Global Stability Properties of Virus Dynamics of a Diffusive Model

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Abstract

A sufficient condition for the global stability of positive equilibrium points of a diffusive differential equation, which appears as a model for basic virus dynamics, is obtained by applying the technique of the strong maximum principle and Liapunov functionals.

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1 Introduction

In this paper, we consider the following differential equation with diffusion

$$\frac{\partial S}{\partial t}(t, x) = k \Delta S(t, x) + \lambda - dS(t, x) - \beta S(t, x)P(t, x) \quad t > 0, x \in \Omega,$$
\[
\begin{align*}
\frac{\partial I}{\partial t}(t, x) &= k\Delta I(t, x) + \beta S(t, x)P(t, x) - aI(t, x) & t > 0, x \in \Omega, \\
\frac{\partial P}{\partial t}(t, x) &= k\Delta P(t, x) + arI(t, x) - bP(t, x) & t > 0, x \in \Omega, \\
\frac{\partial S}{\partial n}(t, x) &= \frac{\partial I}{\partial n}(t, x) = \frac{\partial P}{\partial n}(t, x) = 0 & t > 0, x \in \partial \Omega,
\end{align*}
\]

where \( S(t, x) + I(t, x) + P(t, x) \equiv N_1(t, x) \) denotes the total number of a population at time \( t \) and space \( x \), and \( \Delta \) is the Laplacian in \( \mathbb{R}^m, \Omega \subset \mathbb{R}^m \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \partial/\partial n \) is the outward normal derivative to \( \partial \Omega \). For the equation (1), \( 1 \gg k > 0 \) is the diffusion coefficient of small constant, and we denote uninfected cells population (the susceptible cells) by \( S := S(t, x) \), infected cells by \( I := I(t, x) \) and free virus particles by \( P := P(t, x) \). The cells are assumed to reproduce with a constant rate \( \lambda^*, \lambda = \lambda^*S_c > d, S_c \) is a constant and all the newly produced cells are uninfected (and susceptible). The average life times of the susceptible cells, infected cells and free virus are \( 1/d, 1/a \) and \( 1/b \) respectively, naturally \( a = q + d(\geq d), q \geq 0 \). Free virus is produced from infected cells at the rate \( arI \) and infects the susceptible cells at the rate \( \beta SP \). Naturally, \( S, I \) and \( P \) are individual cell numbers and all the coefficients are assumed positive number.

Equation (1) is a diffusive analogue of the ordinary differential equation

\[
\begin{align*}
S' &= \lambda^* - dS - \beta SP, \\
I' &= \beta SP - aI, \\
P' &= arI - bP.
\end{align*}
\]

which has been proposed by Bonhoeffer et al.[1] and Nowak and Bangham [9] as a model for basic virus dynamics (see also [10]).

We next observe that \( P(t, x) \) can be immediately obtained once \( I(t, x) \) are know and also the cells of free virus \( P(t, x) \) is considered always steady state for time scale of changing cells, so the system (1) can be reduced to

\[
\begin{align*}
\frac{\partial S}{\partial t}(t, x) &= k\Delta S(t, x) + \lambda - dS(t, x) - hS(t, x)I(t, x) & t > 0, x \in \Omega, \\
\frac{\partial I}{\partial t}(t, x) &= k\Delta I(t, x) + hS(t, x)I(t, x) - aI(t, x) & t > 0, x \in \Omega, \\
\frac{\partial S}{\partial n}(t, x) &= \frac{\partial I}{\partial n}(t, x) = 0 & t > 0, x \in \partial \Omega,
\end{align*}
\]

where \( h = \beta ar/b, \beta P = hI \).

Functions \( S, I, P \in C([0, \infty) \times \mathbb{R}, \mathbb{R}) \) is called a (classical) solution of (1) if \( \partial S/\partial t, \partial S/\partial x, \partial^2 S/\partial x^2, \partial I/\partial t, \partial I/\partial x, \partial^2 I/\partial x^2, \partial P/\partial t, \partial P/\partial x \) and \( \partial^2 P/\partial x^2 \) belong to the space \( C((0, \infty) \times \Omega) \), \( \partial S/\partial n, \partial I/\partial n \) and \( \partial P/\partial n \) exist on \( (0, \infty) \times \mathbb{R} \).
∂Ω and (1) is identically satisfied. From [11, Chapter 6] and [13], we can show that the existence of solution is guaranteed for (1) whenever the initial function 

\[ S(0, x) = \phi_1(x) \geq 0, x \in \bar{\Omega}, \quad I(0, x) = \phi_2(x) \geq 0, x \in \bar{\Omega} \]

and 

\[ P(0, x) = \phi_3(x) \geq 0, x \in \bar{\Omega} \]

belong to \( \phi_i \in C^1(\bar{\Omega}) \) for \( i = 1, 2, 3 \). (4)

For any parameters \( \beta, d, a, b, r \) and \( \lambda \), it is easy to check that the equilibrium solution \( (S(t, x), I(t, x), P(t, x)) \) of (1) with the initial condition (4) exists and is a unique for all \( t \geq t_0 \).

