International Journal of Mathematical Analysis
Vol. 10, 2016, no. 25, 1211 - 1220
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2016.6357

On a Subclass of Goodman-Ronning Type Harmonic Univalent Functions Defined by Dziok-Srivastava Operator

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Abstract

The aim of this paper is to investigate a subclass of Goodman-Ronning type of harmonic univalent functions defined by the modified Dziok-Srivastava operator. The properties for this class such as the coefficient conditions, distortions bounds and extreme points are investigated.

Mathematics Subject Classifications: Primary 30C45

Keywords: Harmonic univalent, Dziok-Srivastava operator, coefficient conditions, distortion bounds, extreme points

1 Introduction

Let U denote the open unit disk and S_H denote the class of harmonic univalent functions. Chandrashekar [5] stated that the continuous function $f = u + iv$ is considered to be a univalent complex-valued harmonic function in a domain $D \subseteq \mathbb{C}$ if both u and v are real harmonic. A harmonic univalent function, $f = h + \overline{g}$ for any connected domain D where h and g are analytic in D can be expressed as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

where h and g are identified as the analytic part and co-analytic part of f correspondingly and normalized by the condition $f(0) = h(0) = f'(0) - 1 = 0.$ The family S_H could be reduced to the class S of normalized analytic univalent functions if the co-analytic part of f is identically zero. Clunie and Sheil-Small [6] mentioned that there is a necessary and sufficient condition for f in order to be locally univalent and orientation preserving in U where $|h'(z)| - |g'(z)| > 0.$ They also let $S_{\overline{H}}$ denote the subclass of S_H where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \qquad |b_1| < 1. \tag{1.2}$$

There have been several related papers on S_H and its subclasses, see [1], [2], [4], [8], [9].

Dziok and Srivastava [7] introduced a differential operator by using the generalized hypergeometric functions for positive real value of $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ where $\beta_j \neq 0, -1, -2, \ldots;\ j = 1, 2, 3, \ldots, s$ defined as

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k\, z^k}{(\beta_1)_k \ldots (\beta_s)_k\, k!}, q \leq s+1; q, s \in N_0 = N \cup \{0\}; z \in U$$

where $(a)_m$ is the Pochhamer symbol defined in terms of Gamma function, Γ, by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & m = 0 \\ a(a+1)\ldots(a+m-1) & m \in N \end{cases}.$$

Hence, the Dziok and Srivastava operator, $H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ defined by convolution can be written as

$$H_{q,s}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) = {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$$
$$= z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}\, a_k z^k}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(k-1)!}$$
$$= H_{q,s}[\alpha_1]f(z)$$

where $*$ stands for convolution. Al-Kharsani and Al-Khal [3] introduced the modified Dziok-Srivastava operator of the harmonic function $f = h + \overline{g}$ given by (1.1) as

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}. \qquad (1.4)$$

Motivated from the work of Al-Khal [2], a new class of harmonic univalent function of the form (1.1) denoted as $S_H(\alpha_1;\rho;\beta)$ is introduced such that the functions in this class satisfy the condition

$$\text{Re}\left\{(1+\rho e^{i\gamma})\frac{z\,(H_{q,s}[\alpha_1]f(z))'}{(H_{q,s}[\alpha_1]f(z))} - \rho e^{i\gamma}\right\} \geq \beta, \qquad (1.5)$$

where $z = re^{i\theta}$, $(H_{q,s}[\alpha_1]f(z))' = \frac{\partial}{\partial\theta}H_{q,s}[\alpha_1]f(re^{i\theta})$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 \leq \beta < 1$, $0 \leq \rho \leq 1$, $\gamma \in R$ and $H_{q,s}[\alpha_1]f(z)$ is defined by (1.4).

We also let $S_{\overline{H}}(\alpha_1;\rho;\beta)$ denote the subclass of harmonic univalent functions, $S_H(\alpha_1;\rho;\beta)$ so that h and g are in the form (1.2).

2 Main Results

We begin by giving the sufficient condition for functions in $S_H(\alpha_1;\rho;\beta)$.

