Absolute Summability Factor $\varphi - |C, 1; \delta|_k$

of Infinite Series

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Abstract

In this paper, we established a generalized theorem on absolute summability factors by applying a recently defined absolute Cesàro summability $\varphi - |C, 1; \delta|_k$ and the concept of a quasi-$f$-power increasing sequence for infinite series. We further obtained a well-known result under suitable conditions.

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1 Introduction

Let \( \{s_n\} \) be a sequence of partial sums of the infinite series \( \sum_{n=0}^{\infty} a_n \) and \( n^{th} \) sequence to sequence transformation (mean) of \( \{s_n\} \) is given by \( t_n \) s.t.

\[
t_n = \sum_{k=0}^{\infty} t_{nk}s_k
\]

where \( \{t_{nk}\} \) is the sequence of the coefficients of the matrix. The series \( \sum_{n=0}^{\infty} a_n \) is said to be absolute summable, if

\[
\lim_{n \to \infty} t_n = s,
\]

and

\[
\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty.
\]

If \( \tau_n \) represent the \( n^{th} \) \((C,1)\) means of the sequence \((na_n)\), then series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \( |C, 1|_k, k \geq 1 \) [11], if

\[
\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n|^k < \infty.
\]

Let \( (\varphi_n) \) be a sequence of positive real numbers, then the series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \( \varphi - |C, 1|_k, k \geq 1 \), if

\[
\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |\tau_n|^k < \infty
\]

and is also summable \( \varphi - |C, 1; \delta|_k, k \geq 1, \delta \geq 0 \), if

\[
\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k - \delta k} |\tau_n|^k < \infty.
\]

**Note:** If we take \( \varphi = n \), then \( \varphi - |C, 1, \delta|_k \) summability reduces to \( |C, 1|_k \) summability and if \( \delta = 0 \), then \( |C, 1, \delta|_k \) reduces to \( |C, 1|_k \).

Bor gave a number of theorems on absolute summability. In 1991 [1], he established a theorem for infinite series with the help of \( \varphi - |C, 1|_k \) summability. In 1994 [2], he estimated the result by using the absolute summability of index \( k \) for infinite series which is generalization of its own result [3]. Özarslan [4] generalized the result of Bor [1] by a more general absolute summability \( \varphi - |C, \alpha|_k \) and in [5], he used absolute matrix summability \( \varphi - |A, \delta|_k \) and improve some known results. In 2013, Suciu [6] used the Cesáro mean of higher order for extending the result of Lin et. al. [7] and obtained the boundness conditions for Cesáro mean. Concerning the \( \varphi - |N, p_n|_k \) summability factors, Saxena [8] gave a general theorem for infinite series.
2 Known results

By using $\varphi - |C, 1|_k$ summability and a positive non-decreasing sequence $X_n$, Saxena [9] generalized the results of Mazhar [10] and gave the following results.

**Theorem 2.1** Let $\varphi_n$ be a sequence of positive real numbers and satisfy

$$\lambda_m = O(1), \ m \to \infty,$$

$$\sum_{n=v}^{m} \varphi_n^{k-1} = O\left(\frac{\varphi_{v}^{k-1}}{v^k}\right),$$

and $X_n$ be a positive non-decreasing sequence and $(\lambda_n)$ a sequence such that

$$|\lambda_n|X_n = O(1), \ as \ n \to \infty,$$

$$\sum_{n=1}^{m} n|\Delta^2 \lambda_n|X_n = O(1), \ m \to \infty,$$

$$\sum_{v=1}^{m} \varphi_v^{k-1} |t_v|^k = O(X_m \mu_m), \ as \ m \to \infty,$$

where $(\mu_m)$ is a positive non-decreasing sequence such that

$$nX_m \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \ m \to \infty.$$

Then the series $\sum a_n \lambda_n / \mu_n$ is summable $\varphi - |C, 1|_k$, $k \geq 1$.

