Gracefulness of Graphs Obtained from Vertex Duplication

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Abstract

A graceful labeling of a graph $G$ with $n$ edges is an injection $f : V(G) \to \{0, 1, 2, \ldots, n\}$ with the property that the resulting edge labels are distinct where an edge incident with vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. The characterization of graceful graphs is one of most difficult problems in graph theory. In this paper, we study the effectiveness of the graph operation, the duplication of all the vertices of graphs in obtaining graceful graphs. More precisely, we prove the following results.

1. If $G$ admits a stronger version of graceful labeling, $\alpha$-labeling, then the graph $D(G)$ obtained by duplication of all the vertices of $G$ admits $\alpha$-labeling.

2. The graphs obtained by the duplication of all the vertices of the non-graceful graphs, $C_4 \cup K_{1,n}$, for all $n \geq 1$ and $n \neq 2$ and $C_3 \cup P_n$, for all $n \geq 1$ are graceful.

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1 Introduction

All the graphs considered in this paper are finite and simple graphs. The terms that are not defined here can be referred from [9]. Graph labeling is one of the fastest growing areas in Graph Theory. A graph labeling is an assignment of numbers to the vertices or edges, or both, subject to certain conditions. In 1963, at the Smolience symposium, Ringel posed his celebrated conjecture which states that $K_{2n+1}$ the complete graph on $2n + 1$ vertices, can be decomposed into $2n + 1$ isomorphic copies of a given tree with $n$ edges. In 1965, Kotzig also conjectured that the complete graph $K_{2n+1}$ can be cyclically decomposed into $2n + 1$ copies of a given tree with $n$ edges. Motivated by these two conjectures, Rosa introduced ‘classical’ graph labelings in 1967 and used these labelings to investigate the cyclic decomposition of complete graphs. One of the labelings, called the $\beta$-labeling, introduced by Rosa [5] was later called graceful labeling by Golomb [3] and now this is the term most widely used. A graceful labeling of a graph $G$ with $n$ edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots , n\}$ with the property that the resulting edge labels are distinct, where an edge incident with vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph.

The graceful labeling of graphs is primarily perceived to be a theoretical subject in the field of graph theory and discrete mathematics. But gracefully labeled graphs often serve as models in a wide range of applications such as X-ray crystallography, radar, astronomy, circuit design, communication network addressing, etc. Bloom and Golomb [1] give a detailed explanation of some of these applications of gracefully labeled graphs.

Both theoretically and in the context of applications, determining which graph is graceful is important. The characterization of graceful graphs remains one of the most difficult problems in graph theory. In fact, a very few general results are proved towards characterization of graceful graphs. The following two results are the necessary conditions for a graph to be graceful.

**Theorem 1.1.** (Golomb [3]) Let $G$ be a graceful graph with $n$ vertices and $e$ edges. Let the vertices be partitioned into two sets $E$ and $O$ having respectively the vertices with even and odd labels. Then the number of edges connecting vertices in $E$ with vertices in $O$ is exactly $\frac{e+1}{2}$.

**Theorem 1.2.** (Rosa [5]) Let $G$ be an Eulerian graph. If $|E(G)|$ is equivalent to 1 or $2 \pmod{4}$ then $G$ does not have a graceful labeling.

Sethuraman and Elumalai [6] have shown that any graph can be embedded in a graceful graph. This result implies that it is not possible to characterize graceful graphs by forbidding any graph.

Number of sufficient conditions on the graceful graphs are proved. Majority of the sufficient conditions are obtained using graph operations. For
an exhaustive survey on graph labeling refer excellent survey by Gallian [2].
In this paper we consider the graph operation, duplication of vertices and we
study its effect in obtaining graceful graphs.

The duplication of a vertex $v$ of a graph $G$ is the graph $G'$ obtained from
$G$ by adding a new vertex $v'$ to $G$ such that $N(v') = N(v)$. For a graph $G$,
the graph obtained by duplication of all the vertices of $G$ is denoted by $D(G)$.