Theorem 2.1. *Let a function* $f = h + \overline{g}$ *be given by* (1.1). *If,*

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)k-\rho-\beta}{1-\beta}|a_k| + \frac{(1+\rho)k+\rho+\beta}{1-\beta}|b_k|\right]\Gamma(\alpha_1,k) \leq 2 \qquad (2.1)$$

where $a_1 = 1$, $0 \leq \beta < 1, 0 \leq \rho \leq 1$ *and* $\Gamma(\alpha_1,k) = \left|\frac{(\alpha_1)_{k-1},\ldots,(\alpha_q)_{k-1}}{(\beta_1)_{k-1},\ldots,(\beta_s)_{k-1}}\right|$, *then* f *is a sense- preserving harmonic univalent function in* U *and* $f \in S_H(\alpha_1;\rho;\beta)$.

Proof. The inequality $|h'(z)| \geq |g'(z)|$ is sufficient to show that f is sense preserving. Notice that

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k\,|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} k\,|a_k|$$

$$\geq 1-\sum_{k=2}^{\infty}\frac{(1+\rho)k-\rho-\beta}{1-\beta}\,\Gamma(\alpha_1,k)\,|a_k|$$
$$\geq \sum_{k=1}^{\infty}\frac{(1+\rho)k+\rho+\beta}{1-\beta}\,\Gamma(\alpha_1,k)\,|b_k|$$
$$\geq \sum_{k=1}^{\infty}k|b_k||z|^{k-1}$$
$$\geq |g'(z)|.$$

Whenever the co-analytic part is identically zero, then $f(z)$ will become an analytic univalent function. Conversely, if the co-analytic part is not equal to zero, then $f(z_1)\neq f(z_2)$ whenever $z_1\neq z_2$ must be shown. Notice that U is simply connected and convex hence we have $z(t)=(1-t)z_1+t\,z_2\in U$ with the condition $0\leq t\leq 1$ and if $z_1,z_2\in U$ hence $z_1\neq z_2$. Thus,

$$H_{q,s}[\alpha_1]f(z_1)-H_{q,s}[\alpha_1]f(z_2)=\int_0^1\left[(z_1-z_2)[H_{q,s}[\alpha_1]h(z(t))]' + \overline{(z_1-z_2)[H_{q,s}[\alpha_1]\,g(z(t))]'}\right]dt.$$

Dividing by $z_1-z_2\neq 0$, and taking the real parts so that we have

$$\operatorname{Re}\frac{H_{q,s}[\alpha_1]f(z_1)-H_{q,s}[\alpha_1]f(z_2)}{z_1-z_2}=\int_0^1\left(\operatorname{Re}[H_{q,s}[\alpha_1]h(z(t))]' + \frac{\overline{(z_1-z_2)}}{(z_1-z_2)}\,\overline{[H_{q,s}[\alpha_1]g(z(t))]'}\right)dt$$
$$>\int_0^1\left[\operatorname{Re}[H_{q,s}[\alpha_1]h(z(t))]'-\left|[H_{q,s}[\alpha_1]g(z(t))]'\right|\right]dt.$$

On the other hand,

$$\operatorname{Re}\left[[H_{q,s}[\alpha_1]h(z(t))]'-\left|[H_{q,s}[\alpha_1]g(z(t))]'\right|\right]$$
$$\geq \operatorname{Re}\,[H_{q,s}[\alpha_1]h(z)]'-\sum_{k=1}^{\infty}k\,\Gamma(\alpha_1,k)|b_k|$$
$$=1-\sum_{k=2}^{\infty}k\,\Gamma(\alpha_1,k)\,|a_k|-\sum_{k=1}^{\infty}k\,\Gamma(\alpha_1,k)\,|b_k|$$
$$\geq 2-\sum_{k=1}^{\infty}\frac{(1+\rho)k-\rho-\beta}{1-\beta}\Gamma(\alpha_1,k)|a_k|-\sum_{k=1}^{\infty}\frac{(1+\rho)k+\rho+\beta}{1-\beta}\Gamma(\alpha_1,k)|b_k|$$
$$\geq 2-\left\{\sum_{k=1}^{\infty}\frac{(1+\rho)k-\rho-\beta}{1-\beta}\Gamma(\alpha_1,k)|a_k|+\sum_{k=1}^{\infty}\frac{(1+\rho)k+\rho+\beta}{1-\beta}\Gamma(\alpha_1,k)|b_k|\right\}\geq 0.$$

(by (2.1))

This leads to the univalence of f.