3 Main results

A quasi-$f$-power increasing sequence is a positive sequence $\chi = (\chi_m)$ which satisfy the following

$$K f_n \chi_n \geq f_m \chi_m,$$

where $K = K(f, \chi) \geq 1$ and $n \geq m \geq 1$. With the help of generalized Cesáro summability $\varphi - |C, 1|_k$ and quasi-$f$-power increasing sequence, we established the following theorem.

**Theorem 3.1** Let $\varphi_n$ be a sequence of positive real numbers. Let $(\Upsilon_n)$ be a quasi-$f$-power increasing sequence, $f = (f_n)$, $f_n = n^\beta (\log n)^\gamma$, $0 < \beta \leq 1$, $\gamma \geq 0$, and $(\lambda_n)$ & $(\mu_n)$ be sequences of numbers such that $(\mu_n)$ is positive non-decreasing sequence satisfying the following

$$\sum_{n=1}^{m} \varphi_n^{k-1} = O\left(\frac{\varphi_{v}^{k-1}}{v^k}\right),$$

where

$$nX_m \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \ m \to \infty.$$
\[
\sum_{n=1}^{\infty} n^{\beta+1} (\log n)^{\gamma} \sum_{n} |\Delta^2 \lambda_n| < \infty,
\]

(13)

\[
\lambda_m = O(1), \quad m \to \infty,
\]

(14)

\[
n^{1+\beta} (\log n)^{\gamma} \sum_{n} \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \quad n \to \infty,
\]

(15)

\[
\sum_{n=2}^{m} \frac{\varphi_{n,k-1}^{-1}}{n^{k-\delta} (n^{\beta} (\log n)^{\gamma} \sum_{n})^{k-1}} = O(m^\beta (\log m)^{\gamma} \sum_{m,\mu_m}), \quad m \to \infty,
\]

(16)

\[
\sum_{n=1}^{\infty} \frac{\lambda_n}{n} < \infty,
\]

(17)

\[
\mu_n \Delta^2 \left(\frac{1}{\mu_n}\right) = O\left(\frac{|\Delta \lambda_n|}{n|\lambda_{n+1}|}\right).
\]

(18)

Then the series \( \sum a_n \lambda_n/\mu_n \) is summable \( \varphi - |C, 1; \delta|_k, \ k \geq 1, \ \delta \geq 0 \).

4 Proof of the Theorem

Let \( T_n \) be the \( n^{th} \) \( (C, 1) \) mean of the sequence \( (na_n \lambda_n/\mu_n) \). The series is \( \varphi - |C, 1; \delta|_k \) summable, if

\[
\sum_{n=1}^{\infty} \frac{\varphi_{n,k-1}^{-1}}{n^{k-\delta} k} |T_n|^k < \infty.
\]

(19)

Applying Able’s transformation, we have

\[
T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v/\mu_v
\]

\[
= \frac{1}{n+1} \left( \sum_{v=1}^{n-1} \left( \sum_{r=1}^{v} r a_r \right) \Delta\left(\frac{\lambda_v}{\mu_v}\right) + \left(\frac{\lambda_n}{\mu_v}\right) \sum_{v=1}^{n} v a_v \right)
\]

\[
= \frac{1}{n+1} \left( \sum_{v=1}^{n-1} (v+1) t_v \Delta\left(\frac{1}{\mu_v}\right) \lambda_v \right) + \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1) t_v \Delta\lambda_v + \frac{t_n \lambda_n}{\mu_n}
\]

\[
= T_{n,1} + T_{n,2} + T_{n,3}.
\]

(20)

Using Minkowski’s inequality,

\[
|T_n|^k = |T_{n,1} + T_{n,2} + T_{n,3}|^k < 3^k \left( |T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k \right).
\]

(21)

In order to complete the proof of the theorem, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{\varphi_{n,k-1}^{-1}}{n^{k-\delta} k} |T_{n,r}|^k < \infty, \ for \ r = 1, 2, 3.
\]

