Fuzzy Coalition Values for Fuzzy Games

Jae Duck Kim

BangMok College of Basic Studies
Myongji University,Kyunggi 449-728, South Korea

Eun Jin Jeong

Department of Mathematics
Myongji University,Kyunggi 449-728, South Korea

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Abstract

In this paper, we introduce some classes of fuzzy coalitions that have special effects on other fuzzy coalitions and study their properties. We also propose fuzzy coalition values which express the fuzzy coalition influence of fuzzy games and prove that such fuzzy coalition values have null and dummy properties.

Mathematics Subject Classification: 91A12, 94D05

Keywords: cooperative game, fuzzy game, coalition value, fuzzy coalition

1 Introduction

The basis of cooperative game theory was introduced by von Neumann and Morgenstern [13]. Since then many authors have proposed several solution concepts for cooperative games and introduced several interesting subclasses of games [[3], [4], [8], [9]]. In classical cooperative game theory, the characteristic function describes the maximal gain or minimal cost each subset of players can achieve when they decide to cooperate, regardless of the actions of the other players. Thus, it is one of the principal problems of group decision making
theories to compare the maximal gain or minimal cost of the two coalitions. From such comparisons, we can estimate what final decision is likely to be made and what coalitions are likely to be formed. The comparability of a desirability relation in some classes of games was discussed by many authors [[4], [5], [11], [12], [14]]. In particular, Kojima and Inohara [6] introduced the methods known as blockability and viability for comparison of coalition influence in games.

The concept of cooperative fuzzy games was introduced by Aubin [1] in 1981. It is based on fuzzy coalitions that reflect situations where agents have the possibility of cooperating at different participation levels ranging from non-cooperation to full cooperation and where the reward obtained depends on the levels of participation. Since the cooperation fuzzy game is an extension of cooperative games for fuzzy theory, we shall compare the fuzzy coalition influence for fuzzy games in cooperative fuzzy game theory.

For this purpose, we extend the definition of some players and coalitions introduced in [[6], [7], [10]] to cooperative fuzzy games and study the properties of some fuzzy players and fuzzy coalitions. Also, we extend the concepts of blockability and viability introduced by Kojima and Inohara [6] to cooperative fuzzy games. Moreover, we shall prove some properties of fuzzy coalitions.

Given a finite set \( N = \{1, 2, \ldots, n\} \) of players, a fuzzy coalition is a vector \( s = (s_1, s_2, \ldots, s_n) \in [0, 1]^n \) where \( s_i \) represents the participation level of player \( i \) in situation \( s \). For convenience, we use the notation \( F^N \) to denote the set of all fuzzy coalitions instead of \([0, 1]^n\). A crisp coalition \( S \subseteq N \) can also be regarded as the fuzzy coalition \( e^S = (s_1, s_2, \ldots, s_n) \in F^N \) in which \( s_i = 1 \) if \( i \in S \) and \( s_i = 0 \) if \( i \notin S \). The fuzzy coalition \( e^N = (1, 1, \ldots, 1) \) is the grand coalition, and the fuzzy coalition \( e^0 = (0, 0, \ldots, 0) \) is the empty crisp coalition.

For \( s \in F^N \) we define the carrier of \( s \) by \( \text{car}(s) = \{i \in N | s_i > 0\} \) and the complement of \( s \) by \( s^- = (1-s_1, \ldots, 1-s_n) \). For \( s, t \in F^N \), \( s \leq t \) means \( s_i \leq t_i \) for every \( i \in N \), and \( s < t \) means that \( s \leq t \) but \( s \neq t \). We define \( s \wedge t = (\min(s_1, t_1), \ldots, \min(s_n, t_n)) \), \( s \vee t = (\max(s_1, t_1), \ldots, \max(s_n, t_n)) \) and \( s \ominus t = s \wedge t^- \). The operations \( \vee \) and \( \wedge \) play the same roles as the union and intersection for fuzzy coalitions as they do crisp coalitions.

A fuzzy game \((N, v)\) is defined by a characteristic function \( v : F^N \rightarrow \mathbb{R} \) with \( v(e^0) = 0 \). The map \( v \) assigns each fuzzy coalition to a number that indicates what such a coalition can achieve in cooperation. The map \( v \) defined in the fuzzy game \((N, v)\) can also be regarded as the payoff for a specific fuzzy coalition \( s \).

2 Fuzzy players and Fuzzy coalitions

In this section, we recall some types of players and coalitions in classical cooperative games [[5], [6], [7]] and extend these concepts in cooperative games
to cooperative fuzzy games.

