On a Subclass of Goodman-Ronning Type

Harmonic Univalent Functions Defined by

Dziok-Srivastava Operator

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Abstract

The aim of this paper is to investigate a subclass of Goodman-Ronning type of harmonic univalent functions defined by the modified Dziok-Srivastava operator. The properties for this class such as the coefficient conditions, distortions bounds and extreme points are investigated.

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1 Introduction

Let $U$ denote the open unit disk and $S_H$ denote the class of harmonic univalent functions. Chandrashekhar [5] stated that the continuous function $f = u + iv$ is considered to be a univalent complex-valued harmonic function in a domain $D \subseteq \mathbb{C}$ if both $u$ and $v$ are real harmonic. A harmonic univalent function, $f = h + \overline{g}$ for any connected domain $D$ where $h$ and $g$ are analytic in $D$ can be expressed as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_k| < 1. \quad (1.1)$$

where $h$ and $g$ are identified as the analytic part and co-analytic part of $f$ correspondingly and normalized by the condition $f(0) = h(0) = f'(0) - 1 = 0$. The family $S_H$ could be reduced to the class $S$ of normalized analytic univalent functions if the co-analytic part of $f$ is identically zero. Clunie and Sheil-Small [6] mentioned that there is a necessary and sufficient condition for $f$ in order to be locally univalent and orientation preserving in $U$ where $|h'(z)| - |g'(z)| > 0$. They also let $S_{\Pi}$ denote the subclass of $S_H$ where

$$h(z) = z - \sum_{k=2}^{\infty} \left| a_k \right| z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} \left| b_k \right| z^k, \quad |b_k| < 1. \quad (1.2)$$

There have been several related papers on $S_H$ and its subclasses, see [1], [2], [4], [8], [9].

Dziok and Srivastava [7] introduced a differential operator by using the generalized hypergeometric functions for positive real value of $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ where $\beta_j \neq 0, -1, -2, \ldots; j = 1, 2, 3, \ldots, s$ defined as

$$qF_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{q} (\alpha_j)_k \prod_{j=1}^{s} (\beta_j)_k}{k!} z^k, \quad q \leq s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U$$

where $(a)_m$ is the Pochhammer symbol defined in terms of Gamma function, $\Gamma$, by

$$(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)} = \begin{cases} 1 & m = 0 \\ a(a+1)(a+m-1) & m \in \mathbb{N} \end{cases}.$$ 

Hence, the Dziok and Srivastava operator, $H_{q,s} (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ defined by convolution can be written as
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\[ H_{q,s}(\alpha_1,\ldots,\alpha_s; \beta_1,\ldots,\beta_s) f(z) = q F_s(\alpha_1,\ldots,\alpha_s; \beta_1,\ldots,\beta_s; z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}\ldots(\alpha_s)_{k-1} a_k z^k}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1} (k-1)!} = H_{q,s}[\alpha_1] f(z) \]

where * stands for convolution. Al-Kharsani and Al-Khal [3] introduced the modified Dziok-Srivastava operator of the harmonic function \( f = h + \bar{g} \) given by (1.1) as

\[ H_{q,s}[\alpha_1] f(z) = H_{q,s}[\alpha_1] h(z) + H_{q,s}[\alpha_1] g(z). \tag{1.4} \]

Motivated from the work of Al-Khal [2], a new class of harmonic univalent function of the form (1.1) denoted as \( S_H(\alpha_1; \rho, \beta) \) is introduced such that the functions in this class satisfy the condition

\[ \text{Re} \left( (1 + \rho e^{i\theta}) z \left( H_{q,s}[\alpha_1] f(z)' - \rho e^{i\theta} \right) \right) \geq \beta, \tag{1.5} \]

where \( z = re^{i\theta}, (H_{q,s}[\alpha_1] f(z)') = \frac{\partial}{\partial \theta} H_{q,s}[\alpha_1] f(re^{i\theta}), 0 \leq r < 1, 0 \leq \theta < 2\pi, 0 \leq \beta < 1, 0 \leq \rho \leq 1, \gamma \in \mathbb{R} \) and \( H_{q,s}[\alpha_1] f(z) \) is defined by (1.4).

We also let \( S_H(\alpha_1; \rho, \beta) \) denote the subclass of harmonic univalent functions, \( S_H(\alpha_1; \rho, \beta) \) so that \( h \) and \( g \) are in the form (1.2).

