Remarks on ”Smooth” Functions

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Abstract

The purpose of this paper is to study a class of continuous functions \( \hat{\lambda}_s(\gamma), \gamma > 0 \) satisfying a certain ”smooth”ness condition dependent on a parameter \( \gamma \). For \( \gamma > \frac{1}{2} \), it is shown in [3] that the class \( \hat{\lambda}_s(\gamma) \) is a subset of the class of absolute continuous functions. In this paper, for \( \gamma \in (0, \frac{1}{2}] \), we provide an example which is differentiable almost nowhere. And for \( \gamma > 1 \), we prove that the class \( \hat{\lambda}_s(\gamma) \) is a subset of \( C^1 \). Moreover, for \( \gamma \in (0, 1] \) we get a precise estimate for the modulus of continuity of the functions of \( \hat{\lambda}_s(\gamma) \).

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1 Introduction

Consider a function \( f : I \to \mathbb{R} \) defined on an open finite interval \( I \). Denote by \( \Delta^1_s f(x_0, \tau) \) and \( \Delta^2_s f(x_0, \tau) \) the first and second symmetric differences of \( f \), respectively that is,

\[
\Delta^1_s f(x_0, \tau) = f(x_0 + \tau) - f(x_0 - \tau),
\]

\[
\Delta^2_s f(x_0, \tau) = f(x_0 + \tau) + f(x_0 - \tau) - 2f(x_0)
\]

where \( x_0 \in I \) and \( \tau \in [0, |I|/2] \). Then the ordinary difference \( \Delta f(x_0, \tau) = f(x_0 + \tau) - f(x_0) \) can be written as

\[
\Delta f(x_0, \tau) = \frac{1}{2} \Delta^1_s f(x_0, \tau) + \frac{1}{2} \Delta^2_s f(x_0, \tau).
\]

It is natural in many contexts to examine the continuity and differentiability properties of \( f \) by studying those properties in the two parts \( \Delta^1_s f(x_0, \tau) \) and \( \Delta^2_s f(x_0, \tau) \). The continuity of \( f \) at a point \( x_0 \), that is the requirement that \( \Delta f(x_0, \tau) \to 0 \) as \( \tau \to 0 \), is equivalent to requiring both \( \Delta^1_s f(x_0, \tau) \to 0 \) and \( \Delta^2_s f(x_0, \tau) \to 0 \) as \( \tau \to 0 \). And these two requirements are called in analysis odd and even continuity of \( f \) at \( x_0 \) respectively. Similarly, differentiability of \( f \) at \( x_0 \) is equivalent to the existence of both of limits

\[
\lim_{\tau \to 0} \frac{\Delta^1_s f(x_0, \tau)}{\tau} \quad \text{and} \quad \lim_{\tau \to 0} \frac{\Delta^2_s f(x_0, \tau)}{\tau}.
\]

If the first limit exists then this limit is called symmetric derivative of \( f \) at \( x_0 \).

And if the second limit exists and zero that is

\[
\Delta^2_s f(x_0, \tau) = o(\tau) \quad \text{as} \quad \tau \to 0
\]

then \( f \) is called ”smooth” function at \( x_0 \). It follows immediately that if \( f' \) exists and is finite then \( f \) is ”smooth” at \( x_0 \). The converse is obviously false, but if \( f \) is ”smooth” at \( x_0 \) and if a one-sided derivative \( f \) at \( x_0 \) exists the derivative on the other side also exists and both are equal. If \( f \) is ”smooth” at every point of \( I \), we say that \( f \) is a ”smooth” function on \( I \). If \( f \) is continuous and satisfies (1) uniformly in \( x_0 \) i.e.,

\[
\|\Delta^2_s f(\cdot, \tau)\|_\infty = o(\tau) \quad \text{as} \quad \tau \to 0
\]

we shall say that \( f \) is uniformly ”smooth”, where \( \| \cdot \|_\infty \) is the supremum norm on \( I \). The class of such functions is denoted by \( \lambda_* \). Similarly, denote by \( \Lambda_* \) the
class of continuous functions $f$ which satisfy the following relation uniformly in $x_0$

$$\|\Delta^2_\gamma f(\cdot, \tau)\|_\infty = O(\tau) \text{ as } \tau \to 0.$$  

(3)