Consider a game \((N,v)\) and \(i, j \in N\). A player \(i\) is said to be a null player if and only if \(v(S \cup \{i\}) = v(S)\) for all \(S \subseteq N \setminus \{i\}\), player \(i\) is said to be a dummy player if and only if \(v(T \cup \{i\}) = v(T) + v(\{i\})\) for all \(S \subseteq N \setminus \{i\}\), and player \(i\) and player \(j\) are said to be symmetric players if and only if \(v(T \cup \{i\}) = v(T \cup \{j\})\) for all \(T \subseteq N \setminus \{i, j\}\). A player is a null coalition if and only if \(v(S \cup \{i\}) = v(S)\) for all \(S \subseteq N \setminus \{i\}\), a coalition is a dummy coalition if and only if \(i\) is a dummy player in \((N,v)\) for all \(i \in S\), and \(S^1\) and \(S^2\) are said to be symmetric if and only if there exists bijection \(f : S^1 \to S^2\) such that \(i\) and \(f(i)\) are symmetric players in \((N,v)\).

From now on, we take a closer look at the meanings of null player, dummy player, and symmetric players.

A null player is a player who does not affect payoff when it is contained within other coalition. The dummy player has no additional contribution through cooperation when it is contained in other coalition. And symmetric players do not affect payoff even if they are contained in any other coalition.

The next example shows that such fuzzy players are in fuzzy cooperation game.

**Example 2.1.** Consider a pair \((N,v)\) such that \(N = \{1, 2, 3\}\). For \(s = (s_1, s_2, s_3) \in \mathcal{F}^N\) a characteristic function \(v\) is defined by

\[
v(s) = \begin{cases} 
5s_1 + s_2 + s_1s_2s_3, & \text{if } 0.2 \leq s_3 \leq 0.8, \\
5s_1 + s_2, & \text{otherwise.}
\end{cases}
\]

Then for all \(s = (s_1, s_2, s_3) \in \mathcal{F}^N\)

\[
v((s_1, s_2, s_3) \lor (0, 0, \alpha)) = v((s_1, s_2, \max\{s_3, \alpha\})) = \begin{cases} 
v((s_1, s_2, s_3)), & \text{if } s_3 \geq \alpha, \\
v((s_1, s_2, \alpha)), & \text{otherwise},
\end{cases}
\]

and

\[
v((s_1, s_2, s_3) \ominus (0, 0, \alpha)) = v((s_1, s_2, \min\{s_3, 1 - \alpha\})) = \begin{cases} 
v((s_1, s_2, s_3)), & \text{if } s_3 \leq 1 - \alpha, \\
v((s_1, s_2, \alpha)), & \text{otherwise},
\end{cases}
\]

for all \(\alpha \in [0, 1]\). This means that for all \(s = (s_1, s_2, s_3) \in \mathcal{F}^N\)

\[
v((s_1, s_2, s_3) \lor (0, 0, 0.1)) = v((s_1, s_2, s_3) \ominus (0, 0, 0.1)) = v((s_1, s_2, s_3))
\]

and

\[
v((s_1, s_2, s_3) \lor (0, 0, 0.9)) = v((s_1, s_2, s_3)).
\]
And
\[ v((1, 1, 0.9) \ominus (0, 0, 0.9)) = v((1, 1, 0.1)) = 6.1 \neq v((1, 1, 0.9)) = 6.9. \]

In case of player 3 in Example 2.1, the participation level affects payoff of coalition. But it is not necessarily otherwise in fuzzy cooperative games. Thus we suppose to define special players with participation level as follows.

**Definition 2.2.** (Null fuzzy player) Let \((N, v)\) be a fuzzy game, and \(i \in N\) and \(\alpha \in (0, 1]\). Player \(i\) is said to be a null fuzzy player with level \(\alpha\) if and only if \(v(t \lor \alpha e^{(i)}) = v(t)\) for all \(t \in \mathcal{F}^N\).

Even if a null fuzzy player joins with any other fuzzy coalitions with participation level \(\alpha\), the null fuzzy player does not contribute toward other fuzzy coalitions.

Consider a fuzzy game \((N, v)\). If player \(i\) is a null fuzzy player with level \(\alpha\), then \(v(\beta e^{(i)}) = 0\) for all \(\beta \in [0, \alpha]\). This shows that a null fuzzy player receives no payoff with participation level less than \(\alpha\).

**Definition 2.3.** (Strong null fuzzy player) Let \((N, v)\) be a fuzzy game, and \(i \in N\) and \(\alpha \in (0, 1]\). Player \(i\) is said to be a strong null fuzzy player with level \(\alpha\) if and only if \(v(t \lor \alpha e^{(i)}) = v(t \ominus \alpha e^{(i)}) = v(t)\) for all \(t \in \mathcal{F}^N\).

**Note.** Let \((N, v)\) be a fuzzy game in Example 2.1. Then player 3 is a null fuzzy player with level \(\alpha\) for all \(\cup (0.8, 1]\) but is not a strong null fuzzy player with level \(\alpha\).

We suppose to define dummy fuzzy player and symmetric fuzzy players with the same participation level as we did for the null fuzzy player.

**Definition 2.4.** (Dummy fuzzy player) Let \((N, v)\) be a fuzzy game and \(i \in N\) and \(\alpha \in (0, 1]\). Player \(i\) is said to be a dummy fuzzy player with level \(\alpha\) if and only if \(v(t \lor \alpha e^{(i)}) = v(t) + v(\alpha e^{(i)})\) for all \(t \in \mathcal{F}^N\) with \(t_i = 0\).