2 Main Results

We begin by giving the sufficient condition for functions in \( S_H(\alpha_1; \rho, \beta) \).

Theorem 2.1. Let a function \( f = h + \bar{g} \) be given by (1.1). If,

\[ \sum_{k=1}^{\infty} \left[ \frac{(1+\rho)k - \rho - \beta}{1-\beta} |a_k| + \frac{(1+\rho)k + \rho + \beta}{1-\beta} |p_k| \right] \Gamma(\alpha_1,k) \leq 2 \tag{2.1} \]

where \( a_k = 1, 0 \leq \beta < 1, 0 \leq \rho \leq 1 \) and \( \Gamma(\alpha_1,k) = \frac{(\alpha_1)_{k-1}\ldots(\alpha_s)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}} \), then \( f \) is a sense-preserving harmonic univalent function in \( U \) and \( f \in S_H(\alpha_1; \rho, \beta) \).

Proof. The inequality \( |h'(z)| \geq |g'(z)| \) is sufficient to show that \( f \) is sense preserving. Notice that

\[ |h'(z)| \geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} > 1 - \sum_{k=2}^{\infty} k |a_k| \]


According to the condition (1.5), we just need to show that if (2.1) holds, then

\[ \text{Re} \left\{ \frac{(1 + \rho e^{i\gamma}) z [H_{q,s}[\alpha_1]h(z)]' - (H_{q,s}[\alpha_1]g(z))'}{H_{q,s}[\alpha_1]h(z) + H_{q,s}[\alpha_1]g(z)} - \rho e^{i\gamma} \right\} = \text{Re} \left\{ \begin{array}{c} A(z) \\ B(z) \end{array} \right\} \geq \beta, \]

where
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\[ z = r e^{i\theta}, \quad 0 \leq r < 1, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \beta < 1, \quad 0 \leq \rho \leq 1, \quad \gamma \in R, \quad (H_{q,s} [\alpha_1] f(z))' = \frac{\partial}{\partial \theta} H_{q,s} [\alpha_1] f(re^{i\theta}). \]

Note that

\[ A(z) = (1 + \rho e^{i\theta}) z [(H_{q,s} [\alpha_1] h(z))' - \overline{(H_{q,s} [\alpha_1] g(z))'} - \rho e^{i\theta} (H_{q,s} [\alpha_1] h(z) + \overline{H_{q,s} [\alpha_1] g(z)}) \]

and \( B(z) = H_{q,s} [\alpha_1] h(z) + \overline{H_{q,s} [\alpha_1] g(z)} \).

Considering the fact that \( \text{Re } w \geq \beta \) if and only if \( |1 - \beta + w| \geq |1 + \beta - w| \), it suffices to show that

\[ |A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)| \geq 0. \] (2.2)

Substituting \( A(z) \) and \( B(z) \) in (2.2) yields

\[
(1 - \beta - \rho e^{i\theta}) H_{q,s} [\alpha_1] h(z) + (1 + \rho e^{i\theta}) (z (H_{q,s} [\alpha_1] h(z))') + (1 - \beta - \rho e^{i\theta}) \overline{H_{q,s} [\alpha_1] g(z)} - (1 + \rho e^{i\theta}) (\overline{(H_{q,s} [\alpha_1] g(z))'})
\]

\[
-((1 + \beta + \rho e^{i\theta}) H_{q,s} [\alpha_1] h(z) - (1 + \rho e^{i\theta}) z (H_{q,s} [\alpha_1] h(z))') + (1 + \beta + \rho e^{i\theta}) (H_{q,s} [\alpha_1] g(z) + \overline{(H_{q,s} [\alpha_1] g(z))'})
\]

\[
\geq (2 - \beta)|z| - \sum_{k=2}^{\infty} [1 - \beta + k + \rho k - \rho] \Gamma(\alpha_1, k) |a_k| z^k
\]

\[
- \sum_{k=2}^{\infty} [k + \beta - 1 + \rho k - \rho] \Gamma(\alpha_1, k) |b_k| z^k - \beta|z|
\]

\[
- \sum_{k=2}^{\infty} [k - \beta - 1 + \rho k - \rho] \Gamma(\alpha_1, k) |a_k| z^k - \sum_{k=2}^{\infty} [k + 1 + \beta + \rho k + \rho] \Gamma(\alpha_1, k) |b_k| z^k
\]

\[
= 2(1 - \beta)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{(k + \beta + \rho k - \rho)}{(1 - \beta)} \Gamma(\alpha_1, k) |a_k| z^{k-1}
\]

\[
- \sum_{k=1}^{\infty} \frac{(k + \beta + \rho k + \rho)}{(1 - \beta)} \Gamma(\alpha_1, k) |b_k| z^{k-1} \right\} \geq 0 \text{ by (2.1).}
\]

