : X \times X \to [0, \infty)$ be defined by $d(v_n, v_n) = 0$ for all $n \in \mathbb{N}$ and $d(v_n, v_m) = d(v_m, v_n) = \tau_m$ if
$m > n$. Then $d$ is a metric on $X$. Set $A = \{v_1, v_3, v_5, \ldots\}$, $B = \{v_2, v_4, v_6, \ldots\}$. Now we define a mapping $T : A \cup B \to A \cup B$ by

$$Tv_n \overset{\text{def}}{=} \begin{cases} v_2, & \text{if } n = 1, \\ v_{n-1}, & \text{if } n > 1 \end{cases}$$

for $n \in \mathbb{N}$. Define $\varphi : [0, \infty) \to [0, 1)$ as

$$\varphi(t) \overset{\text{def}}{=} \begin{cases} \frac{\tau_{n-1}}{\tau_n}, & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N} \text{ with } n > 2, \\ 0, & \text{otherwise}. \end{cases}$$

Then $T$ is an $MT$-cyclic contraction with respect to $\varphi$.

The following convergence theorem for $MT$-cyclic contractions was proved by Du and Lakzian in [10].

**Theorem 1.7 (see [10, Theorem 2.1]).** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ an $MT$-cyclic contraction with respect to $\varphi$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} d(x_n, Tx_n) = \inf_{n \in \mathbb{N}} d(x_n, Tx_n) = \text{dist}(A, B).$$

In this paper, we establish some new existence and convergence theorems for best proximity points which improve and generalize some results in [10] and some known results in the literature.

## 2. Main results

The following convergence theorem for new nonlinear mappings is one of the main results in this paper.

**Theorem 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that

1. \textbf{(H1)} there exists an $MT$-funtion $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, T^2x) \leq \varphi(d(x, Tx))d(x, Tx) + (1 - \varphi(d(x, Tx))) \text{dist}(A, B)$$

for all $x \in A \cup B$.

Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).$$
Proof. Since $T$ is cyclic, we know that $T(A) \subset B$ and $T(B) \subset A$. Given $x_1 \in A$. Define $x_{n+1} = Tx_n$ for all $n \in N$. Then $\{x_{2n-1}\}_{n \in N} \subset A$ and $\{x_{2n}\}_{n \in N} \subset B$. We claim

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in N} d(x_n, x_{n+1}) = \text{dist}(A, B).
$$

By (H1), it is easy to see that $d(Tx, T^2x) \leq d(x, Tx)$ for all $x \in A$. So the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing in $[0, \infty)$. Then

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in N} d(x_n, x_{n+1}) \geq 0 \text{ exists.} \quad (2.1)
$$

Since $\varphi$ is an $MT$-function, applying Theorem 1.2, we get

$$
0 \leq \varphi(d(x_n, x_{n+1})) < 1.
$$

Let $\gamma := \sup_{n \in N} \varphi(d(x_n, x_{n+1}))$. So $0 \leq \varphi(d(x_n, x_{n+1})) \leq \gamma < 1$ for all $n \in N$. By (H1), we have

$$
d(x_2, x_3) = d(Tx_1, T^2x_1) \\
\leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B) \\
\leq \gamma d(x_1, x_2) + \text{dist}(A, B)
$$

and

$$
d(x_3, x_4) = d(Tx_2, T^2x_2) \\
\leq \varphi(d(x_2, x_3))d(x_2, x_3) + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \\
\leq \varphi(d(x_2, x_3))\gamma d(x_1, x_2) + \text{dist}(A, B) \\
= \varphi(d(x_2, x_3))\gamma d(x_1, x_2) + \text{dist}(A, B) \\
\leq \gamma^2 d(x_1, x_2) + \text{dist}(A, B).
$$

Hence, by induction, we obtain

$$
\text{dist}(A, B) \leq d(x_{n+1}, x_{n+2}) \leq \gamma^n d(x_1, x_2) + \text{dist}(A, B) \text{ for all } n \in N. \quad (2.2)
$$

Since $\gamma \in [0, 1)$, $\lim_{n \to \infty} \gamma^n = 0$. From (2.2), we get

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = \text{dist}(A, B). \quad (2.3)
$$

Combining (2.1) with (2.3), we obtain

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in N} d(x_n, x_{n+1}) = \text{dist}(A, B).
$$
The proof is completed.