It is easy to see that equation (1) has two equilibrium points. Apart from an infection free equilibrium \( E_0 = (\lambda/d, 0, 0) \), equation (1) has a positive equilibrium \( E^* = (S^*, I^*, P^*) \), where

\[
S^* = \frac{b}{r\beta}, \quad I^* = \frac{r\beta \lambda - bd}{ar\beta}, \quad P^* = \frac{r\beta \lambda - bd}{b\beta}.
\]

The positive equilibrium exists if the basic production number \( R_0^* \) of the following assumption \( (H_1) \) is greater than 1, that is

\[
(H_1) \quad R_0^* := \frac{r\lambda \beta}{bd} > 1
\]

(cf [5],[6]). It is clear that, for equation (3), this assumption \( (H_1) \) is equivalent to the following condition \( (H_2) \)

\[
(H_2) \quad H_0^* := \frac{\lambda h}{ad} > 1.
\]

**Remark.** 1. Equation (1) is equivalent to the SEIR epidemiological model with a constant host population size assumption. Indeed, if the equation for the recovered population \( R \) is omitted (the constant population size assumption allows us to do so), equation (1) is equivalent to the SEIR model, that is \( S(t, x) \) corresponds to susceptible population \( S \), \( I(t, x) \) to exposed population \( E \), and \( P(t, x) \) to infective population \( I \). This equivalence implies that the dynamics of this equation is also similarly, and that most of the results known for the SEIR model which has been well studied can be straightforwardly extended to equation (1).

We discuss the large time behavior of the solution of equation (1) (cf.[4]).

**Definition.** 1. The equation (1) is said to be permanence if there are positive constants \( \nu_i \) and \( M_i(i = 1, 2, 3) \) such that

\[
\nu_1 \leq \lim \inf_{t \to +\infty} \left( \inf_{x \in \bar{\Omega}} S(t, x) \right) \leq \lim \sup_{t \to +\infty} \left( \sup_{x \in \bar{\Omega}} S(t, x) \right) \leq M_1,
\]

\[
\nu_2 \leq \lim \inf_{t \to +\infty} \left( \inf_{x \in \bar{\Omega}} I(t, x) \right) \leq \lim \sup_{t \to +\infty} \left( \sup_{x \in \bar{\Omega}} I(t, x) \right) \leq M_2,
\]

\[
\nu_3 \leq \lim \inf_{t \to +\infty} \left( \inf_{x \in \bar{\Omega}} R(t, x) \right) \leq \lim \sup_{t \to +\infty} \left( \sup_{x \in \bar{\Omega}} R(t, x) \right) \leq M_3.
\]
hold for any solution of (1) with the initial condition (4). Here $\nu_i$ and $M_i (i = 1, 2, 3)$ are independent of (4).

Before main theorem, we mention the following theorem (the strong maximum principle in [12]), and then the main results of our paper are stated as follows.

**Theorem A.** Let $w \in C^{1,2}(D_T)$ and that
\[
\begin{align*}
\frac{\partial w}{\partial t} - d \nabla^2 w + cw & \geq 0 \quad \text{in } D_T = (0, T] \times \Omega, \\
Bw &= 0 \quad \text{on } S_T = (0, T] \times \partial \Omega, \\
w(0, x) &\geq 0 \quad \text{in } \bar{\Omega},
\end{align*}
\]
where $B$ is Neumann type boundary condition and $c \equiv c(t, x)$ is a bounded function in $D_T$. If $w$ attains a maximum value $M$ at some point in $D_T$, then $w \equiv M$ throughout $D_T$.

**Theorem 1.** Under the above assumptions of parameters, if the assumption $(H_1)$ holds;
\[
R^*_0 \equiv \frac{r \lambda \beta}{bd} > 1,
\]
then, for each nonnegative continuous initial function, equation (1) is permanence.

In the rest of this paper, we will report results only for system (3). Before proofs of Theorem 1, we prepare lemmas.

**Lemma 1.** The solution $(S(t, x), I(t, x))$ of equation (3) with (4) except for $P(0, x)$ satisfies for $t \geq 0$, the following inequality
\[
0 < N(t, x) \leq \max \{\sup_{x \in \bar{\Omega}} N(0, x), \frac{\lambda}{d} \} := K, \quad t > 0, x \in \bar{\Omega},
\]
where $N(t, x) = S(t, x) + I(t, x)$ and $N(0, x) = S_0(x) + I_0(x)$.

**Proof.** For the first inequality of (7), it is sufficient to prove that if for any small $\epsilon > 0$, $N(t_0, x) > \epsilon$ for some $t_0 > 0$ and $x \in \bar{\Omega}$, then $N(t, x) > \epsilon/2$ for $t > t_0, x \in \bar{\Omega}$. If it is not true, then
\[
N(t, x) < \frac{\epsilon}{2} \quad \text{for } t > t_1, x \in \bar{\Omega} \quad \text{and} \quad N(t_1, x_1) = \frac{\epsilon}{2} \quad \text{for some } t_1 > t_0, x_1 \in \bar{\Omega}
\]
with $t_1$ being the smallest among all such points $(t_1, x_1)$. If we set $w_0(t, x) = N(t, x) - \epsilon/2$, then $w_0(t, x) < 0$ (for $t > t_1, x \in \bar{\Omega}$), $w_0(t_1, x_1) = 0$ and $\sup_{x \in \bar{\Omega}} w_0(t_0, x) >$
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0, hence the function \( w_0(t, x) \) takes a nonnegative minimum on \([t_0, t_1] \times \bar{\Omega}\). On the other hand, we have

\[
\frac{\partial w_0}{\partial t} = \frac{\partial N}{\partial t} = k \Delta N - dS(t, x) - aI(t, x) + \lambda = k \Delta w_0 - d(w_0 + \frac{\epsilon}{2}) - qI + \lambda
\]

and consequently

\[
k \Delta w_0 - \frac{\partial w_0}{\partial t} - dw_0 = qI + d\frac{\epsilon}{2} - \lambda \leq qN + d\frac{\epsilon}{2} - \lambda \leq (d + q)\frac{\epsilon}{2} - \lambda < 0
\]