(22)
By using Hölder’s inequality and Abel’s transformation, we have

\[
\sum_{n=2}^{m} \frac{\varphi_{n}^{k-1}}{n^{k-\delta k}} |T_{n,1}|^{k} = \sum_{n=2}^{m} \frac{\varphi_{n}^{k-1}}{n^{k-\delta k}} \left| \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1)t_v \Delta \left( \frac{1}{\mu_v} \right) \lambda_v \right|^{k}
\]

\[
= O(1) \sum_{n=2}^{m} \frac{\varphi_{n}^{k-1}}{n^{2k-\delta k}} \sum_{v=1}^{n-1} v^k |t_v|^k \Delta \left( \frac{1}{\mu_v} \right) |\lambda_v|^k \left( \sum_{v=1}^{n-1} \Delta \left( \frac{1}{\mu_v} \right) \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} |t_v|^k \Delta \left( \frac{1}{\mu_v} \right) |\lambda_v|^k \varphi_{n}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \frac{v^{1+k-\delta k} |t_v|^k |\lambda_v|^k \varphi_{n}^{k-1}}{(v^\beta (log v)^\gamma \ Y_v)^{k-1}} \Delta \left( \frac{1}{\mu_v} \right) (|\lambda_v|^v^\beta (log v)^v^\gamma \ Y_v)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \frac{v^{1+k-\delta k} |t_v|^k |\lambda_v|^k \varphi_{n}^{k-1}}{(v^\beta (log v)^\gamma \ Y_v)^{k-1}} \Delta \left( \frac{1}{\mu_v} \right) \left( \sum_{r=v}^{\infty} r^\beta (log r)^\gamma \ Y_r |\Delta \lambda_r| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \frac{v^{1+k-\delta k} |t_v|^k |\lambda_v|^k \varphi_{n}^{k-1}}{(v^\beta (log v)^\gamma \ Y_v)^{k-1}} \Delta \left( \frac{1}{\mu_v} \right) \left( \sum_{r=v}^{\infty} r^\beta (log r)^\gamma \ Y_r |\Delta \lambda_r| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \frac{|t_v|^k |\lambda_v|^k \varphi_{n}^{k-1}}{(v^\beta (log v)^\gamma \ Y_v)^{k-1}} \Delta \left( \frac{1}{\mu_v} \right)
\]

\[
= O(1) \sum_{v=1}^{m} \left( \sum_{r=1}^{v} r^{k-\delta k} (v^\beta (log r)^\gamma \ Y_r)^{k-1} \right) \Delta \left( |\lambda_v| \Delta \left( \frac{1}{\mu_v} \right) \right)
\]

\[+ O(1) \left( \sum_{v=1}^{m} \frac{|t_v|^k |\lambda_v|^k \varphi_{n}^{k-1}}{(v^\beta (log v)^\gamma \ Y_v)^{k-1}} \right) v |\lambda_v| \Delta \left( \frac{1}{\mu_v} \right)
\]

\[
= O(1) \sum_{v=1}^{m} \frac{|\lambda_v|}{v} + O(1) \sum_{v=1}^{m-1} v^\beta (log v)^\gamma \ Y_v |\Delta \lambda_v| + O(|\lambda_m|)
\]

\[
= O(1).
\]