A graceful labeling $f$ on a graph $G$ is called an $\alpha$-labeling if there exists
a positive integer $\lambda$ such that for each edge $uv$ of $G$, either $f(u) \leq \lambda < f(v)$
or $f(v) \leq \lambda < f(u)$. Note that, $\alpha$-labeling is a stronger version of graceful
labeling. $\alpha$-labeled graph are necessarily bipartite. Number of interesting
results are proved on $\alpha$-labeled graphs (Refer [2]). Here we show that, the
graph obtained by duplication of all the vertices of a graph $G$, $D(G)$ preserves
$\alpha$-labeling. More precisely, we prove the following result.

**Theorem 1.3.** If $G$ is an $\alpha$-labeled graph, then the graph $D(G)$, the graph
obtained by duplication of all the vertices of $G$ admits $\alpha$-labeling.

On the other hand, it is also interesting to note that the graph $D(G)$,
the graph obtained by duplication of all the vertices of the graph $G$, becomes
graceful even if $G$ is not graceful. To illustrate this, in this paper we also prove
that the graphs obtained by duplication of all the vertices of the well-known
non-graceful graphs $C_3 \cup P_n$, for all $n \geq 2$ and $C_4 \cup K_{1,n}$, for all $n \geq 1$ and
$n \neq 2$ are graceful.

## 2 Main Result

In this section, we prove that the graph obtained by duplication of all the
vertices of an $\alpha$-labeled graph admits $\alpha$-labeling. Also we prove that, the
graphs obtained by duplication of all the vertices of the non-graceful graphs
$C_3 \cup P_n$ for all $n \geq 2$, $C_4 \cup K_{1,n}$ where $n \geq 1$ but $n \neq 2$ are graceful. The
following lemma will help to understand the graph $D(G)$.

**Lemma 2.1.** Let $G$ be a graph with $n$ vertices and $m$ edges. The graph
obtained by duplication of all the vertices of the graph $G$, $D(G)$ has $2n$ vertices
and $3m$ edges.

**Proof.** Let $u_1, u_2, \ldots, u_n$ denote the $n$ vertices of the graph $G$. Consider the
graph $D(G)$, obtained by the duplication of every vertex $u_i$, for $i$, $1 \leq i \leq n$
of $G$.

Then, by the definition of $D(G)$, $D(G)$ has $2n$ vertices in which $u_1, u_2, \ldots, u_n$
are originally the vertices of $G$ and $v_1, v_2, \ldots, v_n$ are the new vertices, where $v_i$
is the corresponding duplication of $u_i$ in $G$, for $1 \leq i \leq n$. Let $u_{i1}, u_{i2}, \ldots, u_{id}$
be the vertices adjacent to $u_i$ in $G$. Then, by the definition of $D(G)$, there
exists vertices \( v_{i1}, v_{i2}, \ldots, v_{id} \) in \( D(G) \), which are the copies of \( u_{i1}, u_{i2}, \ldots, u_{id} \) respectively and the vertices \( v_{i1}, v_{i2}, \ldots, v_{id} \) are adjacent to \( u_i \) in \( D(G) \). Hence, \( \deg_{D(G)}(u_i) = 2\deg_G(u_i) \). Refer Figure 1.

![Diagram](attachment:image.png)

Figure 1: Local structure of \( D(G) \)

Also observe that, the vertex \( v_i \) is adjacent to all the vertices \( u_{i1}, u_{i2}, \ldots, u_{id} \) which are the adjacent vertices of \( u_i \) in \( G \). Hence, \( \deg_{D(G)}(v_i) = \deg_G(u_i) \). Thus, for \( 1 \leq i \leq n \), \( \deg_{D(G)}(u_i) = 2\deg_G(u_i) \) and \( \deg_{D(G)}(v_i) = \deg_G(u_i) \). Now,

\[
\sum_{i=1}^{n} \deg_{D(G)}(u_i) + \sum_{i=1}^{n} \deg_{D(G)}(v_i) = \sum_{i=1}^{n} 2\deg_G(u_i) + \sum_{i=1}^{n} \deg_G(u_i)
\]

\[
= 3 \sum_{i=1}^{n} \deg_G(u_i)
\]

\[
= 3(2m) \text{ where } m \text{ denotes the number of edges of the graph } G.
\]

That is, \( \sum_{i=1}^{n} (\deg_{D(G)}(u_i) + \deg_{D(G)}(v_i)) = 2(3m) \).