on \((t_1, \infty) \times \Omega\). Then there arises a contradiction by the strong maximum principle (cf.[3],[4],[8],[12]). Indeed, if \(x_1 \in \Omega\), then \(k \Delta w_0 - \frac{\partial w_0}{\partial t} - dw_0 \) must be nonnegative at \((t_1, x_1)\). This is a contradiction. We thus obtain that \(x_1 \in \partial \Omega\) and \(w_0(t, x) > w_0(t_1, x_1)\) for all \((t, x) \in [t_0, t_1] \times \Omega\), and hence \(\frac{\partial w_0}{\partial \nu} \leq 0\) at \((t_1, x_1)\). This is a contradiction, again (cf.[12]). It is clear that, by the initial point \(N(0, x) \geq 0\) and the reduction of the above, \(N(t, x) > 0\) for \((t, x) \in (0, t_0) \times \bar{\Omega}\). Therefore we must have the first inequality.

Let \(K < K_0\) for some \(K_0 > 0\). We claim that \(N(t, x) \leq K_0, [0, \infty) \times \bar{\Omega}\). If it is true, by letting \(K_0 \rightarrow K\), we hold the second inequality of this lemma. If now this is not true, then there exists \((t_2, x_2) \in (0, \infty) \times \bar{\Omega}\) such that \(N(t_2, x_2) > K_0\). If we set \(w(t, x) = N(t, x) - K_0\), then \(w(t_2, x_2) > 0\) and \(\sup_{x \in \Omega} w(0, x) \leq 0, (t_2 > 0)\), hence the function \(w(t, x)\) takes a positive maximum on \([0, t_2] \times \Omega\). On the other hand, we have

\[
\frac{\partial w}{\partial t} = \frac{\partial N}{\partial t} \leq k \Delta N - dN + \lambda = k \Delta w - d(w + K_0) + \lambda
\]

and consequently

\[
k \Delta w - \frac{\partial w}{\partial t} - dw \geq dK_0 - \lambda \geq 0.
\]

Then there arises a contradiction by the strong maximum principle (cf.[3],[4],[8],[12]). Indeed, if \(x_2 \in \Omega\), then \(k \Delta w - \frac{\partial w}{\partial t} - dw \) must be negative at \((t_2, x_2)\). This is a contradiction. We thus obtain that \(x_2 \in \partial \Omega\) and \(w(t, x) < w(t_2, x_2)\) for all \((t, x) \in [0, t_2] \times \Omega\), and hence \(\frac{\partial w}{\partial \nu} > 0\) at \((t_2, x_2)\). This is a contradiction, again (cf.[12]). Therefore we must have (7).
Lemma 2. Under the assumption \((H_2)\), the solution \((S(t,x), I(t,x))\) of equation (3) with (4) except for \(P(0,x)\) satisfies the following inequality
\[
\liminf_{t \to \infty} \inf_{x \in \bar{\Omega}} S(t,x) \geq \frac{d\lambda}{d^2 + h\lambda} \equiv \nu_1 > 0. \tag{8}
\]

Proof. For some \(t_3 > t_0\), we can show that
\[
\hat{S}(t) \leq S(t,x), \quad t > t_3, \quad x \in \bar{\Omega}, \tag{9}
\]
where \(\hat{S}(t)\) is the solution of ordinary differential equation
\[
\frac{d}{dt} \hat{S}(t) = -(d + h\frac{\lambda}{d})\hat{S}(t) + \lambda - \epsilon, \quad t > t_3, \tag{10}
\]

\[
\hat{S}(t_3) = \hat{S}_3 \quad \text{and} \quad \hat{S}_3 \geq \inf_{x \in \Omega} S(t_2, x) \geq \frac{d\lambda}{d^2 + h\lambda}.
\]

To see this, we consider the function \(w_1(t,x) := S(t,x) - \hat{S}(t)\) on \([t_3, \infty) \times \bar{\Omega}\). Then \(w_1(0,x) = S(0,x) - \hat{S}(0) \leq 0\) for \(x \in \bar{\Omega}\), and moreover, since \(S + I(= N) \leq S + \frac{\lambda}{d}\) by Lemma 1,

\[
\frac{\partial w_1}{\partial t} = \frac{\partial S}{\partial t} - d\hat{S}(t)/dt \\
\geq k\Delta S - hS\frac{\lambda}{d} - dS + \lambda \\
- \left( -d\hat{S} - h\frac{\lambda}{d}\hat{S} + \lambda - \epsilon \right) \\
\geq k\Delta w_1 - \hat{S}(w_1 + \hat{S})\frac{\lambda}{d} - d(w_1 + \hat{S}) + \lambda \\
- \left( -d\hat{S} - h\frac{\lambda}{d}\hat{S} + \lambda - \epsilon \right) \\
\geq k\Delta w_1 - \left( h\frac{\lambda}{d} + d \right) w_1 + \epsilon.
\]

Hence,

\[
k\Delta w_1 - \partial w_1/\partial t - \left( h\frac{\lambda}{d} + d \right) w_1 \leq -\epsilon \leq 0 \quad \text{on} \quad [t_3, \infty) \times \bar{\Omega}.
\]

Therefore, by the same reasoning as the one for \(w_0(t,x)\) of Lemma 1, one can see that \(w_1(t,x) \leq 0\) on \([t_3, \infty) \times \bar{\Omega}\). Thus, we must have (9).