The harmonic function

\[
f(z) = z + \sum_{k=2}^{\infty} \frac{1 - \beta}{(1 + \rho) k - \beta} \Gamma(\alpha_1, k) \chi_k z^k
\]

\[
+ \sum_{k=1}^{\infty} \frac{1 - \beta}{(1 + \rho) k + \beta} \Gamma(\alpha_1, k) \psi_k z^k \] (2.3)
where \( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1 \), shows that the coefficient bound given by (2.1) is sharp. Thus, the functions in the form of the harmonic function \( f(z) \) above are in the class of \( S_H(\alpha; \rho; \beta) \) because

\[
\sum_{k=1}^{\infty} \left[ \frac{(1+\rho)k - \rho - \beta}{1 - \beta} \Gamma(\alpha, k) |a_k| + \frac{(1+\rho)k + \rho + \beta}{1 - \beta} \Gamma(\alpha, k) |b_k| \right] = 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.
\]

Next, we show the condition (2.1) is also necessary for functions in \( S_H(\alpha; \rho; \beta) \).

**Theorem 2.2.** Let \( f = h + \bar{g} \) be given by (1.3). Hence, \( f \in S_H(\alpha; \rho; \beta) \) if and only if

\[
\sum_{k=1}^{\infty} \left[ \frac{(1+\rho)k - \rho - \beta}{1 - \beta} |a_k| + \frac{(1+\rho)k + \rho + \beta}{1 - \beta} |b_k| \right] \Gamma(\alpha, k) \leq 2
\]

(2.4)

where \( \alpha_1 = 1, 0 \leq \beta < 1, 0 \leq \rho \leq 1 \).

**Proof.** Since \( S_H(\alpha; \rho; \beta) \subset S_H(\alpha; \rho; \beta) \), we just need to prove the “only if” part of the theorem. A necessary and sufficient bound for \( f = h + \bar{g} \) given by (1.2) to be in the class \( S_H(\alpha; \rho; \beta) \) is that

\[
\text{Re} \left\{ (1+\rho e^{i\gamma} ) \frac{z(H_{\alpha_1}[\alpha_1]f(z))' - \rho e^{i\gamma}}{H_{\alpha_1}[\alpha_1]f(z)} \right\} \geq \beta.
\]

This is equivalent to

\[
\text{Re} \left\{ (1+\rho e^{i\gamma} ) \frac{(z H_{\alpha_1}[\alpha_1]h(z) - (z H_{\alpha_1}[\alpha_1]g(z))' - \rho e^{i\gamma} (H_{\alpha_1}[\alpha_1]h(z) + H_{\alpha_1}[\alpha_1]g(z)) - \beta}{H_{\alpha_1}[\alpha_1]h(z) + H_{\alpha_1}[\alpha_1]g(z)} \right\}
\]

\[
= \text{Re} \left\{ z - \sum_{k=2}^{\infty} \Gamma(\alpha_1, k) k |a_k| z^k + \rho e^{i\gamma} z - \rho e^{i\gamma} \sum_{k=2}^{\infty} \Gamma(\alpha_1, k) k |a_k| z^k - \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) k |b_k| z^k - \rho e^{i\gamma} \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k
\]

\[
- \rho e^{i\gamma} \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k - \beta z + \beta \sum_{k=2}^{\infty} \Gamma(\alpha_1, k) |a_k| z^k + \beta \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k
\]

\[
= \text{Re} \left\{ z - \sum_{k=2}^{\infty} \Gamma(\alpha_1, k) |a_k| z^k + \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k
\]

\[
- \rho e^{i\gamma} \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k - \beta z + \beta \sum_{k=2}^{\infty} \Gamma(\alpha_1, k) |a_k| z^k + \beta \sum_{k=1}^{\infty} \Gamma(\alpha_1, k) |b_k| z^k
\]
Let 

