(2, 1)-Total Labeling of
Cycle with Parallel Paths

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Abstract

A (p, 1)-total labeling of a graph G is an assignment of integers to V(G) ∪ E(G) such that
(i) any two adjacent vertices of G receive distinct integers,
(ii) any two adjacent edges of G receive distinct integers, and
(iii) a vertex and an edge incident receive integers that differ by at least p in absolute value.

The span of a (p, 1)-total labeling is the maximum difference between two labels. The minimum of span of all possible (p, 1)-total labeling of G is called the (p, 1)-total number and denoted by \( \lambda_{p}^{T}(G) \). The well known Havet and Yu Conjecture [6] states that for any connected graph G with \( \Delta(G) \leq 3 \) and \( G \neq K_4 \), \( \lambda_{2}^{T}(G) \leq 5 \). In this paper, we determine the (2, 1)-total number of cycle with parallel paths. This result supports the Havet and Yu conjecture.

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Keywords: (2, 1)-total labeling, cycle with parallel paths, Havet and Yu conjecture on (2, 1)-total labeling

1 Introduction

In the channel/frequency assignment problem we need to assign different frequencies to ‘close’ transmitters so that they can avoid interference. Motivated
by this problem, Griggs and Yeh [3] introduced $L(2,1)$-labeling. Its natural
generalization $L(p,1)$-labeling of a graph $G$ is an integer assignment $f$ to the
vertex set $V(G)$ such that, $|f(u) - f(v)| \geq p$ if $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$
if $d(u,v) = 2$. Whittlesey et al.[14] studied the $L(2,1)$-labeling number of in-
cidence graphs, where the incidence graph of a graph $G$ is the graph obtained
from $G$ by replacing each edge $(v_i, v_j)$ with two edges $(v_i, v_{ij})$ and $(v_{ij}, v_j)$
introducing one new vertex $v_{ij}$. Observe that an $L(2,1)$-labeling of the incidence
graph of a given graph $G$ can be regarded as an assignment $f$ from $V(G)$ to
the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if $x$ is a vertex
and $y$ is an edge incident to $x$ and $|f(x) - f(y)| \geq 1$ if $x$ and $y$ are a pair of
adjacent vertices or a pair of adjacent edges for all $x, y$ in $V(G)$ or for all $x, y$
in $E(G)$. Havet and Yu [4] called such a labeling a “$(2,1)$-total labeling of $G$”.
A generalization of $(2,1)$-total labeling called $(p,1)$-total labeling is defined in
the following manner. Let $p \geq 1$ be an integer. A $k$-$(p,1)$-total labeling of a
graph $G$ is a function $f$ from $V(G) \cup E(G)$ to the set $\{0, 1, \ldots, k\}$ such that
$f(u) \neq f(v)$ if $u$ and $v$ are two adjacent vertices, $f(e) \neq f(e')$ if $e$ and $e'$
are two adjacent edges and $|f(u) - f(e)| \geq p$ if a vertex $u$ is incident to an edge $e$.
The $(p,1)$-total number denoted by $\lambda^T_p(G)$, is the smallest integer $k$ such that
$G$ has a $k$-$(p,1)$-total labeling. Note that $(1,1)$-total labeling of $G$ is equivalent
to total colouring of $G$. As a generalization of the Total Colouring Conjecture,
Havet and Yu [6] posed the following conjecture on $(p,1)$-total labeling number
called $(p,1)$-Total Labeling Conjecture.

$(p,1)$-Total Labeling Conjecture: For any graph $G$, $\lambda^T_p(G) \leq \Delta + 2p - 1$,
where $p \geq 1$ and $\Delta$ denotes the maximum degree of a vertex in $G$.