**Remark 2.2.** It is obvious that [10, Theorem 2.1] is a special case of Theorem 2.1.

The following convergence theorems can be given immediately from Theorem 2.1.

**Corollary 2.3.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that

(h1) there exists a nondecreasing (or nonincreasing) $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, T^2x) \leq \varphi(d(x, Tx))d(x, Tx) + (1 - \varphi(d(x, Tx))) \text{dist}(A, B)$$

for all $x \in A \cup B$.

Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).$$

**Corollary 2.4.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that

(h2) there exists $\lambda \in [0, 1)$ such that

$$d(Tx, T^2x) \leq \lambda d(x, Tx) + (1 - \lambda) \text{dist}(A, B)$$

for all $x \in A \cup B$.

Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).$$

Finally, we give a new best proximity point theorem which generalizes [10, Theorem 2.4].

**Theorem 2.5.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that (H1) holds as in Theorem 2.1 and
(H2) \(d(Tx, Ty) \leq d(x, y)\) for any \(x \in A\) and \(y \in B\).

Let \(x_1 \in A\) be given. Define an iterative sequence \(\{x_n\}_{n \in \mathbb{N}}\) by \(x_{n+1} = Tx_n\) for \(n \in \mathbb{N}\). Suppose that \(\{x_{2n-1}\}\) has a convergent subsequence in \(A\), then there exists \(v \in A\) such that \(d(v, Tv) = \text{dist}(A, B)\).

**Proof.** By the same argument as the proof of Theorem 2.1, we know that \(x_{2n-1} \in A\) and \(x_{2n} \in B\) for all \(n \in \mathbb{N}\). By Theorem 2.1, we obtain

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B). \tag{2.4}
\]

By our hypothesis, \(\{x_{2n-1}\} \subset A\) has a convergent subsequence \(\{x_{2n_k-1}\}\) and \(x_{2n_k-1} \to v\) as \(k \to \infty\) for some \(v \in A\). Since

\[
\text{dist}(A, B) \leq d(v, x_{2n_k}) \leq d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \quad \text{for all } k \in \mathbb{N},
\]

and \(\lim_{n \to \infty} d(v, x_{2n_k}) = 0\), by (2.4), we get

\[
\lim_{n \to \infty} d(v, x_{2n_k}) = \text{dist}(A, B). \tag{2.5}
\]

By (H2), we have

\[
\text{dist}(A, B) \leq d(Tv, x_{2n_k+1}) \leq d(v, x_{2n_k}) \quad \text{for all } k \in \mathbb{N}. \tag{2.6}
\]

Hence, combining (2.5) with (2.6), we obtain \(d(v, Tv) = \text{dist}(A, B)\). \qed

**Corollary 2.6 [10, Theorem 2.4].** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\) be an \(\mathcal{MT}\)-cyclic contraction with respect to \(\varphi\). Let \(x_1 \in A\) be given. Define an iterative sequence \(\{x_n\}_{n \in \mathbb{N}}\) by \(x_{n+1} = Tx_n\) for \(n \in \mathbb{N}\). Suppose that \(\{x_{2n-1}\}\) has a convergent subsequence in \(A\), then there exists \(v \in A\) such that \(d(v, Tv) = \text{dist}(A, B)\).

**References**


Generalizations of Du-Lakzian’s convergence theorem

https://doi.org/10.1186/s13660-015-0931-x


Received: September 15, 2016; Published: November 4, 2016