Even if dummy fuzzy players join other fuzzy coalitions with participation level \(\alpha\), there is no additional payoff.

**Definition 2.5.** (Strong dummy fuzzy player) Let \((N, v)\) be a fuzzy game and \(i \in N\) and \(\alpha \in (0, 1]\). Player \(i\) is said to be a strong dummy fuzzy player with level \(\alpha\) if and only if \(v(t \lor \alpha e^{(i)}) = v(t) + v(\alpha e^{(i)})\) for all \(t \in \mathcal{F}^N\) with \(t_i = 0\) and \(v(s \ominus \alpha e^{(i)}) = v(s) - v(\alpha e^{(i)})\) for all \(s \in \mathcal{F}^N\) with \(s_i = 1\).

**Definition 2.6.** (Symmetric fuzzy players) Let \((N, v)\) be a fuzzy game and \(i, j \in N\) and \(\alpha, \beta \in (0, 1]\). Players \(i\) and \(j\) are said to be symmetric fuzzy players with levels \((\alpha, \beta)\) if and only if \(v(t \lor \alpha e^{(i)}) = v(t \lor \beta e^{(j)})\) for all \(t \in \mathcal{F}^N\) with \(t_i = t_j = 0\).
By extending definitions of special fuzzy players to fuzzy coalition, we define special coalition as follows.

**Definition 2.7.** (Null fuzzy coalition) Let \((N, v)\) be a fuzzy game and \(t \in \mathcal{F}^N\). \(t\) is said to be a null fuzzy coalition if and only if \(v(s \lor t) = v(s)\) for all \(s \in \mathcal{F}^N\).

**Definition 2.8.** (Strong null fuzzy coalition) Let \((N, v)\) be a fuzzy game and \(t \in \mathcal{F}^N\). \(t\) is said to be a strong null fuzzy coalition if and only if \(v(s \lor t) = v(s \ominus t) = v(s)\) for all \(s \in \mathcal{F}^N\).

As a null fuzzy player, a null fuzzy coalition makes no additional contribution toward other fuzzy coalitions through cooperation. The following lemma shows that no fuzzy coalitions obtain payoff with lower participation level than a null fuzzy coalition.

**Lemma 2.9.** Let \((N, v)\) be a fuzzy game and \(t\) be a null fuzzy coalition. Then \(v(s) = 0\) for all \(s \in \mathcal{F}^N\) such that \(s \leq t\).

**Proof.** Let \(t \in \mathcal{F}^N\) be a null fuzzy coalition in fuzzy game \((N, v)\). Then

\[
v(t) = v(e^0 \lor t) = v(e^0) = 0.
\]

Hence \(v(s) = v(s \lor t) = v(t) = 0\) for all \(s \in \mathcal{F}^N\) such that \(s \leq t\).

**Proposition 2.10.** Let \((N, v)\) be a fuzzy game and \(t\) be a null fuzzy coalition. Then \(t\) is a null fuzzy coalition in \((N, v)\) if and only if player \(i\) is a null fuzzy player with level \(t_i\) in \((N, v)\) for all \(i \in \text{car}(t)\).

**Proof.** Let \(t \in \mathcal{F}^N\) be a null fuzzy coalition in fuzzy game \((N, v)\). Then, for all \(i \in \text{car}(t)\) and for all \(s \in \mathcal{F}^N\),

\[
v(s \lor t_ie^{(i)}) = v((s \lor t_ie^{(i)}) \lor t) = v(s \lor (t_ie^{(i)} \lor t)) = v(s \lor t) = v(s).
\]

Hence player \(i\) is a null fuzzy player with level \(t_i\).

Assume that \(t \in \mathcal{F}^N\) and that for all \(i \in \text{car}(t)\) player \(i\) is a null fuzzy player with level \(t_i\). Let \(\text{car}(t) = \{i_1, i_2, \ldots, i_k\}\), where \(k = |\text{car}(t)|\). Then for all \(s \in \mathcal{F}^N\)

\[
v(s) = v(s \lor t_{i_1}e^{(i_1)}) = v(s \lor t_{i_1}e^{(i_1)} \lor t_{i_2}e^{(i_2)}) = \cdots = v(s \lor t_{i_1}e^{(i_1)} \lor t_{i_2}e^{(i_2)} \ldots \lor t_{i_k}e^{(i_k)}) = v(s \lor t)
\]

Hence \(t\) is a null fuzzy coalition in \((N, v)\).
Note. Proposition 2.10 shows that the null fuzzy coalition is the extension of the null fuzzy player.

**Definition 2.11.** (Dummy fuzzy coalition) Let \((N, v)\) be a fuzzy game and \(t \in F^N\). Then, \(t\) is said to be a dummy fuzzy coalition if and only if player \(i\) is a dummy fuzzy player with level \(t_i\) for all \(i \in \text{car}(t)\).