Moreover, by setting \(M = d + \beta\frac{\lambda}{d}\) in (10), we have
\[
\frac{d}{dt}\hat{S} = -MS + \lambda - \epsilon. \tag{11}
\]
By solving equation (11), we obtain that
\[ \hat{S} = \frac{\lambda - \epsilon}{M} + \hat{C}e^{-Mt} \]
and
\[ \hat{C} = e^{Mt}(\hat{S} - \frac{\lambda - \epsilon}{M}). \]
Therefore, we have
\[ \frac{d(\lambda - \epsilon)}{d^2 + h\lambda} \leq \hat{S}, \quad t \geq t_3 \]
for small $\epsilon$. Thus, we obtain
\[ \frac{d(\lambda - \epsilon)}{d^2 + h\lambda} \leq S \]
on $t \geq t_3$, $x \in \Omega$. By taking infimum, $t \to \infty$ and later letting $\epsilon \to 0$ in the above inequality, we obtain
\[ \frac{d\lambda}{d^2 + h\lambda} \leq \lim inf_{t \to \infty} [\inf_{x \in \Omega} S(t, x)]. \]
This completes the proof of Lemma 2.

**Lemma 3.** Under the assumption $(H_2)$, the solution $(S(t, x), I(t, x))$ of equation (3) with (4) except for $P(0, x)$ satisfies the following inequality
\[ \lim sup_{t \to \infty} [\sup_{x \in \Omega} I(t, x)] \leq M_2, \quad (12) \]
for some $M_2 > 0$.

**Proof.** For some $t_4 > t_0$, we can show that
\[ I(t, x) \leq \hat{I}(t) \quad t > t_4, x \in \bar{\Omega}. \quad (13) \]
Here, $\hat{I}(t)$ is the solution of ordinary differential equation
\[ \frac{d}{dt} \hat{I}(t) = h\left(\frac{\lambda}{d} - \hat{I}(t)\right)(\frac{\lambda}{d} - \frac{d\lambda}{d^2 + h\lambda}) - a\hat{I}(t) + \epsilon, \quad t > t_4, \quad (14) \]
\[ \hat{I}(t_4) = \hat{I}_4 \quad \text{and} \quad 0 < \hat{I}_4 \leq \sup_{x \in \Omega} I(t_4, x) \leq \frac{B}{A}, \]
where $A = h\frac{\lambda}{d} - h\nu_1 + a > 0$ and $B = h\frac{\lambda}{d}(\frac{\lambda}{d} - \nu_1) + \epsilon > 0$ by $\frac{\lambda}{d} - \nu_1 > 0$. Here $\nu_1 = \frac{d\lambda}{d^2 + h\lambda}$. To see this, we consider the function $w_2(t, x) := I(t, x) - \hat{I}(t)$ on
\[ [t_4, \infty) \times \bar{\Omega}. \] Then \( w_2(0, x) = I(0, x) - \hat{I}(0) \leq 0 \) for \( x \in \bar{\Omega} \), and moreover, since \( hSI(t, x) \leq h(N - I)I(t, x) = h(N - I)(N - S) \),

\[
\frac{\partial w_2}{\partial t} = \partial I/\partial t - d\hat{I}(t)/dt = k\Delta I + hSI(t, x) - aI - (h(\frac{\lambda}{d} - \hat{I}(t))(\frac{\lambda}{d} - \frac{d\lambda}{d^2 + h\lambda}) - a\hat{I}(t) + \epsilon) \leq k\Delta I + h(\frac{\lambda}{d} - I)(\frac{\lambda}{d} - \frac{d\lambda}{d^2 + h\lambda}) - aI - (h(\frac{\lambda}{d} - \hat{I}(t))(\frac{\lambda}{d} - \frac{d\lambda}{d^2 + h\lambda}) - a\hat{I}(t) + \epsilon).
\]

Here, for the simplicity we write to use \( \nu_1 \) and since \( \frac{\lambda}{d} - \nu_1 > 0 \),

\[
\frac{\partial w_2}{\partial t} \leq k\Delta I + h(\frac{\lambda}{d} - I)(\frac{\lambda}{d} - \nu_1) - aI - (h(\frac{\lambda}{d} - \hat{I}(t))(\frac{\lambda}{d} - \nu_1) - a\hat{I}(t) + \epsilon) = k\Delta I - h\frac{\lambda}{d}I + h\nu_1I - aI + h\frac{\lambda}{d}\hat{I} - h\nu_1\hat{I} + a\hat{I} - \epsilon.
\]

Since \( w_2(t, x) = I(t, x) - \hat{I}(t) \),

\[
\frac{\partial w_2}{\partial t} \leq k\Delta w_2 - h\frac{\lambda}{d}(w_2 + \hat{I}) + h\nu_1(w_2 + \hat{I}) - a(w_2 + \hat{I}) + h\frac{\lambda}{d}\hat{I} - h\nu_1\hat{I} + a\hat{I} - \epsilon = k\Delta w_2 - h\frac{\lambda}{d}w_2 + h\nu_1w_2 - aw_2 - \epsilon.
\]

Hence,

\[
k\Delta w_2 - \frac{\partial w_2}{\partial t} - (h\frac{\lambda}{d} - h\nu_1 + a)w_2 \geq \epsilon > 0.
\]

Therefore, by the same reasoning as the one for \( w_0(t, x) \), of Lemma 1, one can see that \( w_2(t, x) \leq 0 \) on \( [t_4, \infty) \times \bar{\Omega} \). Thus, we must have (13). Moreover, from (14),

\[
\frac{d}{dt} \hat{I}(t) = -h(\frac{\lambda}{d} - h\nu_1 + a)\hat{I} + h\frac{\lambda^2}{d^2} - h\frac{\lambda}{d}\nu_1 + \epsilon.
\]