\[
(1-\beta)z - \sum_{k=2}^{\infty} \left[ k + kpe^{i\gamma} - \rho e^{j\gamma} - \beta \right] \Gamma(\alpha, k) |a_k|^2 z^k
- \sum_{k=2}^{\infty} \left[ k + kpe^{i\gamma} + \rho e^{j\gamma} + \beta \right] \Gamma(\alpha, k) |b_k|^2 z^k
\]

\[
z - \sum_{k=2}^{\infty} \Gamma(\alpha, k) |a_k|^2 z^k + \sum_{k=2}^{\infty} \Gamma(\alpha, k) |b_k|^2 z^k
\]

\[
x - \sum_{k=2}^{\infty} \Gamma(\alpha, k) |a_k|^2 z^k + \sum_{k=2}^{\infty} \Gamma(\alpha, k) |b_k|^2 z^k
\]

\[
\geq 0
\]

By choosing \( z \) on the positive real axis \( 0 \leq z = r < 1 \) and \( \Re (-e^{i\gamma}) \geq -|e^{i\gamma}| = -1 \),

\[
\Re \left[ \frac{1}{1} \left( \frac{1}{1} \right) \right] \geq 0.
\]

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for \( r \) sufficiently close to 1. Hence there exist a \( z_0 = r_0 \) in \((0, 1)\) for which the quotient in (2.5) is negative. This contradicts the condition for \( f \in S_{\Pi}(\alpha; \rho; \beta) \). Hence, the proof is complete.

The following theorem is on the distortion bounds for the class \( S_{\Pi}(\alpha; \rho; \beta) \).

**Theorem 2.3:** If \( f \in S_{\Pi}(\alpha; \rho; \beta) \), \(|z| < 1\), then

\[
|f(z)| \leq (1 + |b_1|) r + \frac{1}{\Gamma(\alpha, 2)} \left[ \frac{1-\beta}{1+2\rho+\beta} |b_1|^2 \right] r^2, \quad |z| = r < 1
\]

and

\[
|f(z)| \geq (1 - |b_1|) r - \frac{1}{\Gamma(\alpha, 2)} \left[ \frac{1-\beta}{1+2\rho+\beta} |b_1|^2 \right] r^2, \quad |z| = r < 1.
\]

**Proof.** Let \( f \in S_{\Pi}(\alpha; \rho; \beta) \), \(|z| < 1\). Taking the absolute value of \( f \), we have

\[
|f(z)| \leq (1 + |b_1|) r + \sum_{k=2}^{\infty} \left[ |a_k|^2 + |b_k|^2 \right] r^k
\]

\[
= (1 + |b_1|) r + \frac{1-\beta}{(2+\rho-\beta) \Gamma(\alpha, 2)} \left[ \sum_{k=2}^{\infty} \left( \frac{2+\rho-\beta}{1-\beta} |a_k|^2 + \frac{2\rho+\beta}{1-\beta} |b_k|^2 \right) \Gamma(\alpha, k) r^k \right]
\]

\[
\leq (1 + |b_1|) r + \frac{1-\beta}{(2+\rho-\beta) \Gamma(\alpha, 2)} \left[ \sum_{k=2}^{\infty} \left[ (1+p)k - \rho - \beta \right] |a_k|^2 + \frac{1+p+k+\rho+\beta}{1-\beta} |b_k|^2 \right] \Gamma(\alpha, k) r^k
\]
\[ \leq (1+|b_1|)r + \frac{1-\beta}{(2+\rho-\beta)\Gamma(\alpha,2)} \left( 1 - \frac{1+2\rho+\beta}{1-\beta} |b_1| \right) r^2 \]

\[ = (1+|b_1|)r + \frac{1}{\Gamma(\alpha,2)} \left( \frac{1-\beta}{(2+\rho-\beta)} - \frac{1+2\rho+\beta}{2+\rho-\beta} |b_1| \right) r^2. \]

and

\[ |f(z)| \geq (1-|b_1|)r - \sum_{k=2}^{\infty} \left[ |a_k| + |b_k| \right] r^k \]

\[ \geq (1-|b_1|)r - \sum_{k=2}^{\infty} \left[ |a_k| + |b_k| \right] r^k \]