Havet and Yu [6, 7] have obtained some interesting bounds on $\lambda^T_p(G)$ sup-
porting $(p,1)$-total labeling conjecture. Also, $(p,1)$-total number is determined
for classes of special graphs. For example, the $(p,1)$-total number is determined
for complete graphs [6], planar graphs [1], graphs with a given maximum aver-
age degree [8], outer planar graphs [2, 11], etc. The case $p = 1$ corresponds
to the usual notion of total colouring, which is NP-hard to compute even for
cubic bipartite graphs. Havet et al [7] have completely settled the computa-
tional complexity of deciding whether $\lambda^T_p(G)$ is equal to $\Delta + p - 1$ or $\Delta + p$, for
$p \geq 2$, when $G$ is bipartite and the remaining cases are NP-complete. When
$p = 2$, Havet and Yu [6] posed the following stronger version of the $(p,1)$-total
labeling conjecture.
**Havet and Yu Conjecture** : If $G$ is any connected graph with $\Delta(G) \leq 3$ and $G \neq K_4$, then $\lambda_2^T(G) \leq 5$.

In a graph $G$, a set of $r$ paths are called parallel paths if the origin and the terminus of each of the $r$ paths must be non-adjacent, all the $r$ paths have no common vertex, no common edge and edges of all the $r$ paths do not cross. In this paper, we determine $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})) \leq 5$, where $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})$ denote the graph cycle $C_n$ with $\lfloor n/2 \rfloor - 1$ parallel paths. As $\Delta(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})) = 3$, our result that $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})) \leq 5$ supports the Havet Conjecture.

### 2 Main Result

In this section we prove our main result. Consider the graph $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})$, cycle $C_n$ with $\lfloor n/2 \rfloor - 1$ parallel paths $P_{k_i}$’s. For the convenience, we describe the cycle $C_n$ in the graph $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})$ as $u_0, u_1, \ldots, u_{\lfloor n/2 \rfloor - 1}$, $v_0, v_{\lfloor n/2 \rfloor - 1}$, $\ldots$, $v_{1}$, $u_0$ when $n$ is even and $u_0, u_1, \ldots, u_{\lfloor n/2 \rfloor - 1}$, $u_{\lfloor n/2 \rfloor}$, $v_{\lfloor n/2 \rfloor}$, $v_{\lfloor n/2 \rfloor - 1}$, $\ldots$, $v_{1}$, $u_0$ when $n$ is odd and the path $P_{k_i}$ as $w_{i,1}, w_{i,2}, \ldots, w_{i,k_i - 1}, v_i$, for $1 \leq i \leq \lfloor n/2 \rfloor - 1$, where $k_i$ is the size of the path and $k_i \geq 1$. Note that $w_{i,j}$, for $1 \leq j \leq k_i - 1$ are not the vertices of the cycle $C_n$. Figure 1 and Figure 2 shows that the cycle $C_8$ with 3 parallel paths and cycle $C_{11}$ with 4 parallel paths respectively.

![Figure 1: $C_8 \oplus (P_3, P_5, P_6)$](image_url)
Figure 2: $C_{11} \oplus (P_4, P_7, P_6, P_3)$

We use the following lemma to prove our main result.

**Lemma 2.1.** Let $G(V, E)$ be a connected graph with maximum degree $\Delta = 3$. If there exist three vertices $u, v, w$ in $G$ with each of degree $\Delta$ such that $u$ is adjacent to $v$ and $w$, then $\lambda_{T_2}(G) \geq 5$.

**Proof.** Let $G$ be a connected graph with maximum degree $\Delta = 3$. Let $u, v$ and $w$ be three vertices of $G$ with each of degree $\Delta$ such that $u$ is adjacent to $v$ and $w$.

Suppose there exist $(2,1)$-total labeling of $G$ with $\lambda_{T_2}(G) = 4$. Let $f : V \cup E \to \{0,1,2,3,4\}$ be a $(2,1)$-total labeling of $G$ with $\lambda_{T_2}(G) = 4$. Then, Table 1 shows all the possible vertex labels for the vertices of $G$ under the $(2,1)$-total labeling $f$ as well as all the possible labels for the edges incident at a vertex with the corresponding vertex label under $f$.