Hence, the number of edges in the graph \( D(G) \) is \( 3m \).

**Theorem 2.2.** The graph obtained by duplication of all the vertices of an \( \alpha \)-labeled graph admits \( \alpha \)-labeling.

**Proof.** Let \( G \) be an \( \alpha \)-labeled graph with \( n \) vertices and \( m \) edges.
Let \( u_1, u_2, \ldots, u_n \) denote the vertices of the graph \( G \). Since \( G \) admits \( \alpha \)-labeling, there exists an \( \alpha \)-labeling \( f : V(G) \to \{0, 1, 2, \ldots, m\} \) and a positive integer \( \lambda \) such that, for every edge \( uv \) in \( G \), either \( f(u) \leq \lambda < f(v) \) or \( f(v) \leq \lambda < f(u) \).

Hence, the vertex set of the graph \( G \), \( V(G) \) can be partitioned into two sets say \( V_1 = \{x \in V(G) / f(x) \leq \lambda\} \) and \( V_2 = \{x \in V(G) / f(x) > \lambda\} \).

Consider the graph \( D(G) \) obtained by the duplication of every vertex \( u_i \), for \( i, 1 \leq i \leq n \, \text{of} \, G \).

Then, by definition of \( D(G) \), \( D(G) \) has \( 2n \) vertices in which \( u_1, u_2, \ldots, u_n \) are originally the vertices of \( G \) and \( v_1, v_2, \ldots, v_n \) are the new vertices, where \( v_i \) is the duplication of \( u_i \) in \( G \), for \( 1 \leq i \leq n \), which is exactly adjacent to all those adjacent vertices of \( u_i \) in \( G \) and no two \( v_i \)'s are adjacent.

For \( i = 1, 2 \), let \( V_i' \) denote the set of all vertices in \( D(G) \) which are obtained by the duplication of the vertices in the set \( V_i \).

Then, \( V_1 \cup V_1' \) and \( V_2 \cup V_2' \) are independent and hence \((V_1 \cup V_1', V_2 \cup V_2')\) is a bipartition of \( D(G) \).

The above description can be visualized in the following figure.

![Figure 2: The graph \( D(G) \)](image)

By Lemma 2.1, \( D(G) \) has \( 3m \) edges.

Let \( M = |E(D(G))| = 3m \).

For the convenience, we describe the vertices of \( V_1 \) as \( u_1, u_2, \ldots, u_\ell \) and the vertices of \( V_2 \) as \( u_{\ell+1}, u_{\ell+2}, \ldots, u_n \).

Also, for the convenience, we describe the vertices of \( V_1' \) as \( v_1, v_2, \ldots, v_\ell \) and the vertices of \( V_2' \) as \( v_{\ell+1}, v_{\ell+2}, \ldots, v_n \).

Define \( g : V(D(G)) \to \{0, 1, 2, \ldots, M\} \) in the following manner.

\[
g(u_i) = \begin{cases} 
  f(u_i) & \text{for } 1 \leq i \leq \ell, \\
  f(u_i) + m & \text{for } \ell + 1 \leq i \leq n,
\end{cases}
\]

\[
g(v_i) = \begin{cases} 
  f(v_i) + m & \text{for } 1 \leq i \leq \ell, \\
  f(v_i) + 2m & \text{for } \ell + 1 \leq i \leq n.
\end{cases}
\]

From the definition of \( g \) on \( V(D(G)) \), if the labels of vertices of \( D(G) \) are arranged as a sequence, \( g(u_1), g(u_2), \ldots, g(u_\ell), g(v_1), g(v_2), \ldots, g(v_\ell), g(u_{\ell+1}), \ldots, \),
$g(u_{\ell+2}), \ldots, g(u_n), g(v_{n-1}), \ldots, g(u_{\ell+1})$, then it forms a monotonically increasing sequence.