The conditions (2) and (3) are called small Zygmund and Zygmund conditions, respectively. We denote by $\lambda_\gamma$ and $\Lambda_\gamma$ the classes of functions defined on $I$ and satisfying (2) and (3) respectively. Furthermore, denote by $\Lambda_\alpha$ the class of functions satisfying the $\alpha$-Hölder condition in $I$. Next we compare the classes $\lambda_\gamma$ and $\Lambda_\gamma$ with well known classes of analysis in order to be well understood for the reader the importance of these classes. It is easy to verify that if a function $g$ is continuously differentiable then it belongs to both of the classes $\lambda_\gamma$, $\Lambda_\gamma$ and if $g$ is Lipschitz then it belongs only to the class $\Lambda_\gamma$. The reverse is not true. For example, the function $g(x) = x \log(x) + Ax$, $x > 0$ satisfies (3) but it is not Lipschitz. It is, however, $\alpha$-Hölder for each $\alpha < 1$. If $g$ has bounded variation then it also does not necessarily belong to the class $\Lambda_\gamma$ and vice versa. For example, the function $g(x) = \sqrt{x}$ has bounded variation but does not belong to the class $\Lambda_\gamma$, and on the other hand the function $g(x) = x^2 \sin(1/x^2)$ has unbounded variation but it belongs to the class $\Lambda_\gamma$. The following theorem was proved in [4].

Theorem 1.1. ([4], p.44) $f$ be defined in a finite interval $I$. If $f \in \Lambda_\gamma$ then

$$\omega(\delta; f) = O\left(\delta \log \frac{1}{\delta}\right)$$

and in particular $f \in \Lambda_\alpha$ for every $\alpha \in (0, 1)$. If $f \in \lambda_\gamma$ then

$$\omega(\delta; f) = o\left(\delta \log \frac{1}{\delta}\right)$$

where $\omega(\cdot; f)$ is the modulus of continuity of $f$.

The purpose of this work is to generalize the above theorem for a subclass of $\lambda_\gamma$ which is defined as follows. Consider the function $Z_\gamma : [0, 1) \to (0, +\infty)$, given

$$Z_\gamma(x) = \frac{1}{(\log \frac{1}{x})^\gamma}, \quad x \in (0, 1)$$

and $Z_\gamma(0) = 0$, where $\gamma > 0$. Denote by $\hat{\lambda}_\gamma(\gamma)$ the class of continuous functions $f : I \to \mathbb{R}$ satisfying

$$\|\Delta^2_\gamma f(\cdot, \tau)\|_\infty \leq C_\tau Z_\gamma(\tau)$$  

(4)

for some constant $C := C(f) > 0$. Note that $\hat{\lambda}_\gamma(\gamma) \subset \lambda_\gamma$ for all $\gamma > 0$, since $Z_\gamma(\tau) \to 0$ as $\tau \to 0$ and this class was investigated by Weiss and Zygmund [3]. They proved the following theorem.
Theorem 1.2. [3] Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic and satisfies (4) for some $\gamma > \frac{1}{2}$. Then $f$ is absolute continuous and $f \in L^p[0, 2\pi]$ for every $p > 1$.

In this work we study the class of functions $\hat{\lambda}_*(\gamma)$ for different $\gamma$'s. More precisely, first we show that the modulus of continuity of the functions of $\hat{\lambda}_*(\gamma)$ for $\gamma \in (0, 1)$ is $O(\delta(\log \frac{1}{\delta})^{1-\gamma})$ and for $\gamma = 1$ is $O(\delta(\log \log \frac{1}{\delta}))$. And then we provide an example which is differentiable almost nowhere. These results extend Theorem 1.1. Moreover, we prove the differentiability of the functions of $\hat{\lambda}_*(\gamma)$ for $\gamma > 1$. This result generalizes Theorem 1.2 for $\gamma > 1$.

2 Main results

Let $\mathcal{P}_\gamma : (0, 1) \to \mathbb{R}$ defined as

$$\mathcal{P}_\gamma(x) = \sum_{n=1}^{\infty} \mathcal{Z}_\gamma(x2^{-n}) \text{ where } x \in (0, 1) \text{ and } \gamma > 1. \quad (5)$$

It is clear that $\mathcal{P}_\gamma$ is continuous and $\lim_{x \to 0} \mathcal{P}_\gamma(x) = 0$. This function will be needed in the proof of main theorems. The first main result is the following.

**Theorem 2.1.** Let $f : I \to \mathbb{R}$ be continuous and satisfies the inequality (4) on $I$. Then

$$\omega(\delta, f) \leq C \begin{cases} \delta(\log \frac{1}{\delta})^{1-\gamma} & \text{if } \gamma \in (0, 1); \\ \delta(\log \log \frac{1}{\delta}) & \text{if } \gamma = 1; \end{cases}$$

where $\omega(\cdot, f)$ is the modulus of continuity of $f$. Moreover, $f$ is a Lipschitz function if $\gamma > 1$.