Here, for the simplicity, we use \( A \) and \( B \) \( > 0 \) in the above equation, then we obtain

\[
\frac{d}{dt} \hat{I} = -A\hat{I} + B.
\]

(15)
By solving equation (15), we have
\[
\dot{I} = \frac{B}{A} + \hat{C} e^{-At}
\]
and
\[
\hat{C} = e^{At}(\hat{I}_4 - \frac{B}{A}).
\]
Therefore, we obtain
\[
\hat{I}(t) \leq \hat{I}(t_4)
\]
for \(t \geq t_4\). By (13), (17), we have
\[
I(t, x) \leq \hat{I}(t_4)
\]
on \([t_4, \infty) \times \tilde{\Omega}\). By taking supremum, \(t \to \infty\) and later letting \(\epsilon \to 0\) in the above inequality, we obtain (12)
\[
\lim sup_{t \to \infty} [\sup_{x \in \tilde{\Omega}} I(t, x)] \leq M_2,
\]
where the positive number \(M_2 = \frac{B'}{A}, B' = \frac{h_1}{d} (\frac{\lambda}{d} - \nu_1) > 0\). This completes the proof of Lemma 3.

**Proof of Theorem 1.** From Lemma 1, we have
\[
N(t, x) \leq \frac{\lambda}{d},
\]
where \(N(t, x) = S(t, x) + I(t, x)\). Since, by Lemma 3,
\[
\lim sup_{t \to \infty} [\sup_{x \in \tilde{\Omega}} I(t, x)] \leq M_2
\]
for \(M_2 > 0\), we hold that the solution \((S(t, x), I(t, x))\) of equation (3) with initial condition (4) except for \(P(0, x)\) satisfies
\[
\lim sup_{t \to \infty} [\sup_{x \in \tilde{\Omega}} S(t, x)] \leq M_1
\]
for some \(M_1 > 0\).

We can show that the solution \((S(t, x), I(t, x))\) of equation (3) with initial condition (4) except for \(P(0, x)\) satisfies
\[
\lim inf_{t \to \infty} [\inf_{x \in \tilde{\Omega}} I(t, x)] \geq \nu_2
\]
for some $\nu_2 > 0$ which does not depend on the initial function in (4). To see this, it is sufficient to prove
\[ I(t, x) \to \frac{h}{a} \left( \frac{\lambda}{d} \right)^2 \text{ as } t \to \infty, \ x \in \Omega. \]  

We define the function
\[ f(t, x) = \frac{I(t, x)}{a} - \frac{h}{a^2} \left( \frac{\lambda}{d} \right)^2. \]

Then
\[
\frac{\partial f}{\partial t} \leq \frac{1}{a} (k \Delta I + h \left( \frac{\lambda}{d} \right)^2 - aI) = \frac{k}{a} \Delta I - I + \frac{h}{a} \left( \frac{\lambda}{d} \right)^2.
\]

We thus have the following differential inequality:
\[
\frac{\partial f}{\partial t} \leq k \Delta f - af.
\]

Then, we can see that
\[ f(t, x) \to 0 \text{ as } t \to \infty, x \in \bar{\Omega}. \]

If we set $W(t) := W(f)(t) = \int_{\Omega} f^2(t, x) dx, t \geq 0$, then, $W(t) \geq 0$ and we have
\[
\frac{dW(t)}{dt} = 2 \int_{\Omega} f \frac{\partial f}{\partial t} dx \\
\leq 2 \int_{\Omega} f (k \Delta f - af) dx \\
\leq -2k \int_{\Omega} \left( \frac{\partial f}{\partial x} \right)^2 dx - 2a \int_{\Omega} f^2 dx.
\]

$H_1$ and $H_2$ be defined by the following
\[ H_1 = 2k, \quad H_2 = 2a. \]

Then $H_1 > 0$ and $H_2 > 0$. It follows from (21) that for $t > 0$,
\[ W(t) + H_1 \int_0^t \int_{\Omega} \left( \frac{\partial f(s, x)}{\partial x} \right)^2 dx ds + H_2 \int_0^t \int_{\Omega} f^2(s, x) dx ds \leq W(0). \]  

Since $W(t) \geq 0$, we have from (22) that
\[
\int_0^t \int_{\Omega} \left( \frac{\partial f(s, x)}{\partial x} \right)^2 dx ds < \frac{W(0)}{H_1}, \quad \int_0^t \int_{\Omega} f^2(s, x) dx ds < \frac{W(0)}{H_2}.
\]
Thus, we conclude from (21), (22), and (23) that $W(t) \in L^1[0, \infty)$ and $\frac{dW(t)}{dt} \in L^1[0, \infty)$. By Barbalate’s lemma [2, Lemma 1.2.2], we obtain $W(t) \to 0$ and thus, $f \to 0$ in $L^2$ as $t \to \infty$, that is

$$\|f(t, \cdot)\|_{L^2} \to 0 \text{ as } t \to \infty,$$

(24)

where $\| \cdot \|_{L^2}$ denotes the $L^2$-norm of functions on $\Omega$. We next prove that

$$\sup_{x \in \bar{\Omega}} |f(t, x)| \to 0 \text{ as } t \to \infty.$$  

(25)