Hence, the vertex labels of $D(G)$ are distinct.

By the definition of $g$ on $V(D(G))$, there exists an integer $\lambda + m$ such that for each edge $uv$ of $D(G)$, $f(u) \leq \lambda + m < f(v)$.

The labels of the $m$ edges joining the vertices of the sets $V_1$ and $V_2'$ can be arranged as $M, M - 1, \ldots, M - (m - 1)$.

The labels of the $m$ edges joining the vertices of the sets $V_1$ and $V_2$ can be arranged as $M - (m - 2), M - (m - 3), \ldots, m + 1$ and the labels of the $m$ edges joining the vertices of $V_2$ and $V_1'$ can be arranged as $m, m - 1, \ldots, 1$. Thus, the labels of edges of $D(G)$ can be arranged as the disjoint union of three sets as $\{M, M - 1, \ldots, M - (m - 1)\} \cup \{M - (m - 2), M - (m - 3), \ldots, m + 1\} \cup \{m, m - 1, \ldots, 1\}$.

Thus the edge labels are distinct and it ranges from 1 to $M$.

Hence, $D(G)$ admits $\alpha$-labeling. □

Illustration

$\alpha$-labeling of the graph $D(P_3 \times P_4)$ is illustrated below.

![Figure 3: α-labeling of the grid $P_3 \times P_4$](image)

Figure 3: $\alpha$-labeling of the grid $P_3 \times P_4$

![Figure 4: The graph $D(P_3 \times P_4)$](image)

Figure 4: The graph $D(P_3 \times P_4)$
Figure 5: $\alpha$-labeling of $D(P_3 \times P_4)$ as obtained in the proof of Theorem 2.2

Figure 6: $\alpha$-labeling of $D(P_3 \times P_4)$ as in original structure
The graph obtained by duplication of all the vertices of the non-graceful graph $C_3 \cup P_n$, for all $n \geq 2$ admits graceful labeling.

**Proof.** Let $G$ denote the graph $C_3 \cup P_n$, the disjoint union of the cycle

$C_3 : u_1u_2u_3u_1$ and the path $P_n : v_1v_2 \ldots v_n$.

For $1 \leq i \leq 3$, let $u'_i$ denote the duplication of the vertices $u_i$ and for $1 \leq j \leq n$, let $v'_j$ denote the duplication of the vertices $v_j$. Let $D(C_3)$ denote the graph obtained from the duplication of all the vertices of $C_3$ and let $D(P_n)$ denote the graph obtained from the duplication of all the vertices of $P_n$.

Let $D(G)$ denote the duplication of the graph $G$.

Then, $D(G) = D(C_3) \cup D(P_n)$. Since $|V(G)| = |V(C_3)| + |V(P_n)| = n + 3$ and $|E(G)| = |E(C_3)| + |E(P_n)| = n + 2$, by definition of $D(G)$, we have $|V(D(G))| = 2n + 6$ and $|E(D(G))| = 3n + 6$.

Let $M = |E(D(G))| = 3n + 6$.

Then define $f : V(D(G)) \to \{0, 1, 2, \ldots, M\}$ in the following manner.

For $n = 2$, define $f(v_1) = M - 3$, $f(v_2) = 3$, $f(v'_1) = 8$ and $f(v'_2) = 6$.