**Proof.** The proof follows closely that of [4] (p.44). Let us consider the function $\Delta f(x, \tau) = f(x + \tau) - f(x)$. Take any $x \in I$ we fix. The inequality (4) implies

$$|\Delta f(x, \tau) - 2\Delta f(x, \tau 2^{-1})| \leq C \tau \mathcal{Z}_\gamma(\tau)$$

for small enough $\tau \in (0, \zeta]$. Replacing here $\tau$ successively by $\tau 2^{-1}$, $\tau 2^{-2}$, ... , $\tau 2^{-(n-1)}$ and in each step multiplying $2, 2^2, ..., 2^{n-1}$ we obtain

$$|\Delta f(x, \tau) - 2\Delta f(x, \tau 2^{-1})| \leq C \tau \mathcal{Z}_\gamma(\tau 2^{-1}),$$

$$|2\Delta f(x, \tau 2^{-1}) - 2^2\Delta f(x, \tau 2^{-2})| \leq C \tau \mathcal{Z}_\gamma(\tau 2^{-2}),$$

$$|2^n \Delta f(x, \tau 2^{-(n-1)}) - 2^n \Delta f(x, \tau 2^{-n})| \leq C \tau \mathcal{Z}_\gamma(\tau 2^{-n}).$$
Below $n$ and $\tau$ will be chosen. By termwise addition above inequalities we get

$$|\Delta f(x, \tau) - 2^n \Delta f(x, \tau 2^{-n})| \leq C \tau \sum_{k=1}^{n} Z_\gamma(\tau 2^{-k}).$$  \hfill (6)

Suppose $\delta > 0$ be sufficiently small. Let $0 < \delta \leq \frac{1}{2} \zeta$ and $n$ be a natural such that $\frac{\zeta}{2} \leq 2^n \delta < \zeta$. This implies $n < \log_2 \frac{\zeta}{\delta}$. Make substitution $\tau = 2^n \delta$ in (6) we get

$$|\Delta f(x, \delta)| \leq \frac{2 \max_{x \in I} |f(x)| \delta}{2^n \delta} + C \delta \sum_{k=1}^{n} Z_\gamma(\delta 2^{n-k}).$$  \hfill (7)

Easily can be seen

$$\delta \sum_{k=1}^{n} Z_\gamma(\delta 2^{n-k}) = \delta \sum_{k=1}^{n} \frac{2^{-(n-k)} Z_\gamma(\delta 2^{n-k})}{2^{-(n-k)}} \leq C \int_{2^{-n}}^{1} \frac{\delta Z_\gamma(\frac{\delta}{x})}{x} dx. \hfill (8)$$

An easy computation shows that

$$\int_{2^{-n}}^{1} \frac{\delta Z_\gamma(\frac{\delta}{x})}{x} dx \leq C \begin{cases} 
\delta (\log \frac{1}{\delta})^{1-\gamma} & \text{if } \gamma \in (0, 1); \\
\delta (\log \log \frac{1}{\delta}) & \text{if } \gamma = 1.
\end{cases} \hfill (9)$$

This proves the first assertion of Theorem 2.1. To prove the second assertion we use the inequality

$$\sum_{k=1}^{n} Z_\gamma(\delta 2^{n-k}) \leq P_\gamma(\delta) \hfill (10)$$

for $\delta \in [0, \frac{\zeta}{2}]$. The proof of this inequality is simple. Moreover, $P_\gamma$ is bounded since it is continuous on $[0, \frac{\zeta}{2}]$. This and the relation (7) implies the second assertion of Theorem 2.1.

Next we provide an example which belongs to $\hat{\lambda}_* (\gamma)$ but the derivative of this function can be ”bad” as it can for $\gamma \in (0, \frac{1}{2}]$. Consider a class of Weierstrass functions

$$W_\beta(x) = \sum_{n=1}^{\infty} \theta_n b^{-\beta n} \cos(b^n x) \quad \text{where } b > 1 \quad \text{and} \quad \lim_{n \to \infty} \theta_n = 0. \hfill (11)$$

The following facts can be found in [4]. Weierstrass showed that for a small enough $\beta > 0$ the function $W_\beta$ is nowhere differentiable. The extension to $\beta \leq 1$ was first proved by Hardy. For $\beta > 1$ the function $W_\beta^\prime$ exists and continuous. Moreover, if the sum of squares of the sequence $\theta_n$ is divergence then $W_1$ is differentiable in a set of measure zero only. Thus,
Lemma 2.2. Let \( \gamma \in (0, \frac{1}{2}] \). If we choose \( b = 2, \theta_n = n^{-\gamma} \) and \( \beta = 1 \) in (11), then the function \( W_1 \) satisfies (4).