According to the condition (1.5), we just need to show that if (2.1) holds, then

$$\operatorname{Re}\left\{\frac{(1+\rho e^{i\gamma})\,z\,[(H_{q,s}[\alpha_1]h(z))'-\overline{(H_{q,s}[\alpha_1]g(z))'}]}{H_{q,s}[\alpha_1]h(z)+\overline{H_{q,s}[\alpha_1]g(z)}}-\rho e^{i\gamma}\right\}=\operatorname{Re}\left\{\frac{A(z)}{B(z)}\right\}\geq\beta,$$

where

$z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta < 2\pi, 0 \le \beta < 1, 0 \le \rho \le 1$, $\gamma \in R$, $(H_{q,s}[\alpha_1]f(z))' = \frac{\partial}{\partial\theta} H_{q,s}[\alpha_1]f(re^{i\theta})$.

Note that

$A(z) = (1+\rho e^{i\gamma})\, z[(H_{q,s}[\alpha_1]h(z))' - \overline{(H_{q,s}[\alpha_1]\, g(z))'}\,] - \rho e^{i\gamma}(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]\, g(z)})$

and $B(z) = H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}$.

Considering the fact that $\operatorname{Re} w \ge \beta$ if and only if $|1-\beta+w| \ge |1+\beta-w|$, it suffices to show that

$$|A(z)+(1-\beta)B(z)| - |A(z)-(1+\beta)B(z)| \ge 0. \tag{2.2}$$

Substituting $A(z)$ and $B(z)$ in (2.2) yields

$$\begin{aligned}
&\Big|(1-\beta-\rho e^{i\gamma})H_{q,s}[\alpha_1]h(z) + (1+\rho e^{i\gamma})(z(H_{q,s}[\alpha_1]h(z))') \\
&\qquad + (1-\beta-\rho e^{i\gamma})\overline{H_{q,s}[\alpha_1]g(z)} - (1+\rho e^{i\gamma})(z\,\overline{(H_{q,s}[\alpha_1]g(z))'})\Big| \\
&\qquad - \Big|(1+\beta+\rho e^{i\gamma})H_{q,s}[\alpha_1]h(z) - (1+\rho e^{i\gamma})\, z\, (H_{q,s}[\alpha_1]h(z))' \\
&\qquad + (1+\beta+\rho e^{i\gamma})(\overline{H_{q,s}[\alpha_1]\, g(z)} + (1+\rho e^{i\gamma})\, z\,(\overline{H_{q,s}[\alpha_1]g(z))'}\Big| \\
&\ge (2-\beta)|z| - \sum_{k=2}^{\infty} [1-\beta+k+\rho k-\rho]\, \Gamma(\alpha_1,k)\, |a_k||z|^k \\
&\quad - \sum_{k=1}^{\infty} [k+\beta-1+\rho k+\rho]\, \Gamma(\alpha_1,k)\, |b_k||z|^k - \beta|z| \\
&\quad - \sum_{k=2}^{\infty} [k-\beta-1+\rho k-\rho]\, \Gamma(\alpha_1,k)\, |a_k||z|^k - \sum_{k=1}^{\infty} [k+1+\beta+\rho k+\rho]\, \Gamma(\alpha_1,k)\, |b_k||z|^k \\
&= 2(1-\beta)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{(k-\beta+\rho k-\rho)}{(1-\beta)}\, \Gamma(\alpha_1,k)|a_k|\,|z|^{k-1} \right. \\
&\qquad \left. - \sum_{k=1}^{\infty} \frac{(k+\beta+\rho k+\rho)}{(1-\beta)}\, \Gamma(\alpha_1,k)|b_k|\,|z|^{k-1} \right\} \\
&\ge 2(1-\beta)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{(k-\beta+\rho k-\rho)}{(1-\beta)}\, \Gamma(\alpha_1,k)|a_k| \right. \\
&\qquad \left. - \sum_{k=1}^{\infty} \frac{(k+\beta+\rho k+\rho)}{(1-\beta)}\, \Gamma(\alpha_1,k)|b_k| \right\} \ge 0 \text{ by (2.1).}
\end{aligned}$$