**Definition 2.12.** (Strong dummy fuzzy coalition) Let \((N, v)\) be a fuzzy game and \(t \in F^N\). Then, \(t\) is said to be a strong dummy fuzzy coalition if and only if player \(i\) is a strong dummy fuzzy player with level \(t_i\) for all \(i \in \text{car}(t)\).

Since a dummy fuzzy coalition contains only dummy fuzzy players, it seems reasonable that there exists no additional payoff even if the dummy fuzzy coalition joins another fuzzy coalition. The following proposition shows that:

**Proposition 2.13.** Let \((N, v)\) be a fuzzy game and \(t\) be a dummy fuzzy coalition in \((N, v)\). Then, \(v(s \lor t) = v(s) + v(t)\) for all \(s \in F^N\) where \(\text{car}(s) \cap \text{car}(t) = \emptyset\).

**Proof.** Let \(t \in F^N\) be a dummy fuzzy coalition in fuzzy game \((N, v)\), and let \(\text{car}(t) = \{i_1, i_2, \ldots, i_k\}\) where \(k = |\text{car}(t)|\). Then

\[
v(t) = v\left(t_{i_1}e^{\{i_1\}} \lor t_{i_2}e^{\{i_2\}} \lor \ldots \lor t_{i_k}e^{\{i_k\}}\right) \\
= v\left(t_{i_1}e^{\{i_1\}} \lor t_{i_2}e^{\{i_2\}} \lor \ldots \lor t_{i_{k-1}}e^{\{i_{k-1}\}}\right) + v\left(t_{i_k}e^{\{i_k\}}\right) \\
\vdots \\
= v\left(t_{i_1}e^{\{i_1\}}\right) + v\left(t_{i_2}e^{\{i_2\}}\right) + \ldots + v\left(t_{i_k}e^{\{i_k\}}\right) \\
= \sum_{j=1}^{k} v\left(t_{i_j}e^{\{i_j\}}\right).
\]

And so

\[
v(s \lor t) = v\left(s \lor t_{i_1}e^{\{i_1\}} \lor t_{i_2}e^{\{i_2\}} \lor \ldots \lor t_{i_k}e^{\{i_k\}}\right) \\
= v\left(s \lor t_{i_1}e^{\{i_1\}} \lor t_{i_2}e^{\{i_2\}} \lor \ldots \lor t_{i_{k-1}}e^{\{i_{k-1}\}}\right) + v\left(t_{i_k}e^{\{i_k\}}\right) \\
\vdots \\
= v(s) + \sum_{j=1}^{k} v\left(t_{i_j}e^{\{i_j\}}\right) \\
= v(s) + v(t)
\]

for all \(s \in F^N\) where \(\text{car}(s) \cap \text{car}(t) = \emptyset\). \(\square\)

**Definition 2.14.** (Symmetric fuzzy coalition) Let \((N, v)\) be a fuzzy game and \(p, q \in F^N\). \(p\) and \(q\) are said to be symmetric fuzzy coalitions if and only if there exists bijection \(\phi : \text{car}(p) \rightarrow \text{car}(q)\) such that for \(i \in \text{car}(p)\), \(i\) and \(\phi(i)\) are symmetric fuzzy players with level \((p_i, q_{\phi(i)})\) in \((N, v)\).
Note. For all symmetric fuzzy coalitions \( p \) and \( q \), each player of \( p \) is matched with a symmetric fuzzy player of \( q \).

Although symmetric fuzzy coalitions may appear to make the same contributions to other fuzzy coalitions, the next example shows that this is not necessarily true.

**Example 2.15.** Let \((N,v)\) be a fuzzy game, such that \( N = \{1, 2, 3\} \) and a characteristic function \( v \) is defined for \( s = (s_1, s_2, s_3) \in \mathcal{F}^N \)

\[
v(s) = \begin{cases} 1, & \text{if } 0 = s_1 < s_2 \leq 1 \text{ or } 0 = s_2 < s_1 \leq 1 \\ s_1s_2s_3, & \text{otherwise.} \end{cases}
\]

Then, players 1 and 2 are symmetric fuzzy players with level \((\alpha, \beta)\) for all \(\alpha, \beta \in (0, 1]\). So are players 2 and 1. Hence fuzzy coalitions \( p = (0.3, 0.5, 0) \) and \( q = (0.7, 0.3, 0) \) are symmetric fuzzy coalitions. But for \( t = (0, 0, 1) \in \mathcal{F}^N \), \( v(t \lor p) = v(0.3, 0.5, 1) = 0.15 \neq v(t \lor q) = v(0.7, 0.3, 1) = 0.21. \)

Note. From Example 2.15, it is established that symmetric fuzzy coalitions \( p \) and \( q \) do not make the same contributions to fuzzy coalition \( s \).

The next proposition provides sufficient conditions for symmetric fuzzy coalitions to make the same contributions to other fuzzy coalitions.