Table 1: Possible labels for the vertices and for their incident edges under $(2,1)$-total labeling $f$

<table>
<thead>
<tr>
<th>Vertex label</th>
<th>Labels of the corresponding incident edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2, 1, 0</td>
</tr>
<tr>
<td>3</td>
<td>1, 0</td>
</tr>
<tr>
<td>2</td>
<td>0, 4</td>
</tr>
<tr>
<td>1</td>
<td>3, 4</td>
</tr>
<tr>
<td>0</td>
<td>2, 3, 4</td>
</tr>
</tbody>
</table>
Case 1: Suppose the vertices $v$ and $w$ are adjacent. Then, the vertices $u$, $v$, and $w$ are mutually adjacent. From Table 1 observe that the only possible labels for the vertices $u$, $v$, and $w$ satisfying the property of $(2, 1)$-total labeling are 4 and 0. As the vertices $u$, $v$, and $w$ are mutually adjacent, it is not possible to assign three distinct labels to the three mutually adjacent vertices $u$, $v$, and $w$. (Refer Figure 3).

![Figure 3: Choice of the labels for the vertices $u$, $v$, and $w$ under the Case 1](image)

Case 2: Suppose $v$ and $w$ are not adjacent. Then from Table 1 we observe that 4 and 0 are the only possible labels for the vertices $u$, $v$, and $w$ under $f$. Since the vertices $v$ and $w$ are not adjacent, the same label for the vertices $v$ and $w$ could have been assigned under $f$. Suppose $f(u) = 4$, $f(v) = 0$ and $f(w) = 0$. From Table 1 observe that the only possible edge label for the edge having end vertices labeled 0 and 4 is 2. As $uv$ and $uw$ are adjacent edges, the label 2 cannot be assigned to both the edges $uv$ and $uw$. Consequently, only one of the edges $uv$ or $uw$ can get the edge label 2. Thus one of the edge $uv$ or $uw$ cannot be assigned any label. (Refer Figure 4).

![Figure 4: Choice of the labels for the edges $uw$ and $uv$ under the Case 2](image)

Similar argument hold good if the labels $f(u) = 0$, $f(v) = 4$ and $f(w) = 4$. This implies that there cannot be any possible $(2, 1)$-total labeling with span 4. Thus $\lambda^T_2(G) > 4$. Hence $\lambda^T_2(G) \geq 5$. □
Theorem 2.2. Let $n$ be an integer and $n \geq 8$, then
$$\lambda_T^2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n^{2}/2 \rfloor-1}})) = \Delta + p = 5.$$ 

Proof. Consider $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n^{2}/2 \rfloor-1}})$ with $n \geq 8$. Let $V$ and $E$ denote the vertex set and edge set of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n^{2}/2 \rfloor-1}})$ respectively.

Define $f : V \cup E \to \{0, 1, 2, 3, 4, 5\}$ in the following way. First we define $f$ for all the vertices as given below.

I. Labeling the vertices of $C_n$

Define $f(u_0) = 3,$

$f(v_0) = 3$, if $n$ is even,

$f(u_i) = 5$, $f(u_{i+1}) = 4$, $f(v_i) = 4$, $f(v_{i+1}) = 5$, for $i = 1, 3, \ldots, \alpha$, where

$$\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l, \text{ } l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 3, & \text{if } n = 2l, \text{ } l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \text{ } l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l + 1, \text{ } l \text{ even and } l \geq 4 
\end{cases}$$

$f(u_{\lfloor n^{2}/2 \rfloor-1}) = 5$, $f(v_{\lfloor n^{2}/2 \rfloor-1}) = 4$, if $n = 2l$, $l$ even and $l \geq 4$

$f(u_{\lfloor n^{2}/2 \rfloor}) = 5$, $f(v_{\lfloor n^{2}/2 \rfloor}) = 4$, if $n = 2l + 1$, $l$ odd and $l \geq 5$.

II. Labeling the vertices of $P_{k_i}$ with $i = 1, 3, \ldots, \alpha$,

where

$$\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l, \text{ } l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l, \text{ } l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \text{ } l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l + 1, \text{ } l \text{ even and } l \geq 4 
\end{cases}$$

Define $f(w_{i,k_{i-1}}) = 0$, if $k_i$ is even and $k_i \geq 2$,

$f(w_{i,j}) = 4$, $f(w_{i,j+1}) = 5$, for $j = 1, 3, \ldots, \beta$, where

$$\beta = \begin{cases} 
{k_i - 3, & \text{if } k_i \text{ is even and } k_i \geq 4} \\
{k_i - 2, & \text{if } k_i \text{ is odd and } k_i \geq 3} 
\end{cases}$$
III. Labeling the vertices of $P_k_i$ with $i = 2, 4, \ldots, \alpha$,
where

$$
\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l, \ l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l + 1, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \ l \text{ even and } l \geq 4
\end{cases}
$$