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\beta}{[(1+\rho)k-\rho-\beta]\Gamma(\alpha_1,k)}\, x_k\, z^k + \sum_{k=1}^{\infty} \frac{1-\beta}{[(1+\rho)k+\rho+\beta]\Gamma(\alpha_1,k)}\, \overline{y}_k\, \overline{z}^k \tag{2.3}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp. Thus, the functions in the form of the harmonic function $f(z)$ above are in the class of $S_H(\alpha_1;\rho;\beta)$ because

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)k-\rho-\beta}{1-\beta}\Gamma(\alpha_1,k)\,|a_k|+\frac{(1+\rho)k+\rho+\beta}{1-\beta}\Gamma(\alpha_1,k)\,|b_k|\right]$$
$$=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

□

Next, we show the condition (2.1) is also necessary for functions in $S_{\overline{H}}(\alpha_1;\rho;\beta)$.

Theorem 2.2. *Let* $f=h+\overline{g}$ *be given by* (1.3). *Hence,* $f\in S_{\overline{H}}(\alpha_1;\rho;\beta)$ *if and only if*

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)k-\rho-\beta}{1-\beta}|a_k|+\frac{(1+\rho)k+\rho+\beta}{1-\beta}|b_k|\right]\Gamma(\alpha_1,k)\le 2 \tag{2.4}$$

where $a_1=1,\ 0\le\beta<1, 0\le\rho\le1.$

Proof. Since $S_{\overline{H}}(\alpha_1;\rho;\beta)\subset S_H(\alpha_1;\rho;\beta)$, we just need to prove the "only if" part of the theorem. A necessary and sufficient bound for $f=h+\overline{g}$ given by (1.2) to be in the class $S_{\overline{H}}(\alpha_1;\rho;\beta)$ is that

$$\mathrm{Re}\left\{(1+\rho e^{i\gamma})\frac{z(H_{q,s}[\alpha_1]f(z))'}{H_{q,s}[\alpha_1]f(z)}-\rho e^{i\gamma}\right\}\ge\beta.$$

This is equivalent to

$$\mathrm{Re}\left\{\frac{(1+\rho e^{i\gamma})\,[(zH_{q,s}[\alpha_1]h(z)'-(\overline{zH_{q,s}[\alpha_1]g(z)})']-\rho e^{i\gamma}\,(H_{q,s}[\alpha_1]h(z)+\overline{H_{q,s}[\alpha_1]g(z)})}{H_{q,s}[\alpha_1]h(z)+\overline{H_{q,s}[\alpha_1]g(z)}}-\beta\right\}$$

$$=\mathrm{Re}\left\{\frac{z-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)k|a_k|z^k+\rho e^{i\gamma}z-\rho e^{i\gamma}\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)k|a_k|z^k-\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)k|b_k|\bar{z}^k}{z-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|z^k+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)|b_k|\bar{z}^k}\right.$$
$$\left.\frac{-\rho e^{i\gamma}\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)|b_k|\bar{z}^k-\beta z+\beta\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|z^k-\beta\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)|b_k|\bar{z}^k}{z-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|z^k+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)|b_k|\bar{z}^k}\right\}$$

$$= \operatorname{Re}\left\{\frac{(1-\beta)z-\sum_{k=2}^{\infty}[k+k\rho e^{i\gamma}-\rho e^{i\gamma}-\beta]\,\Gamma(\alpha_1,k)\,|a_k|z^k}{z-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)\,|a_k|z^k+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)\,|b_k|\bar{z}^k}-\frac{\sum_{k=1}^{\infty}[k+k\rho e^{i\gamma}+\rho e^{i\gamma}+\beta]\,\Gamma(\alpha_1,k)\,|b_k|\bar{z}^k}{z-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)\,|a_k|z^k+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)\,|b_k|\bar{z}^k}\right\}$$

$$\geq 0$$

By choosing z on the positive real axis $0 \leq z = r < 1$ and $\operatorname{Re}(-e^{i\gamma}) \geq -|e^{i\gamma}| = -1$,

$$\operatorname{Re}\left\{\frac{(1-\beta)-\sum_{k=2}^{\infty}[(1+\rho)k-\rho-\beta]\,\Gamma(\alpha_1,k)\,|a_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)\,|b_k|\,r^{k-1}}-\frac{\sum_{k=1}^{\infty}[(1+\rho)k+\rho+\beta]\Gamma(\alpha_1,k)|b_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}\Gamma(\alpha_1,k)\,|b_k|\,r^{k-1}}\right\}$$

$$\geq 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist a $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative This contradicts the condition for $f \in S_{\overline{H}}(\alpha_1;\rho;\beta)$. Hence, the proof is complete. □