\[ = (1-|b_1|)r - \frac{1-\beta}{(2+\rho-\beta)\Gamma(\alpha,2)} \left\{ \sum_{k=2}^{\infty} \left( \frac{2+p-\beta}{1-\beta} |a_k| + \frac{2+p-\beta}{1-\beta} |b_k| \right) \Gamma(\alpha,2) r^2 \right\} \]

\[ \geq (1-|b_1|)r - \frac{1}{(2+\rho-\beta)\Gamma(\alpha,2)} \left[ \sum_{k=2}^{\infty} \left( \frac{1+\rho k-p-\beta}{1-\beta} |a_k| + \frac{1+\rho k+p+\beta}{1-\beta} |b_k| \right) \Gamma(\alpha,k) r^2 \right] \]

\[ \geq (1-|b_1|)r - \frac{1-\beta}{(2+\rho-\beta)\Gamma(\alpha,2)} \left( 1 - \frac{1+2\rho+\beta}{1-\beta} |b_1| \right) r^2 \]

\[ = (1-|b_1|)r - \frac{1}{\Gamma(\alpha,2)} \left( \frac{1-\beta}{(2+\rho-\beta)} - \frac{1+2\rho+\beta}{(2+\rho-\beta)} |b_1| \right) r^2. \]

Now, the extreme points will be examined. The closed convex cover (or hull) is denoted as \(\text{cloc}\) of \(S_\Gamma(\alpha; \rho; \beta)\).

**Theorem 2.4** \(f \in \text{cloc} S_\Gamma(\alpha; \rho; \beta)\) if and only if \(f\) can be expressed as

\[ f(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_k(z) \right), \] (2.6)

where \(h_k(z) = z + \frac{1-\beta}{[(1+\rho)k-\rho-\beta]\Gamma(\alpha,k)} z^k, (k = 2, 3, \ldots),\)

\[ g_k(z) = z + \frac{1-\beta}{[(1+\rho)k+\rho+\beta]\Gamma(\alpha,k)} z^k, (k = 1, 2, 3, \ldots) \sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0.\]

In particular, the extreme points of \(f \in S_\Gamma(\alpha; \rho; \beta)\) are \(\{h_k\}\) and \(\{g_k\}\).

**Proof.** For the function \(f\) of the form (2.6), we have

\[ f(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_k(z) \right) \]
\[
= \sum_{k=1}^{\infty} \left[ X_k + Y_k \right] z - \sum_{k=2}^{\infty} \left[ \frac{1-\beta}{(1+\rho)k - \rho - \beta} \Gamma(\alpha, k) \right] X_k z^k + \sum_{k=1}^{\infty} \left[ \frac{1-\beta}{(1+\rho)k + \rho + \beta} \Gamma(\alpha, k) \right] Y_k z^k.
\]

Then,
\[
\sum_{k=2}^{\infty} \frac{(1+\rho)k - \rho - \beta}{1-\beta} \Gamma(\alpha, k) \left( \frac{1-\beta}{(1+\rho)k - \rho - \beta} \Gamma(\alpha, k) X_k \right) + \sum_{k=1}^{\infty} \frac{(1+\rho)k + \rho + \beta}{1-\beta} \Gamma(\alpha, k) \left( \frac{1-\beta}{(1+\rho)k - \rho - \beta} \Gamma(\alpha, k) Y_k \right) = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1.
\]

Hence, \( f \in c\ell_{c} S_{\frac{\alpha}{\rho}}(\alpha_1; \rho, \beta) \). Conversely, suppose that \( f \in c\ell_{c} S_{\frac{\alpha}{\rho}}(\alpha_1; \rho, \beta) \). Set \( X_k = \frac{(1+\rho)k - \rho - \beta}{1-\beta} \Gamma(\alpha, k) | a_k | \); \( Y_k = \frac{(1+\rho)k + \rho + \beta}{1-\beta} \Gamma(\alpha, k) | b_k | ; \ k = (2, 3, ...) \).

Note that by Theorem 2.2, \( 0 \leq X_k \leq 1 \) and \( 0 \leq Y_k \leq 1 \).

Define \( X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \) and since \( X_1 \geq 0 \), we have \( f(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_k(z) \right) \).

This complete the proof of the theorem.

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