**Proposition 2.16.** Let \((N,v)\) be a fuzzy game and \( p \) and \( q \) be symmetric fuzzy coalitions in \((N,v)\) such that \( \text{car}(p) \cap \text{car}(q) = \emptyset \). Then, \( v(s \lor p) = v(s \lor q) \) for all \( s \in \mathcal{F}^N \) where \( \text{car}(s) \cap [\text{car}(p) \cup \text{car}(q)] = \emptyset. \)

**Proof.** Suppose that there exists \( \phi : \text{car}(p) \to \text{car}(q) \) such that for all \( i \in \text{car}(p) \), \( i \) and \( \phi(i) \) are symmetric fuzzy players with level \((p_i, q_{\phi(i)})\) in \((N,v)\). Let \( \text{car}(p) = \{i_1, i_2, \ldots, i_k\} \) where \( k = |\text{car}(p)| \). Then for all \( s \in \mathcal{F}^N \) where \( \text{car}(s) \cap [\text{car}(p) \cup \text{car}(q)] = \emptyset \),

\[
v(s \lor p) = v\left(s \lor p_{i_1}e^{\{i_1\}} \lor p_{i_2}e^{\{i_2\}} \lor \cdots \lor p_{i_{k-1}}e^{\{i_{k-1}\}} \lor p_{i_k}e^{\{i_k\}}\right) \\
= v\left(s \lor p_{i_1}e^{\{i_1\}} \lor p_{i_2}e^{\{i_2\}} \lor \cdots \lor p_{i_{k-1}}e^{\{i_{k-1}\}} \lor q_{\phi(i_k)}e^{\{\phi(i_k)\}}\right) \\
= v\left(s \lor q_{\phi(i_1)}e^{\{\phi(i_1)\}} \lor q_{\phi(i_2)}e^{\{\phi(i_2)\}} \lor \cdots \lor q_{\phi(i_k)}e^{\{\phi(i_k)\}}\right) \\
= v(s \lor q)
\]

\(\square\)
3 Fuzzy Coalition values for fuzzy games

In this section, we recall the methods to evaluate coalition influence with numerical values in classical cooperative games \([6]\) and extend these concepts for cooperative games to cooperative fuzzy games.

For a game \((N, v)\), the blockability value of the coalition \(S \subseteq N\) is defined as follows:

\[
\hat{B}_S(N, v) = \frac{\sum_{T \subseteq N} v(T) - B^*(S)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N)
\]

where \(B^*(S) = \sum_{T \subseteq N} v(T \setminus S)\) and \(\sum_{T \subseteq N} v(T) - B^*(N) \neq 0\).

The viability value of coalition \(S \subseteq N\) is defined as follows:

\[
\hat{V}_S(N, v) = \frac{V^*(S)}{V^*(N)} \cdot v(N)
\]

where \(V^*(S) = \sum_{T \subseteq N} v(S \setminus T)\) and \(V^*(S) \neq 0\).

Indeed, \(B^*(N) = 0\) and denominators in both case are the same.

Blockability value and viability value are numerical values to evaluate coalition influence between the two coalitions. In particular, the expression in numerator of blockability is the sum of differences of payoff when the first coalition is eliminated from the given coalition. Hence this means that the bigger numerical value is, the more effect excluding coalition is. However the viability value of coalition has different meaning. Since it is the sum of payoff when other coalitions are eliminated from the given coalition, it means that the bigger number gives the larger valued even when other coalitions are excluded. Thus we suppose to define special values having a similar meaning in fuzzy cooperative game.

**Definition 3.1.** Consider a fuzzy game \((N, v)\). The blockability value of fuzzy coalition \(t \in \mathcal{F}^N\) is defined as follows:

\[
B(t) = \frac{\sum_{U \subseteq N} \left[ v(e^U) - v(e^U \ominus t) \right]}{\sum_{U \subseteq N} \left[ v(e^U) - v(e^U \ominus e^N) \right]} \cdot v(e^N)
\]

where \(\sum_{U \subseteq N} \left[ v(e^U) - v(e^U \ominus e^N) \right] \neq 0\). And the viability value of fuzzy coalition \(t \in \mathcal{F}^N\) is defined as follows:

\[
V(t) = \frac{\sum_{U \subseteq N} v(t \ominus e^U)}{\sum_{U \subseteq N} v(e^N \ominus e^U)} \cdot v(e^N)
\]

where \(\sum_{U \subseteq N} v(e^N \ominus e^U) \neq 0\).
Since the participation level of players joining in contribution of crisp coalition is the highest, we just calculate the sum of difference of their payoff in Definition 3.1. This calculation is simple and it preserves the meaning of fuzzy coalition value.

**Example 3.2.** Consider a pair \((N, v)\) such that \(N = \{1, 2, 3\}\). For \(s = (s_1, s_2, s_3) \in F^N\) a characteristic function \(v\) is defined by

\[
v(s) = 5s_1 + s_2 + s_1s_2s_3.
\]

Then the blockability value and the viability value of \(p = (0.8, 0.2, 0)\) and \(q = (0.2, 0.8, 0)\) are

\[
B(p) = \frac{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus p)]}{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus e^N)]} \cdot v(e^N) = \frac{3087}{625},
\]

\[
V(p) = \frac{\sum_{U \subseteq N} v(p \ominus e^U)}{\sum_{U \subseteq N} v(e^N \ominus e^U)} \cdot v(e^N) = \frac{588}{125},
\]

and

\[
B(q) = \frac{1407}{625}, V(q) = \frac{252}{125}.
\]

In case of \(p = (0.8, 0.2, 0)\) and \(q = (0.2, 0.8, 0)\), both players 1 and 2 joined the game and their participation levels sum are the same, but \(B(p) > B(q)\) and \(V(p) > V(q)\). This means that the player 1 has a larger coefficient of payoff than player 2 has. Thus the coalition value may be used instead of fuzzy coalition values in fuzzy game.