\[\sum_{n=2}^{m} \frac{\varphi_{n}^{k-1}}{n^{k-\delta k}} |T_{n,2}|^{k} = \sum_{n=2}^{m} \frac{\varphi_{n}^{k-1}}{n^{k-\delta k}} \left| \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1)t_v \Delta \frac{\lambda_v}{\mu_v+1} \right|^{k}
\]
\begin{align*}
&= O(1) \sum_{n=2}^{m} \varphi_n^{k-1} n^{2k-\delta k} \sum_{v=1}^{n-1} \frac{v^k |t_v|^k \Delta \lambda_v}{\mu_v^{k+1} (\nu^\beta (\log v)^\gamma \mathcal{Y}_v)^{k-1}} \times \\
&\times \left( \sum_{v=1}^{n-1} \nu^\beta (\log v)^\gamma \mathcal{Y}_v \Delta \lambda_v \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \frac{|t_v|^k \Delta \lambda_v}{\mu_v^{k+1} (\nu^\beta (\log v)^\gamma \mathcal{Y}_v)^{k-1}} \sum_{n=1}^{m} \varphi_n^{k-1} n^{1+k-\delta k} \\
&= O(1) \sum_{v=1}^{m} \frac{|t_v|^k \Delta \lambda_v}{\mu_v^{k+1} (\nu^\beta (\log v)^\gamma \mathcal{Y}_v)^{k-1}} \text{Var}_{\nu} \left( \frac{\nu^\beta (\log v)^\gamma \mathcal{Y}_v}{\mu_v} \right) \\
&= O(1) \sum_{v=1}^{m} \frac{|t_v|^k \Delta \lambda_v}{\mu_v^{k+1} (\nu^\beta (\log v)^\gamma \mathcal{Y}_v)^{k-1}} \text{Var}_{\nu} \left( \frac{\nu^\beta (\log v)^\gamma \mathcal{Y}_v}{\mu_v} \right) + O(1) m \mathcal{Y}_m |\Delta \lambda_m| \\
&= O(1) \sum_{v=1}^{m} \nu^\beta (\log v)^\gamma \mathcal{Y}_v \mu_v |\Delta \lambda_v| + O(1) \sum_{v=1}^{m} \nu^\beta (\log v)^\gamma \mathcal{Y}_v \mu_v |\Delta \lambda_v| \\
&\quad + O(1) \sum_{v=1}^{m} \nu^\beta (\log v)^\gamma \mathcal{Y}_v |\Delta^2 \lambda_v| + O(1) m \mathcal{Y}_m |\Delta \lambda_m| + O(1) \\
&= O(1).
\end{align*}
\begin{align*}
\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |T_{n,3}|^k &= \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left| \frac{t_n \lambda_n}{\mu_n} \right|^k \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left( \nu^\beta (\log n)^\gamma \mathcal{Y}_n \right)^{k-1} \times \\
&\times \left( \nu^\beta (\log n)^\gamma \mathcal{Y}_n |\lambda_n| \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left( \nu^\beta (\log n)^\gamma \mathcal{Y}_n \right)^{k-1} |\lambda_n| \\
&\quad + \sum_{n=1}^{m} \left( \frac{\nu^\beta (\log n)^\gamma \mathcal{Y}_n}{\mu_n} \right)^{k-1} |\lambda_m| \\
&= O(1) \sum_{n=1}^{m} \nu^\beta (\log n)^\gamma \mathcal{Y}_n \mu_n |\lambda_n| + O(1) \mathcal{Y}_m |\lambda_m|
\end{align*}
Collecting (20) - (25), we have

\[
\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |T_n|^k < \infty.
\]

(26)

Hence proof of the theorem is complete.

5 Corollary

**Corollary 5.1** Let \( \varphi_n \) be a sequence of positive real numbers. Let \((\Upsilon_n)\) be a quasi-\(f\)-power increasing sequence, \(f = (f_n), f_n = n^\beta (\log n)^\gamma, 0 < \beta \leq 1, \gamma \geq 0\), and let \((\lambda_n), (\mu_n)\) be sequences of numbers such that \((\mu_n)\) is positive non-decreasing sequence satisfying (13)-(15), (17), (18) and following

\[
\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{1+k}} = O\left(\frac{\varphi_n^{k-1}}{n^k}\right),
\]

(27)

\[
\sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k(n^\beta (\log n)^\gamma \Upsilon_n)^{k-1}} = O(m^\beta (\log m)^\gamma \Upsilon_m \mu_m), \ m \to \infty.
\]

(28)

Then the series \( \sum a_n \lambda_n/\mu_n \) is summable \( \varphi - |C, 1|_k, k \geq 1 \).

**Proof:** On putting \( \delta = 0 \) in Theorem 3.1, we will get (27) and (28). We omit the details as the proof is similar to that of Theorem 3.1 and we use (27) - (28) instead of (12) and (16).

6 Conclusion

The aim of this research article is to formulate the problem of generalization of absolute Cesáro summability factor \( (\varphi - |C, 1; \delta|_k, k \geq 1, \delta \geq 0) \) of infinite series which is a motivation for the researchers, interested in theoretical studies of infinite series.

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