To do this (cf.[8]), we take notice of the boundedness of $f(t, x)$ by (7) in Lemma 1. Thus, we see that the orbit for meaning of differential equation (except for inequality; $<$) in (20), that is $\{f(t, \cdot) | t \geq 0\}$ is relatively compact. The assertion (25) follows from this fact. Indeed, if (25) is not true, then there exist sequences $\{t_n\}, t_n \to \infty$ as $n \to \infty$, and $\{x_n\} \subset \bar{\Omega}$ such that $|f(t_n, x_n)| \geq \epsilon > 0, n = 1, 2, \ldots$ for some $\epsilon > 0$. We can assume that $x_n \to x_0$ and $f(t_n, x_n) \to \tilde{f}(x_0)$ uniformly on $\bar{\Omega}$ for some $x_0 \in \bar{\Omega}$ and $\tilde{f} \in C(\bar{\Omega})$ as $n \to \infty$, if necessary taking a subsequence of these. In particular, we get $|\tilde{f}(x_0)| \geq \epsilon$. This is a contradiction, because $\int_\Omega \tilde{f}^2(x)dx = \lim_{n \to \infty} \|f(t_n, \cdot)\|_{L^2}^2 = 0$ by (24).

Thus, we must have (25). From the definition of $f$, we have (19), that is $\nu_3 > 0$. Thus, (18) holds. Moreover, from Eq.(1) and (18), we easily have that

$$0 < \nu_3 \leq \liminf_{t \to +\infty} \inf_{x \in \Omega} [P(t, x)]$$

for some $\nu_3 > 0$. Thus, equation (1) is permanent by Lemmas 1, 2 and 3. This proves Theorem 1.

**Theorem 2.** If $H_0^* < 1$, the disease free equilibrium $E_0$ of (3) satisfies

$$\lim_{t \to \infty} \sup_{x \in \Omega} I(t, x) = 0$$

and

$$\lim_{t \to \infty} \sup_{x \in \Omega} [S(t, x) - \frac{\lambda}{d}] = 0.$$

**Proof.** If $\lambda/d \leq N(0, x) \leq K$, then we can show that

$$N(t, x) \leq \hat{N}(t) \quad t > 0, x \in \bar{\Omega},$$

(26)

where $\hat{N}(t)$ is the solution of ordinary differential equation

$$\frac{d}{dt} \hat{N}(t) = -d\hat{N}(t) + \lambda, \quad t > 0$$

and

$$\hat{N}(0) = K.$$
To see this, we consider the function \( w_1(t, x) := N(t, x) - \hat{N}(t) \) on \([0, \infty) \times \bar{\Omega}\). Then \( w_1(0, x) = N(0, x) - \hat{N}(0) \leq 0 \) for \( x \in \bar{\Omega} \), and moreover
\[
\frac{\partial w_1}{\partial t} = \frac{\partial N}{\partial t} - d\hat{N}(t)/dt
= k\Delta N - dN - qI + \lambda + d\hat{N} - \lambda
= k\Delta w_1 - d(w_1 + \hat{N}) - qI + d\hat{N}
\]
and hence
\[
k\Delta w_1 - \frac{\partial w_1}{\partial t} - dw_1 = qI \geq 0.\]
Therefore, by the same reasoning as the one for \( w(t, x) \), one can see that \( w_1(t, x) \leq 0 \); Thus, we must have (26). Since \( \hat{N}(t) = C e^{-\lambda t} + \lambda/d, \hat{C} = K - \lambda/d \), by letting \( t \to \infty \) in the above inequality (26), we obtain
\[
\limsup_{t \to \infty} \sup_{x \in \bar{\Omega}} N(t, x) \leq \frac{\lambda}{d}.
\]
Hence for the discussion of the asymptotic behavior of solutions as \( t \to +\infty \) we can (without loss of generality) assume that
\[
N(t, x) \leq \lambda/d \quad t > 0, x \in \bar{\Omega}.
\]
(27)

We next define
\[
f(t, x) = \frac{I(t, x)}{a}.
\]
Then
\[
\frac{\partial f}{\partial t} = \frac{1}{a}(k\Delta I + hSI(t, x) - aI)
\leq \frac{k}{a}\Delta I + (\frac{h}{a} \frac{\lambda}{d} - 1)I
\leq \frac{k}{a}\Delta I - \frac{1}{a}Q^* I,
\]
where \( Q^* = \frac{da - h\lambda}{d} > 0 \). We thus have the following differential inequality of
\[
\frac{\partial f}{\partial t} \leq k\Delta f - Q^* f.
\]
(28)

Then, we can see that
\[
f(t, x) \to 0 \quad \text{as} \quad t \to \infty, x \in \bar{\Omega},
\]
by the same argument of the proof in Theorem 1. From the definition of \( f \), we have
\[
I(t, x) \to 0 \quad \text{as} \quad t \to \infty, x \in \bar{\Omega}.
\]
(29)
Since $S$ is bounded,

$$
\beta SI(t, x) \rightarrow 0 \text{ as } t \rightarrow \infty, \ x \in \bar{\Omega}.
$$

We next claim that

$$
S \rightarrow \frac{\lambda}{d} \text{ as } t \rightarrow \infty, \ x \in \bar{\Omega}. \quad (30)
$$

By (29), for any small $\epsilon > 0$, there exists a large time $t_2 > 0$ such that

$$
I(t, x) \leq \epsilon \text{ for } t \geq t_2, \ x \in \bar{\Omega}.
$$

Then, it is sufficient for (30) to prove

$$
N(t, x) \geq \tilde{N}(t) \quad t \geq t_2, \ x \in \bar{\Omega}, \quad (31)
$$

where $\tilde{N}(t)$ is the solution of ordinary differential equation

$$
\frac{d}{dt} \tilde{N}(t) = -d\tilde{N}(t) + \lambda - q\epsilon \quad t > t_2,
$$

$$
\tilde{N}(t_2) = \tilde{N}_2 \text{ and } 0 < \tilde{N}_2 \leq \sup_{x \in \Omega} N(t_2, x) \leq \frac{\lambda}{d}.
$$