In section 2, we introduced some types of fuzzy coalitions. Since fuzzy coalition values evaluate the influence of fuzzy coalition with numerical values, it is reasonable that fuzzy coalition values represent the concept of fuzzy coalitions. We shall mention some desirable properties for fuzzy coalition values.
Null Fuzzy Coalition Property (NFP). Consider a fuzzy game \((N,v)\) and a fuzzy coalition value \(\phi\). Coalition value \(\phi\) is said to satisfy the null fuzzy property if and only if \(\phi(s) = 0\) for all null coalitions \(s \in \mathcal{F}^N\).

Strong Null Fuzzy Coalition Property (SNFP). Consider a fuzzy game \((N,v)\) and a fuzzy coalition value \(\phi\). Coalition value \(\phi\) is said to satisfy the strong null fuzzy property if and only if \(\phi(s) = 0\) for all strong null coalitions \(s \in \mathcal{F}^N\).

Dummy Fuzzy Coalition Property (DFP). Consider a fuzzy game \((N,v)\) and a fuzzy coalition value \(\phi\). Fuzzy coalition value \(\phi\) is said to satisfy the dummy fuzzy coalition property if and only if for all dummy fuzzy coalitions \(s \in \mathcal{F}^N\),

\[
\phi(s) = \sum_{i \in \text{car}(s)} \phi\left(s_ie\{i\}\right).
\]

Strong Dummy Fuzzy Coalition Property (SDFP). Consider a fuzzy game \((N,v)\) and a fuzzy coalition value \(\phi\). Fuzzy coalition value \(\phi\) is said to satisfy the dummy fuzzy coalition property if and only if for all dummy fuzzy coalitions \(s \in \mathcal{F}^N\),

\[
\phi(s) = \sum_{i \in \text{car}(s)} \phi\left(s_ie\{i\}\right).
\]

From definitions of null fuzzy coalition and dummy fuzzy coalition, it is easy to see that for fuzzy coalition value \(\phi\), NFP implies SNFP and DFP implies SDFP.

Since null fuzzy coalitions make no contributions to other fuzzy coalitions, it may seem that the blockability value of fuzzy coalitions would receive zero evaluation. The next proposition shows how coalition values satisfy some properties for fuzzy coalition values.

Proposition 3.3. Viability value satisfies NFP and DFP.

Proof. Let \((N,v)\) be a fuzzy game.

(NFP) Let \(t \in \mathcal{F}^N\) be a null fuzzy coalition in \((N,v)\). Since \(t \oplus e^U \leq t\) for all \(U \subset N\),

\[
v(t \oplus e^U) = 0
\]

by Lemma 2.9. Hence \(V(t) = 0\).

(DFP) Let \(t \in \mathcal{F}^N\) be a dummy fuzzy coalition in \((N,v)\). Since

\[
(t \oplus e^U)_i = \begin{cases} t_i, & \text{if } i \in \text{car}(t) \setminus U, \\ 0, & \text{otherwise,} \end{cases}
\]

for \(U \in N\),

\[
v(t \oplus e^U) = \sum_{i \in \text{car}(t) \setminus U} v(t_ie\{i\}) = \sum_{i \in \text{car}(t)} v(t_ie\{i\} \oplus e^U).
\]
Fuzzy coalition values for fuzzy games

Hence

\[
V(t) = \frac{\sum_{U \subseteq N} v(t \ominus e^U)}{\sum_{U \subseteq N} v(e^N \ominus e^U)} \cdot v(e^N)
\]

\[
= \frac{\sum_{U \subseteq N} \left[ \sum_{i \in \text{car}(t)} v(t_i e^{\{i\}} \ominus e^U) \right]}{\sum_{U \subseteq N} v(e^N \ominus e^U)} \cdot v(e^N)
\]

\[
= \sum_{i \in \text{car}(t)} \left[ \sum_{U \subseteq N} v(t_i e^{\{i\}} \ominus e^U) \cdot v(e^N) \right]
\]

\[
= \sum_{i \in \text{car}(t)} V(t_i e^{\{i\}}).
\]

\[\square\]

**Proposition 3.4.** Blockability value satisfies SNFP and SDFP.

**Proof.** Let \((N, v)\) be a fuzzy game.

(SNFP) It is trivial by the definitions of strong null fuzzy coalition and blockability.