The following theorem is on the distortion bounds for the class $S_{\overline{H}}(\alpha_1;\rho;\beta)$.

Theorem 2.3: *If* $f \in S_{\overline{H}}(\alpha_1;\rho;\beta)$, $|z|<1$, *then*

$$|f(z)| \leq \left(1+|b_1|\right)r+\frac{1}{\Gamma(\alpha_1,2)}\left[\frac{1-\beta}{2+\rho-\beta}-\frac{1+2\rho+\beta}{2+\rho-\beta}\,|b_1|\right]r^2, \quad |z|=r<1$$

and

$$|f(z)| \geq \left(1-|b_1|\right)r-\frac{1}{\Gamma(\alpha_1,2)}\left[\frac{1-\beta}{2+\rho-\beta}-\frac{1+2\rho+\beta}{2+\rho-\beta}\,|b_1|\right]r^2, \quad |z|=r<1.$$

Proof. *Let* $f \in S_{\overline{H}}(\alpha_1;\rho;\beta)$, $|z|<1$. Taking the absolute value of f, we have

$$|f(z)| \leq \left(1+|b_1|\right)r+\sum_{k=2}^{\infty}\left[|a_k|+|b_k|\right]r^k$$

$$\leq \left(1+|b_1|\right)r+\sum_{k=2}^{\infty}\left[|a_k|+|b_k|\right]r^2$$

$$=\left(1-|b_1|\right)r+\left[\frac{1-\beta}{(2+p-\beta)\,\Gamma(\alpha_1,2)}\right]\left\{\sum_{k=2}^{\infty}\left[\frac{(2+p-\beta)}{1-\beta}|a_k|+\frac{(2+p-\beta)}{1-\beta}|b_k|\right]\Gamma(\alpha_1,2)r^2\right\}$$

$$\leq \left(1+|b_1|\right)r+\left[\frac{1-\beta}{(2+p-\beta)\,\Gamma(\alpha_1,2)}\right]\left\{\sum_{k=2}^{\infty}\left[\frac{(1+p)k-\rho-\beta}{1-\beta}|a_k|+\frac{(1+p)k+\rho+\beta}{1-\beta}|b_k|\right]\Gamma(\alpha_1,k)r^2\right\}$$

$$\leq \left(1+|b_1|\right) r + \frac{1-\beta}{(2+\rho-\beta)\Gamma(\alpha_1,2)} \left(1 - \frac{1+2\rho+\beta}{1-\beta} |b_1|\right) r^2$$

$$= \left(1+|b_1|\right) r + \frac{1}{\Gamma(\alpha_1,2)} \left(\frac{1-\beta}{(2+\rho-\beta)} - \frac{1+2\rho+\beta}{2+\rho-\beta} |b_1|\right) r^2.$$

and

$$|f(z)| \geq \left(1-|b_1|\right) r - \sum_{k=2}^{\infty} \left[|a_k|+|b_k|\right] r^k$$

$$\geq \left(1-|b_1|\right) r - \sum_{k=2}^{\infty} \left[|a_k|+|b_k|\right] r^2$$

$$= \left(1-|b_1|\right) r - \left[\frac{1-\beta}{(2+p-\beta)\,\Gamma(\alpha_1,2)}\right] \left\{ \sum_{k=2}^{\infty} \left[\frac{(2+p-\beta)}{1-\beta} |a_k| + \frac{(2+p-\beta)}{1-\beta} |b_k|\right] \Gamma(\alpha_1,2) r^2 \right\}$$