Then, we have

$$
\tilde{N}(t) \leq N(t, x) \leq \frac{\lambda}{d} \quad t \geq t_2, \ x \in \bar{\Omega}.
$$

Since $\tilde{N}(t) = \hat{C}e^{-dt} + \lambda/d - q\epsilon/d$,

$$
\hat{C} = e^{d\epsilon}(\tilde{N}_2 - \lambda/d + q\epsilon/d),
$$

by letting $t \rightarrow \infty$ and later letting $\epsilon \rightarrow 0$ in the above inequality, we obtain

$$
\frac{\lambda}{d} \leq \liminf_{t \rightarrow \infty} \left[ \sup_{x \in \Omega} N(t, x) \right] \leq \limsup_{t \rightarrow \infty} \left[ \sup_{x \in \Omega} N(t, x) \right] \leq \limsup_{t \rightarrow \infty} \left[ \sup_{x \in \Omega} S(t, x) \right] \leq \frac{\lambda}{d}.
$$

To see (31), we consider the function $w_2(t, x) := \tilde{N}(t) - N(t, x)$ on $[t_2, \infty) \times \bar{\Omega}$.

Then $w_2(t_2, x) = \tilde{N}_2 - N(t_2, x) \leq 0$ for $x \in \bar{\Omega}$, and moreover

$$
\frac{\partial w_2}{\partial t} = d\tilde{N}(t)/dt - \partial N/\partial t
$$

$$
= -d\tilde{N} + \lambda - q\epsilon - k\Delta N + dN - \lambda + qI
$$

$$
= k\Delta w_2 + d(\tilde{N} - w_2) - d\tilde{N} + q(I - \epsilon)
$$

and hence

$$
k\Delta w_2 - \frac{\partial w_2}{\partial t} - dw_2 = q(\epsilon - I) \geq 0.
$$
Therefore, by the same reasoning as the one for \( w(t, x) \), one can see that \( w_2(t, x) \leq 0 \), that is (31) holds. Thus, we have (30). This completes the proof of Theorem 2.

We have show also that the following theorem.

**Theorem 3.** If \( H_0^* > 1 \) in the assumption \((H_2)\) and \( I_0(x) \neq 0 \), then, for each nonnegative continuous initial function, there is a unique positive equilibrium \((S^*, I^*)\) of equation (3) satisfies

\[
\lim_{t \to \infty} \sup_{x \in \Omega} |I(t, x) - I^*| = 0
\]

and

\[
\lim_{t \to \infty} \sup_{x \in \Omega} |S(t, x) - S^*| = 0.
\]

**Proof of Theorem 3.** In order to prove this theorem, we need the following Corollary in [5, pp.148-153]. As there are complete comments and references of this result for ordinary differential equations in [cf. 5, pp.159-160], we omit the proof of this for simplicity.

**Corollary.** If \( H_0^* > 1 \) and \( I_0(x) \neq 0 \), then there is a unique positive endemic equilibrium \((S^*, I^*)\).

With \( N = S + I \), the system (3) drives to

\[
\begin{align*}
\frac{\partial N}{\partial t} & = k\Delta N - dN - qI + \lambda \\
\frac{\partial I}{\partial t} & = k\Delta I + hI[(N - I) - \frac{a}{h}] 
\end{align*}
\]  

(32)  

(33)

This system has the positive equilibrium \((N^*, I^*)\) where \( N^* = S^* + I^* \). We can rewrite (32) in the form

\[
\frac{\partial N}{\partial t} = k\Delta N - d(N - N^*) - q(I - I^*),
\]

because \(-dN^* - qI^* + \lambda = 0\). For also (33),

\[
\frac{\partial I}{\partial t} = k\Delta I + hI\{G(N) - (I - I^*)\}
\]

where

\[
G(N) = N - N^*.
\]
Then, \( G(N) > 0 \) for \( N > N^* \) and \( G(N) < 0 \) for \( N < N^* \). We now define a function \( V(t) \) by

\[
V(t) := V(N, I)(t) = \int_{\Omega} \left( \frac{h}{q} \int_{N^*}^{N} G(s) ds + I - I^* - I^* \log \frac{I}{I^*} \right) dx.
\] (34)

Then \( V(N^*, I^*)(t) = 0 \) and \( V(N, I)(t) > 0 \) for other admissible \((N, I)\). Furthermore, we calculate \( dV/dt \) along the solution of (32) and (33).

\[
\frac{dV(t)}{dt} = \int_{\Omega} \left\{ \frac{h}{q} G(N) \frac{\partial N}{\partial t} + \frac{\partial I}{\partial t} - I^* \frac{\partial I}{\partial t} I \right\} dx
\leq \int_{\Omega} \left\{ \frac{h}{q} G(N)(k \Delta N - d(N - N^*) - q(I - I^*)) \right\} dx
+ k \Delta I + hI\{(N - N^*) - (I - I^*)\} - I^*\{k \frac{\Delta I}{I} + h(G(N) - (I - I^*))\} dx
= \frac{hk}{q} \int_{\Omega} \Delta NG(N) dx - \frac{hd}{q} \int_{\Omega} G(N)(N - N^*) dx - h \int_{\Omega} G(N)(I - I^*) dx
+ k \int_{\Omega} \Delta I \frac{I - I^*}{I} dx + h \int_{\Omega} G(N)(I - I^*) dx - h \int_{\Omega} (I - I^*)^2 dx
\] (35)
< 0

whenever \((N, I) \neq (N^*, I^*)\). To drive this, we continue to estimate for (35) in more detail.