(SDFP) Let \(t \in \mathcal{F}^N\) be a strong dummy fuzzy coalition in \((N, v)\). And let \(U \cap \text{car}(t) = \{i_1, \ldots, i_k\}\) and \(k \in \text{car}(t)\) for \(U \subseteq N\). Then

\[
e^U \ominus t = e^U \land t^-
\]

\[
= e^U \land [(1 - t_{i_1})e^{\{i_1\}} \lor \ldots \lor (1 - t_{i_k})e^{\{i_k\}}]
\]

\[
= [e^U \land [(1 - t_{i_1})e^{\{i_1\}} \lor \ldots \lor (1 - t_{i_{k-1}})e^{\{i_{k-1}\}} \lor e^{\{i_k\}}]] \ominus t_{i_k}e^{\{i_k\}}
\]

\[
= e^U \ominus t_{i_1}e^{\{i_1\}} \ldots \ominus t_{i_k}e^{\{i_k\}}
\]

Since \(t\) is a strong dummy fuzzy coalition, player \(i \in U \cap \text{car}(t)\) is strong dummy fuzzy players with level \(t_i\). This implies that

\[
v (e^U \ominus t) = v \left( e^U \ominus t_{i_1}e^{\{i_1\}} \ldots \ominus t_{i_k}e^{\{i_k\}} \right)
\]

\[
= v (e^U) - \sum_{i \in U \cap \text{car}(t)} v(t_i e^{\{i\}}).
\]

Hence

\[
\sum_{U \subseteq N} \left[ v(e^U) - v (e^U \ominus t) \right] = \sum_{U \subseteq N} \sum_{i \in U \cap \text{car}(t)} v(t_i e^{\{i\}})
\]

\[
= 2^{|N|-1} \sum_{i \in \text{car}(t)} v(t_i e^{\{i\}}).
\]
And for \( i \in \text{car}(t) \)

\[
\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t_i e^{\{i\}})] = \sum_{i \in U \subseteq N} [v(e^U) - v(e^U \ominus t_i e^{\{i\}})] + \sum_{i \notin U \subseteq N} [v(e^U) - v(e^U \ominus t_i e^{\{i\}})] = \sum_{i \in U \subseteq N} v(t_i e^{\{i\}}) = 2^{|N|-1} v(t_i e^{\{i\}}).
\]

This completes the proof. \(\square\)

In general fuzzy cooperative game, the blockability value does not satisfy NFP and DFP. The following example shows that.

**Example 3.5.** Consider a fuzzy game \((N, v)\) such that \(N = \{1, 2, 3\}\). For \(s = (s_1, s_2, s_3) \in \mathcal{F}^N\) a characteristic function \(v\) is defined by

\[
v(s) = \begin{cases} 
5s_1 + s_2 + s_1s_2s_3, & \text{if } s_3 \geq 0.5, \\
5s_1 + s_2, & \text{otherwise}.
\end{cases}
\]

Then \(p = (0, 0, 0.5)\) is a both null and dummy fuzzy coalition, but \(B(p) = 0.14 \neq 0\).

In classical cooperative games, the constant-sum game \((N, v)\) is defined by

\[
v(S) + v(N \setminus S) = v(N) \quad \text{for all } S \subseteq N.
\]

The constant-sum game has been extensively investigated in early works on game theory, and it has already been established that most political games are constant-sum. We extend the concept of constant-sum games to cooperative fuzzy games as following. A fuzzy game \((N, v)\) is said to be a constant-sum game if

\[
v(e^N) = v(s) + v(e^N \ominus s)
\]

for every coalition \(s \in \mathcal{F}^N\).

The following proposition provides sufficient conditions for a blockability value that satisfies NFP and DFP.

**Lemma 3.6.** Let \((N, v)\) be a fuzzy game and player \(i\) be a dummy fuzzy player with level \(\alpha\) in \((N, v)\). Then

\[
\sum_{U \subseteq N} v(e^U \lor \alpha e^{\{i\}}) = \sum_{U \subseteq N} v(e^U) + 2^{|N|-1} \cdot v(\alpha e^{\{i\}}).
\]
Proof. Let player $i$ be a dummy fuzzy player with level $\alpha$ in $(N, v)$ and let $U \subseteq N$. If $i \in U$, then $e^U \lor \alpha e^{\{i\}} = e^U$. Hence $v(e^U \lor \alpha e^{\{i\}}) = v(e^U)$. Otherwise,

$$v(e^U \lor \alpha e^{\{i\}}) = v(e^U) + v(\alpha e^{\{i\}}).$$

Thus

$$\sum_{U \subseteq N} v(e^U \lor \alpha e^{\{i\}}) = \sum_{i \in U \subseteq N} v(e^U \lor \alpha e^{\{i\}}) + \sum_{i \not\in U \subseteq N} v(e^U \lor \alpha e^{\{i\}})$$

$$= \sum_{i \in U \subseteq N} v(e^U) + \sum_{i \not\in U \subseteq N} [v(e^U) + v(\alpha e^{\{i\}})]$$

$$= \sum_{U \subseteq N} v(e^U) + 2^{|N|-1} \cdot v(\alpha e^{\{i\}}).$$

Lemma 3.7. Let $(N, v)$ be a fuzzy game and $t$ be a dummy fuzzy coalition in $(N, v)$. Then

$$\sum_{U \subseteq N} v(e^U \lor t) = \sum_{U \subseteq N} v(e^U) + 2^{|N|-1} \sum_{i \in car(t)} v(t_i e^{\{i\}}).$$