For $n > 2$, define $f(v_1) = M - 3$, $f(v_2) = 5$, $f(v'_1) = n + 7$, $f(v'_2) = 3$, $f(v'_3) = 6$,

$$f(v'_3) = \begin{cases} 8 & \text{for } n = 3 \\ f(v'_1) - 1 & \text{for } n > 3, \end{cases}$$

$$f(v'_5) = \begin{cases} 13 & \text{for } n = 5 \\ f(v'_3) + n - 3 & \text{for } n > 5, \end{cases}$$

for $6 \leq i \leq n$ and $i$ even, define $f(v'_i) = f(v'_{i-2}) + 1$,

for $7 \leq i \leq n$ and $i$ odd, define $f(v'_i) = f(v'_{i-2}) - 1$,

for $4 \leq i \leq n$ and $i$ even, define $f(v_i) = f(v'_i) + n - 3$,

for $3 \leq i \leq n$ and $i$ odd, define $f(v_i) = f(v'_{i-2}) - 1$.

For $n$ odd, if the labels of vertices of $D(P_n)$ are arranged as a sequence of sequences,

$$(f(v_i))_{i=1,odd}^n, (f(v'_i))_{i=5,odd}^n, (f(v'_1), f(v'_3)), (f(v_{n-1}'))_{i=1,odd}^{n-4},$$

$$(f(v''_{n-1}))_{i=0,even}^{n-4}, (f(v_2), f(v'_2)),$$

then it forms a monotonically decreasing sequence.

For $n$ even, if the labels of vertices of $D(P_n)$ are arranged as a sequence of sequences

$$(f(v_i))_{i=1,odd}^n, (f(v'_i))_{i=5,odd}^n, (f(v'_1), f(v'_3)), (f(v_{n-1}'))_{i=0,even}^{n-4},$$

$$(f(v''_{n-1}))_{i=0,even}^{n-4}, (f(v_2), f(v'_2)),$$

then it forms a monotonically decreasing sequence. Further, these labels are different from the labels of vertices of $D(C_3)$, $M, M - 1, 4, 2, 1, 0$.

Hence, the vertex labels of $D(G)$ are distinct.

Let $A_1 = E(D(C_3))$, $A_2 = \{v_1v'_2, v'_2v_3, v_1v_2, v_2v_3\}$, $A_3 = \{v_3v'_4, v'_4v_5, v_5v'_6, v'_6v_7, \ldots, v_nv'_n\}$, where $\alpha = v'_{n-1}$, $v_{\beta} = v_n$ for $n$ odd or $\alpha = v_{n-1}$, $v_{\beta} = v'_n$ for $n$ even, $A_4 = \{v_3v_4, v_4v_5, \ldots, v_{n-1}v_n\}$ and $A_5 = \{v'_4v_5, v_5v'_6, v'_6v_7, \ldots, v_nv_{\beta}\}$, where $\alpha = v_{n-1}$, $v_{\beta} = v'_n$ for $n$ odd or $\alpha = v'_{n-1}$, $v_{\beta} = v_n$ for $n$ even.
We give below the edge labels of the edges in the sets $A_1, A_2, A_3, A_4$ and $A_5$ consecutively and we denote these sets by $A'_1, A'_2, A'_3, A'_4$ and $A'_5$ respectively. $A'_1 = \{M, M - 1, \ldots, M - 5, 4, 2, 1\}$, $A'_2 = \{M - 6, M - 7, M - 8, M - 9\}$, $A'_3 = \{M - 10, M - 11, \ldots, 2n\}$, $A'_4 = \{2n - 1, 2n - 2, \ldots, n + 3\}$ and $A'_5 = \{n + 2, n + 1, 3, n, n - 1, \ldots, 5\}$.

The labels in the above sets can be arranged as a monotonically decreasing sequence and it ranges from 1 to $M$. Thus, the edge labels of $D(G)$ are distinct. Hence, the graph $D(G)$ admits graceful labeling. \hfill $\Box$

**Illustration**

Graceful labeling of the graph $D(C_3 \cup P_7)$ is illustrated in the following Figure.