\[
\frac{hk}{q} \int_{\Omega} \Delta NG(N) dx = \frac{hk}{q} \left\{ \frac{\partial N}{\partial x} G(N) \right\} \Omega - \int_{\Omega} \frac{\partial N}{\partial x} \frac{\partial G(N)}{\partial x} dx
= -\frac{hk}{q} \int_{\Omega} \frac{\partial N}{\partial x} \frac{\partial G(N)}{\partial x} dx.
\] (36)

Here

\[
\frac{\partial G(N)}{\partial x} = \frac{\partial}{\partial x} \{(N - N^*)\} = \frac{\partial N}{\partial x}.
\]

Thus, expression (36) is

\[-\frac{hk}{q} \int_{\Omega} \left( \frac{\partial N}{\partial x} \right)^2 dx < 0.\]

Moreover, we have

\[-\frac{hd}{q} \int_{\Omega} G(N)(N - N^*) dx = -\frac{hd}{q} \int_{\Omega} (N - N^*)^2 dx < 0.\]

Similarly, we can check

\[k \int_{\Omega} \Delta I \frac{I - I^*}{I} dx = k \int_{\Omega} \Delta I(1 - \frac{I^*}{I}) dx = k\left[ \frac{\partial I}{\partial x} (1 - \frac{I^*}{I}) \right]_{\Omega} - kI^* \int_{\Omega} \frac{(\partial I/\partial x)^2}{I^2} dx < 0.\]
Thus, $V(t)$ of (34) is non increasing in $t$ that is there exists a constant $c_1 \geq 0$ such that $V(t) \rightarrow c_1$ as $t \rightarrow \infty$. 

Since $I(t, x) \leq \lambda/d, I(t, x)$ is uniformly bounded on $[0, \infty) \times \Omega$. Thus, we see that for any $\delta > 0$, there exists $C(\delta) > 0$ such that $|I(t + \delta, \cdot) - I(t, \cdot)| \leq C(\delta)$ for $t \geq 0$. From (35), we have $V(t) \leq -W(N, I)(t) \leq 0$ (included equilibrium point case), where $W(N, I)(t)$ is the function of right hand side in (35). Suppose that $V(t) \neq 0$. For any sequence $\{t_k\}, t_k \rightarrow \infty$ as $k \rightarrow \infty$ and some positive number $\gamma$, there exists $\delta > 0$ such that

$$V(t) < -\gamma$$

if $|I(t + t_k, \cdot) - I(t, \cdot)| \leq \delta, 0 \leq t \leq \delta$ and $k$ is sufficient large. For regions $[t_k, t_k + \delta]$, we can see that

$$V(t_k + \delta) \leq V(t_k) - \gamma \delta$$

to integral on $[t_k, t_k + \delta]$ for the both sides of (37). Since (38) is true for all large number $k$ and $\lim_{t \rightarrow \infty} V(t) = c_1 \geq 0$, it contradicts by $\gamma \delta$ is positive. This shows that $V(t) = 0$. Then, we have $W(N, I)(t) = 0$. We thus obtain $N \rightarrow N^*$ and $I \rightarrow I^*$ by continuity of $V$ and $W$. The asymptotic behavior of $S$ now follows from the above result on the behavior of $N$ and $I$. Thus, it is clear from $S = N - I$ that $S \rightarrow S^*$. This completes the proof.

**Example.** We consider the following equation of

$$
\frac{\partial S}{\partial t}(t, x) = 0.1 \Delta S(t, x) + 0.5 - 0.1 S(t, x) - 0.1 S(t, x) I(t, x) \\
\frac{\partial I}{\partial t}(t, x) = 0.1 \Delta I(t, x) + 0.1 S(t, x) I(t, x) - 0.4 I(t, x) 
\quad t > 0, x \in \Omega, \quad (39)
$$

where, in equation (3), $k = 0.1$, $d = 0.1$, $h = 0.1$, $a = 0.4$ and $\lambda = 0.5$. Thus, we have

$$S^*_0 = \frac{\lambda}{d} = \frac{0.5}{0.1} = 5.0,$$

$$S^* = \frac{a}{h} = \frac{0.4}{0.1} = 4.0, \quad \text{then } S^*_0 > S^*,$$

and thence

$$H^*_0 = \frac{\lambda h}{ad} = \frac{0.5 \times 0.1}{0.4 \times 0.1} = 1.25 > 1,$$

$$E_{S^*_0} = (S^*_0, 0) = (4.0, 0) \quad \text{and } E^* = (S^*, I^*) = (4.0, 0.25),$$

where

$$I^* = \frac{\lambda h - ad}{ah} = \frac{0.5 \times 0.1 - 0.4 \times 0.1}{0.4 \times 0.1} = 0.25 > 0.$$
The initial functions are

\[ S(0, x) = \phi_1(x) \equiv 1 > 0, \ x \in \overline{\Omega} \] and
\[ I(0, x) = \phi_2(x) \equiv 1 > 0, \ x \in \overline{\Omega} \]

belong to the \( \phi_i(x) \in C^1(\overline{\Omega}) \) for \( i = 1, 2 \).

Figures illustrate our theorem and suggest that, for small diffusion term \( 0 < k < 1 \), the endemic equilibrium \( E^+ \) of equation (3) is globally asymptotically stable if assumption \( (H_1) \) that is, \( (H_2) \) holds. In figures, the line of x appear \( S \) and \( I \) respectively, the vertical line is meaning ”time” in the graph of the trajectory of equation (39).
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