Proof. Let $t \in F^N$ be a dummy fuzzy coalition in fuzzy game $(N, v)$. Since player $i$ is a dummy fuzzy player with level $t_i$ for all $i \in car(t)$,

$$v(e^U \lor t) = v(e^U) + \sum_{i \in car(t) \setminus U} v(t_i e^{\{i\}}).$$

Hence by Lemma 3.6

$$\sum_{U \subseteq N} v(e^U \lor t) = \sum_{U \subseteq N} \left[ v(e^U) + \sum_{i \in car(t) \setminus U} v(t_i e^{\{i\}}) \right]$$

$$= \sum_{U \subseteq N} v(e^U) + \sum_{U \subseteq N} \left[ \sum_{i \in car(t) \setminus U} v(t_i e^{\{i\}}) \right]$$

$$= \sum_{U \subseteq N} v(e^U) + 2^{|N|-1} \sum_{i \in car(t)} v(t_i e^{\{i\}}).$$

Proposition 3.8. Let $(N, v)$ be a constant-sum fuzzy game. Then the block-ability value satisfies NFP and DFP.
Proof. Let \((N, v)\) be a constant-sum fuzzy game. Then

\[
\begin{align*}
v(e^U \ominus t) &= v(e^N) - v\left([e^U \ominus t]^\varpi\right) \\
&= v(e^N) - v\left([e^U \land t]^\varpi\right) \\
&= v(e^N) - v\left((e^U)^- \lor t\right) \\
&= v(e^N) - v((e^U)^-) \\
&= v(e^U)
\end{align*}
\]

for all \(U \subseteq N\). Hence

\[
B(t) = \frac{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t)]}{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus e^N)]} \cdot v(e^N) = 0.
\]

(DFP) Let \(t \in \mathcal{F}^N\) be a dummy fuzzy coalition in \((N, v)\). Then, for all \(U \subseteq N\)

\[
v(e^U) - v(e^U \ominus t) = v(e^U) - \left[v(e^N) - v\left([e^U \ominus t]^\varpi\right)\right]
\]

\[
= v(e^U) - v(e^N) + v\left([e^U \land t]^\varpi\right)
\]

\[
= [v(e^U) - v(e^N)] + v\left((e^U)^- \lor t\right)
\]

\[
= -v\left((e^U)^-\right) + v\left((e^U)^- \lor t\right).
\]

Similarly,

\[
v(e^U) - v(e^U \ominus t_ie^{\{i\}}) = -v\left((e^U)^-\right) + v\left((e^U)^- \lor t_i e^{\{i\}}\right)
\]

for all \(U \subseteq N\) and for all \(i \in \text{car}(t)\). By Lemma 3.7,

\[
\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t)]
\]

\[
= \sum_{U \subseteq N} \left[-v\left((e^U)^-\right) + v\left((e^U)^- \lor t\right)\right]
\]

\[
= -\sum_{U \subseteq N} v\left((e^U)^-\right) + \sum_{U \subseteq N} v\left((e^U)^- \lor t\right)
\]

\[
= -\sum_{U \subseteq N} v(e^U) + \sum_{U \subseteq N} v(e^U \lor t)
\]

\[
= -\sum_{U \subseteq N} v(e^U) + \sum_{U \subseteq N} v(e^U) + 2^{|N|-1} \sum_{i \in \text{car}(t)} v(t_ie^{\{i\}})
\]

\[
= \sum_{i \in \text{car}(t)} \sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t_ie^{\{i\}})].
\]
Hence

\[
B(t) = \frac{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t)]}{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus e^N)]} \cdot v(e^N)
\]

\[
= \frac{\sum_{i \in \text{car}(t)} \sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t_ie^{(i)})]}{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus e^N)]} \cdot v(e^N)
\]

\[
= \sum_{i \in \text{car}(t)} \left[ \frac{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus t_ie^{(i)})]}{\sum_{U \subseteq N} [v(e^U) - v(e^U \ominus e^N)]} \cdot v(e^N) \right]
\]

\[
= \sum_{i \in \text{car}(t)} B(t_ie^{(i)}).
\]

This completes the proof.

In this paper we introduced some classes of fuzzy players and coalitions that have special effects on other fuzzy coalitions and studied their properties. The concept of these special players and coalitions contain not only the concept of players and coalitions in traditional cooperative game but also the effect of player’s participation level in fuzzy cooperative game. We also proposed fuzzy coalition values which express the fuzzy coalition influence of fuzzy games. We also proved that these fuzzy coalition values have both null and dummy properties, but let for future research the symmetric properties of coalition values.

References


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Received: June 24, 2016; Published: August 15, 2016