Proof. Indeed,

\[
W_1(x + \tau) + W_1(x - \tau) - 2W_1(x) = -4 \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n n^\gamma} \sin^2\left(\frac{2^n \tau}{2}\right) =
\]

\[-4 \sum_{n=1}^{N} \frac{\cos(2^n x)}{2^n n^\gamma} \sin^2\left(\frac{2^n \tau}{2}\right) + (-4) \sum_{n=N+1}^{\infty} \frac{\cos(2^n x)}{2^n n^\gamma} \sin^2\left(\frac{2^n \tau}{2}\right) := I_1 + I_2\]

where \( N := N(\tau) \) is the largest integer satisfying \( 2^N \tau \leq 1 \), so that \( 2^{N+1} \tau > 1 \).

Next we estimate \( I_1, I_2 \) separately. It is obvious

\[
|I_1| \leq 4 \sum_{n=1}^{N} \frac{1}{2^n n^\gamma} \left(\frac{2^n \tau}{2}\right)^2 = \tau^2 \sum_{n=1}^{N} \frac{2^n}{n^\gamma} = \tau^2 2^N \sum_{n=1}^{N} \frac{1}{2^N - n n^\gamma} \leq \tau \sum_{n=1}^{N} \frac{1}{2^N - n n^\gamma}.
\]

On the other hand

\[
\sum_{n=1}^{N} \frac{1}{2^N - n n^\gamma} = \sum_{n=1}^{[\frac{N}{2}]} \frac{1}{2^N - n n^\gamma} + \sum_{n=[\frac{N}{2}] + 1}^{N} \frac{1}{2^N - n n^\gamma} \leq \frac{2}{2^{[\frac{N}{2}]} + (\frac{N}{2} + 1)\gamma} \leq C \left(\sqrt{\tau} + \frac{1}{(\log \frac{1}{\tau})^\gamma}\right) \leq C \mathcal{Z}_\gamma(\tau).
\]

Thus \( I_1 \leq C \tau \mathcal{Z}_\gamma(\tau) \). It is easy to see

\[
|I_2| \leq 4 \sum_{n=N+1}^{\infty} \frac{1}{2^n n^\gamma} \leq 4 \frac{1}{2^N (N+1)^\gamma} \leq C \tau \mathcal{Z}_\gamma(\tau).
\]

Hence

\[
|W_1(x + \tau) + W_1(x - \tau) - 2W_1(x)| \leq C \tau \mathcal{Z}_\gamma(\tau).
\]

The right hand side of this inequality does not depend on \( x \). Therefore \( \|\cdot\|_{\infty} \)-norm of the left hand side bounded with \( C \tau \mathcal{Z}_\gamma(\tau) \). This proves lemma. \( \square \)

Further we investigate the class \( \mathcal{A}_s(\gamma) \) for \( \gamma > 1 \).

**Theorem 2.3.** Let \( f : I \to \mathbb{R} \) satisfies the inequality (4) for some \( \gamma > 1 \). Then \( f \in C^1(I) \) and

\[
|f'(\xi) - f'(\eta)| \leq C \cdot \mathcal{P}_\gamma(|\xi - \eta|)
\]