$$\geq \left(1-|b_1|\right) r - \left[\frac{1-\beta}{(2+p-\beta)\,\Gamma(\alpha_1,2)}\right] \left\{ \sum_{k=2}^{\infty} \left[\frac{(1+p)k-p-\beta}{1-\beta} |a_k| + \frac{(1+p)k+p+\beta}{1-\beta} |b_k|\right] \Gamma(\alpha_1,k) r^2 \right\}$$

$$\geq \left(1-|b_1|\right) r - \frac{1-\beta}{(2+\rho-\beta)\,\Gamma(\alpha_1,2)} \left(1 - \frac{1+2\rho+\beta}{1-\beta} |b_1|\right) r^2$$

$$= \left(1-|b_1|\right) r - \frac{1}{\Gamma(\alpha_1,2)} \left(\frac{1-\beta}{(2+\rho-\beta)} - \frac{1+2\rho+\beta}{(2+\rho-\beta)} |b_1|\right) r^2.$$

□

Now, the extreme points will be examined. The closed convex cover (or hull) is denoted as *clco* of $S_{\overline{H}}(\alpha;\rho;\beta)$.

Theorem 2.4 *$f \in clco\, S_{\overline{H}}(\alpha;\rho;\beta)$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k + Y_k g_k\right), \tag{2.6}$$

where $h_1(z) = z$, $h_k(z) = z - \dfrac{1-\beta}{[(1+\rho)k-\rho-\beta]\Gamma(\alpha_1,k)} z^k$, $(k = 2,3,...)$,

$g_k(z) = z + \dfrac{1-\beta}{[(1+\rho)k+\rho+\beta]\Gamma(\alpha_1,k)} \bar{z}^k$, $(k = 1,2,3,...)$, $\sum_{k=1}^{\infty}\left(X_k + Y_k\right) = 1, X_k \geq 0,\ Y_k \geq 0.$

In particular, the extreme points of $f \in S_{\overline{H}}(\alpha;\rho;\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For the function f of the form (2.6), we have

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z) \right)$$

$$=\sum_{k=1}^{\infty}\left[X_k+Y_k\right]z-\sum_{k=2}^{\infty}\left[\frac{1-\beta}{\left[(1+\rho)k-\rho-\beta\right]\Gamma(\alpha_1,k)}\right]X_k z^k$$
$$+\sum_{k=1}^{\infty}\left[\frac{1-\beta}{\left[(1+\rho)k+\rho+\beta\right]\Gamma(\alpha_1,k)}\right]Y_k \bar{z}^k.$$

Then,

$$\sum_{k=2}^{\infty}\frac{\left[(1+\rho)k-\rho-\beta\right]\Gamma(\alpha_1,k)}{1-\beta}\left(\frac{1-\beta}{\left[(1+\rho)k-\rho-\beta\right]\Gamma(\alpha_1,k)}X_k\right)+$$
$$\sum_{k=1}^{\infty}\frac{\left[(1+\rho)k+\rho+\beta\right]\Gamma(\alpha_1,k)}{1-\beta}\left(\frac{1-\beta}{\left[(1+\rho)k-\rho-\beta\right]\Gamma(\alpha_1,k)}Y_k\right)$$
$$=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k$$
$$=1-X_1\le 1.$$

Hence, $f\in clco\,S_{\overline{H}}(\alpha_1;\rho;\beta)$. Conversely, suppose that $f\in clco\,S_{\overline{H}}(\alpha_1;\rho;\beta)$ .Set

$$X_k=\frac{[(1+\rho)k-\rho-\beta]\,\Gamma(\alpha_1,k)}{1-\beta}\,|a_k|;\;\; Y_k=\frac{[(1+\rho)k+\rho+\beta]\,\Gamma(\alpha_1,k)}{1-\beta}\,|b_k|;\;\; k=(2,3,...)$$

Note that by Theorem 2.2, $0\le X_k\le 1$ and $0\le Y_k\le 1$.

Define $X_1=1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k$ and since $X_1\ge 0$,

we have $f(z)=\sum_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right)$.

This complete the proof of the theorem. □

Acknowledgements. The first author would like to thank Universiti Teknologi MARA and the support of the grant 600-RMI/FRGS5/3/(7/2013) in the peparation of this paper.

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Received: April 12, 2016; Published: October 18, 2016