Define $f(w_{i,k_i-1}) = 0$, if $k_i$ is even and $k_i \geq 2$,
$f(w_{i,j}) = 5$, $f(w_{i,j+1}) = 4$, for $j = 1, 3, \ldots, \beta$, where

$$
\beta = \begin{cases} 
k_i - 3, & \text{if } k_i \text{ is even and } k_i \geq 4 \\
k_i - 2, & \text{if } k_i \text{ is odd and } k_i \geq 3
\end{cases}
$$

Now we label the edges of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\frac{4\alpha}{1}}-1})$.

I. Labeling the edges of $C_n$

Define $f(u_0u_1) = 1$, $f(u_0v_1) = 0$,
$f(u\left\lfloor \frac{n}{2} \right\rfloor v\left\lfloor \frac{n}{2} \right\rfloor) = 2$, if $n$ is odd,
$f(v_0u\left\lfloor \frac{n}{2} \right\rfloor - 1) = 0$, $f(v_0v\left\lfloor \frac{n}{2} \right\rfloor - 1) = 1$, if $n = 2l$, $l$ even and $l \geq 4$,
$f(v_0u\left\lfloor \frac{n}{2} \right\rfloor - 1) = 1$, $f(v_0v\left\lfloor \frac{n}{2} \right\rfloor - 1) = 0$, $f(u\left\lfloor \frac{n}{2} \right\rfloor - 1u\left\lfloor \frac{n}{2} \right\rfloor - 2) = 0$,
$f(v\left\lfloor \frac{n}{2} \right\rfloor - 1v\left\lfloor \frac{n}{2} \right\rfloor - 2) = 1$, if $n = 2l$, $l$ odd and $l \geq 5$,
$f(u\left\lfloor \frac{n}{2} \right\rfloor - 1u\left\lfloor \frac{n}{2} \right\rfloor) = 0$, $f(v\left\lfloor \frac{n}{2} \right\rfloor - 1v\left\lfloor \frac{n}{2} \right\rfloor) = 1$, if $n = 2l + 1$, $l$ even and $l \geq 4$,
$f(u_{i+1}u_{i+1}) = 0$, $f(u_{i+1}u_{i+2}) = 1$, $f(v_{i+1}v_{i+1}) = 1$, $f(v_{i+1}v_{i+2}) = 0$,
for $i = 1, 3, \ldots, \alpha$, where

$$
\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 4, & \text{if } n = 2l, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 3, & \text{if } n = 2l, \ l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 3, & \text{if } n = 2l + 1, \ l \text{ even and } l \geq 4
\end{cases}
$$

II. Labeling the edges of $P_k_i$ with $i = 1, 3, \ldots, \alpha$,
where

$$
\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l, \ l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l + 1, \ l \text{ even and } l \geq 4
\end{cases}
$$
Define $f(u_i v_i) = 2$, if $k_i = 1$.
If $k_i$ is even and $k_i \geq 2$, define

- $f(w_i k_i - 1 v_i) = 2$,
- $f(u_i w_{i,k_i-1}) = 3$, if $k_i = 2$,
- $f(w_{i,k_i-1} w_{i,k_i-2}) = 3$, if $k_i \geq 4$,
- $f(u_i w_{i,1}) = 2$, $f(w_{i,1} w_{i,2}) = 1$,
- $f(w_{i,j} w_{i,j+1}) = 2$, $f(w_{i,j+1} w_{i,j+2}) = 1$, for $j = 2, 4, \ldots, k_i - 4$.

If $k_i$ is odd and $k_i \geq 3$, define

- $f(u_i w_{i,1}) = 2$, $f(v_i w_{i,k_i-1}) = 2$, $f(w_{i,k_i-1} w_{i,k_i-2}) = 1$,
- $f(w_{i,j} w_{i,j+1}) = 1$, $f(w_{i,j+1} w_{i,j+2}) = 2$, for $j = 1, 3, \ldots, k_i - 4$.