for any \( \xi, \eta \in I \), where \( C > 0 \) depends only on \( f \).
Proof. According to Theorem 2.1, \( f \) is at least Lipschitz function in the case of \( \gamma > 1 \). Hence \( f' \) exists almost everywhere and \( f \) is the indefinite integral of \( f' \). Taking any Lebesgue points \( \xi, \eta \in I \) of \( f' \) we set \( \tau := |\xi - \eta| \). By inequality (4) we have
\[
\Delta f(x, \tau) = \Delta f(x - \tau, \tau) + \mathcal{O}(\tau \mathcal{Z}_\gamma(\tau))
\] (12)
for all \( x \in I \) and for all \( \tau \in [0, |I|/2] \). Replacing in (12) \( x \) successively by \( x_n = \eta + \tau 2^{-n} \) and \( \tau \) successively by \( \tau 2^{-n} \), \( n = 1, 2, \ldots \) we obtain
\[
\Delta f(x_n, \tau 2^{-n}) = \Delta f(\eta, \tau 2^{-n}) + \mathcal{O}(\tau 2^{-n} \mathcal{Z}_\gamma(\tau 2^{-n})).
\]
It easily can be seen that for all \( n \geq 1 \) the following identity holds.
\[
\Delta f(x_n, \tau 2^{-n}) - \Delta f(\eta, \tau 2^{-n}) = \Delta f(\eta, \tau 2^{-n+1}) - 2\Delta f(\eta, \tau 2^{-n}).
\]
Thus
\[
\Delta f(\eta, \tau 2^{-n+1}) = 2\Delta f(\eta, \tau 2^{-n}) + \mathcal{O}(\tau 2^{-n} \mathcal{Z}_\gamma(\tau 2^{-n})).
\]
This implies
\[
2^{-n+1}\Delta f(\eta, \tau 2^{-n+1}) = 2^{-n}\Delta f(\eta, \tau 2^{-n}) + \mathcal{O}(\tau \mathcal{Z}_\gamma(\tau 2^{-n})).
\]
By termwise addition from \( n = 1 \) to \( N \) and divide \( \tau \) we obtain
\[
\frac{\Delta f(\eta, \tau)}{\tau} = \frac{2^N}{\tau} \Delta f(\eta, \tau 2^{-N}) + \mathcal{O}\left(\sum_{n=1}^{N} \mathcal{Z}_\gamma(\tau 2^{-n})\right). \tag{13}
\]
Since the point \( \eta \) is the Lebesgue point of \( f' \) and \( \gamma > 1 \) taking limit as \( N \to \infty \) in (13) we get
\[
\lim_{N \to \infty} \frac{2^N}{\tau} \Delta f(\eta, \tau 2^{-N}) = f'(\eta) \quad \text{and} \quad \frac{\Delta f(\eta, \tau)}{\tau} = f'(\eta) + \mathcal{O}(\mathcal{P}_\gamma(\tau)). \tag{14}
\]
Similarly as above, replacing in (12) \( x \) successively by \( x_n = \xi - \tau 2^{-n} \) and \( \tau \) successively by \( \tau 2^{-n} \), \( n = 1, 2, \ldots \) we obtain
\[
\Delta f(x_n, \tau 2^{-n}) = \Delta f(x_{n-1}, \tau 2^{-n}) + \mathcal{O}(\tau 2^{-n} \mathcal{Z}_\gamma(\tau 2^{-n})).
\]
Here the following identity can be also easily checked for all \( n \geq 1 \).
\[
\Delta f(x_n, \tau 2^{-n}) - \Delta f(x_{n-1}, \tau 2^{-n}) = - \left( \Delta f(x_{n-1}, \tau 2^{-n+1}) - 2\Delta f(x_n, \tau 2^{-n}) \right).
\]
This implies
\[
2^{-n+1}\Delta f(x_{n-1}, \tau 2^{-n+1}) = 2^{-n}\Delta f(x_n, \tau 2^{-n}) + \mathcal{O}(\tau \mathcal{Z}_\gamma(\tau 2^{-n})).
\]
The same manner as above we get
\[
\frac{\Delta f(\xi - \tau, \tau)}{\tau} = f'(\xi) + \mathcal{O}(\mathcal{P}_\gamma(\tau)).
\] (15)

So far as the right hand sides of (14) and (15) are same. Therefore we have
\[
|f'(\xi) - f'(\eta)| \leq C \cdot \mathcal{P}_\gamma(\|\xi - \eta\|).
\]

This proves uniform continuity of \( f' \) on the set of Lebesgue points, thus \( f' \) coincides almost everywhere on \( I \) with a some continuous function \( \mathcal{U} : I \to \mathbb{R} \). It is clear \( \int_a^x \mathcal{U}(t)dt \) is a \( C^1(I) \) function and the function \( \mathcal{L}(x) := \int_a^x \mathcal{U}(t)dt - f(x) \) is absolute continuous. However, \( \mathcal{L}'(x) = 0 \) almost everywhere, hence \( \mathcal{L}(x) \equiv \text{const.} \) Theorem 2.3 is therefore completely proved.

\begin{remark}
Note that, the classes \( \lambda_* \) and \( \Lambda_* \) are not only important classes in harmonic analysis, but they have deserved applications in the theory circle maps. For example recently the authors have applied the class \( \hat{\lambda}_*(\gamma) \) to the renormalizations of circle maps [1].
\end{remark}

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\textbf{References}


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