III. Labeling the edges of $P_{k_i}$ with $i = 2, 4, \ldots, \alpha$, where

$$\alpha = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l, \ l \text{ even and } l \geq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l + 1, \ l \text{ odd and } l \geq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l + 1, \ l \text{ even and } l \geq 4 
\end{cases}$$

Define $f(u_i v_i) = 2$, if $k_i = 1$.
If $k_i$ is even and $k_i \geq 2$, define

- $f(w_{i,k_i-1} v_i) = 3$,
- $f(u_i w_{i,k_i-1}) = 2$, if $k_i = 2$,
- $f(w_{i,k_i-1} w_{i,k_i-2}) = 2$, if $k_i \geq 4$,
- $f(u_i w_{i,1}) = 2$, $f(w_{i,1} w_{i,2}) = 1$,
- $f(w_{i,j} w_{i,j+1}) = 2$, $f(w_{i,j+1} w_{i,j+2}) = 1$, for $j = 2, 4, \ldots, k_i - 4$.

If $k_i$ is odd and $k_i \geq 3$, define

- $f(u_i w_{i,1}) = 2$, $f(v_i w_{i,k_i-1}) = 2$, $f(w_{i,k_i-1} w_{i,k_i-2}) = 1$,
- $f(w_{i,j} w_{i,j+1}) = 1$, $f(w_{i,j+1} w_{i,j+2}) = 2$, for $j = 1, 3, \ldots, k_i - 4$.

From the definition of $f$ it follows that adjacent vertices get distinct labels and adjacent edges get distinct labels.

Table 2: Possible labels for the vertices and for their incident edges under (2,1)-total labeling $f$

<table>
<thead>
<tr>
<th>Vertex label</th>
<th>Labels of the corresponding incident edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3, 2, 1, 0</td>
</tr>
<tr>
<td>4</td>
<td>2, 1, 0</td>
</tr>
<tr>
<td>3</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Table 2 we observe that the labels of the incident elements have the difference at least 2. Thus $f$ satisfies all the condition of $(2,1)$-total labeling. Therefore, $\lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})) \leq 5$.

By Lemma 2.1, $\lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})) \geq 5$.

Hence, $\lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})) = 5$. 

\[ \lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})) \leq 5 \] 

**Theorem 2.3.** For $n = 6, 7$.

\[ \lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})) \leq \Delta + p = 5 \]

**Proof.** Let $V$ and $E$ denote the vertex set and edge set of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{(2)j-1}})$ respectively, where $n = 6$ or 7. Define $f : V \cup E \rightarrow \{0, 1, 2, 3, 4, 5\}$ in the following way.

**I. Labeling the vertices and edges of $C_n$**

Define $f(u_0) = 3$,

\[ f(u_1) = 5, \quad f(u_2) = 4, \quad f(v_1) = 4, \quad f(v_2) = 5, \]

\[ f(v_0) = 3, \quad \text{if } n = 6, \]

\[ f(u_3) = 5, \quad f(v_3) = 4, \quad \text{if } n = 7 \]

Define $f(u_0v_1) = 1, \quad f(u_0v_1) = 0$,

\[ f(u_1v_2) = 0, \quad f(v_1v_2) = 1, \]

\[ f(u_2v_0) = 1, \quad f(v_2v_0) = 0, \quad \text{if } n = 6, \]

\[ f(u_2v_3) = 1, \quad f(v_2v_3) = 0, \quad f(u_3v_3) = 2, \quad \text{if } n = 7. \]

**II. Labeling the vertices and edges of $P_{k_1}$**

Define $f(w_{1,k_1-1}) = 0$, if $k_1$ is even and $k_1 \geq 2$,

\[ f(w_{1,j}) = 4, \quad f(w_{1,j+1}) = 5, \quad \text{for } j = 1, 3, \ldots, \beta, \quad \text{where} \]

\[ \beta = \begin{cases} k_1 - 3, & \text{if } k_1 \text{ is even and } k_1 \geq 4 \\ k_1 - 2, & \text{if } k_1 \text{ is odd and } k_1 \geq 3 \end{cases} \]

Define $f(u_1v_1) = 2$, if $k_1 = 1$,

If $k_1$ is even and $k_1 \geq 2$, define

\[ f(w_{1,k_1-1}v_1) = 2, \]

\[ f(u_1w_{1,k_1-1}) = 3, \quad \text{if } k_1 = 2, \]

\[ f(w_{1,k_1-1}w_{1,k_1-2}) = 3, \quad \text{if } k_1 \geq 4, \]

\[ f(u_1w_{1,1}) = 2, \quad f(w_{1,1}w_{1,2}) = 1, \]

\[ f(w_{1,j}w_{1,j+1}) = 2, \quad f(w_{1,j+1}w_{1,j+2}) = 1, \quad \text{for } j = 2, 4, \ldots, k_1 - 4, \]

If $k_1$ is odd and $k_1 \geq 3$, define

\[ f(u_1w_{1,1}) = 2, \quad f(v_{1}w_{1,k_1-1}) = 2, \quad f(w_{1,k_1-1}w_{1,k_1-2}) = 1, \]

\[ f(w_{1,j}w_{1,j+1}) = 1, \quad f(w_{1,j+1}w_{1,j+2}) = 2, \quad \text{for } j = 1, 3, \ldots, k_1 - 4. \]
III. Labeling the vertices and edges of $P_{k_2}$

Define $f(w_{2,k_2-1}) = 0$, if $k_2$ is even and $k_2 \geq 2$,

\[ f(w_{2,j}) = 5, \quad f(w_{2,j+1}) = 4, \quad \text{for } j = 1, 3, \ldots, \beta, \]

where

\[ \beta = \begin{cases} 
    k_2 - 3, & \text{if } k_2 \text{ is even and } k_2 \geq 4 \\
    k_2 - 2, & \text{if } k_2 \text{ is odd and } k_2 \geq 3 
\end{cases} \]

Define $f(u_2v_2) = 2$, if $k_2 = 1$,

If $k_2$ is even and $k_2 \geq 2$, define

\[ f(w_{2,k_2-1}v_2) = 3, \]

\[ f(u_2w_{2,k_2-1}) = 2, \quad f(w_{2,k_2-1}w_{2,k_2-2}) = 2, \quad f(w_{2,k_2-1}w_{2,2}) = 1, \]

\[ f(w_{2,j}w_{2,j+1}) = 2, \quad f(w_{2,j+1}w_{2,j+2}) = 1, \quad \text{for } j = 2, 4, \ldots, k_2 - 4, \]

If $k_2$ is odd and $k_2 \geq 3$, define

\[ f(u_2w_{2,1}) = 2, \quad f(v_2w_{2,k_2-1}) = 2, \quad f(w_{2,k_2-1}w_{2,k_2-2}) = 1, \]

\[ f(w_{2,j}w_{2,j+1}) = 1, \quad f(w_{2,j+1}w_{2,j+2}) = 2, \quad \text{for } j = 1, 3, \ldots, k_2 - 4. \]

From the definition of $f$ it follows that adjacent vertices get distinct labels and adjacent edges get distinct labels and also labels of the incident elements have the difference at least 2. Thus $f$ satisfies all the condition of $(2,1)$-total labeling.

Therefore, $\lambda^T_2(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})) \leq \Delta + p = 5$. 

\[ \square \]

**Corollary 2.4.** Havet and Yu conjecture is true for the family cycles with parallel paths.

**Proof.** Let $G = C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})$. Then, by Theorem 2.2 and Theorem 2.3, $\lambda^T_2(G) \leq \Delta + 2 = 5$, since $\Delta(G) = 3$. Hence, Havet and Yu conjecture is true for $G = C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor n/2 \rfloor - 1}})$. 

\[ \square \]

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**References**


[10] Sumei Zhang, Oiaoling Ma, Jihui Wang, The \((2,1)\)-total labeling of \(S_{n+1} \lor P_m\) and \(S_{n+1} \times P_m\), \textit{Applied Mathematics}, 1 (2010), 366–369. http://dx.doi.org/10.4236/am.2010.15048


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