![Figure 7: Graceful labeling of $D(C_3 \cup P_7)$](image)

**Theorem 2.4.** The graph obtained by duplication of all the vertices of the non-graceful graph $C_4 \cup K_{1,n}$, for all $n \geq 1$ and $n \neq 2$ admits graceful labeling.

**Proof.** Let $G$ denote the graph $C_4 \cup K_{1,n}$, the disjoint union of the cycle $C_4 : u_1u_2u_3u_4u_1$ and the star $K_{1,n} : v, v_1, v_2, \ldots, v_n$ where $n \geq 1$ but $n \neq 2$ with $v$ as the centre of the star.

Let $u'_i, v'_j, v'$ denote the duplication of the vertices $u_i, v_j, v$ respectively for $1 \leq i \leq 4$ and for $1 \leq j \leq n$.

Let $D(C_4)$ denote the graph obtained from the duplication of all the vertices of $C_4$ and $D(K_{1,n})$ denote the graph obtained from the duplication of all the vertices of $K_{1,n}$ where $n \geq 1$ but $n \neq 2$.

Let $D(G)$ denote the duplication of the graph $G$.

Then, $D(G) = D(C_4) \cup D(K_{1,n})$. Since $|V(G)| = |V(C_4)| + |V(K_{1,n})| = n + 5$ and $|E(G)| = |E(C_4)| + |E(K_{1,n})| = n + 4$, by definition of $D(G)$, we have, $|V(D(G))| = 2n + 10$ and $|E(D(G))| = 3n + 12$. Let $M = |E(D(G))| = 3n + 12$.

Then define $f : V(D(G)) \to \{0, 1, 2, \ldots, M\}$ in the following manner.

For $n = 1$, define $f(v) = 6, f(v_1) = 8, f(v') = 7, f(v'_1) = 3,$
for $n > 2$, define $f(v') = M - 1$, $f(v_1) = 11$,
for $2 \leq i \leq n$, define $f(v_i) = f(v_{i-1}) + 1$,
for $3 \leq n \leq 5$, define $f(v) = f(v_n) + 1$, $f(v'_1) = 10$,
for $2 \leq i \leq n$, define $f(v'_i) = f(v'_{i-1}) - 1$,
for $n > 5$, define $f(v'_1) = f(v_n) + 1$,
for $2 \leq i \leq n$, define $f(v'_i) = f(v'_{i-1}) + 1$ and define $f(v) = f(v'_n) + 1$.

The labels of vertices of $D(C_4)$ can be arranged as a sequence, $f(u'_2), f(u'_4), f(u'_3), f(u'_1), f(u'_5), f(u)$.
From the definition of $f$, the vertex and edge labels of $D(K_{1,1})$ are distinct.
For $n \leq 5$, if the labels of vertices of $D(K_{1,n})$ are arranged as a sequence $f(v'), f(v), f(v_n), f(v_{n-1}), \ldots, f(v_1), f(v'_1), f(v'_2), \ldots, f(v'_n)$ and for $n > 5$, if the labels of vertices of $D(K_{1,n})$ are arranged as a sequence $f(v'), f(v), f(v'_n), f(v'_{n-1}), \ldots, f(v'_1), f(v_n), f(v_{n-1}), \ldots, f(v_1)$, then it forms a monotonically decreasing sequence.
Hence, the vertex labels of $D(G)$ are distinct.

Let $A_1 = E(D(C_4))$ and $A_2 = E(D(K_{1,n}))$. Let $A'_1 = \{M, M-1, \ldots, M-11\}$ and $A'_2 = \{M-12, M-13, \ldots, 1\}$ which are the labels of the edges in the set $A_1$, $A_2$ respectively.
The labels in the above set can be arranged as a monotonically decreasing sequence. Hence, the edge labels of $D(G)$ are distinct and it ranges from 1 to $M$. Thus, the graph $D(G)$ admits graceful labeling. \hfill \Box

**Illustration**

Graceful labeling of the graph $D(C_4 \cup K_{1,6})$ is illustrated in the following figure.

![Graceful Labeling of Graph](image)

**Figure 8:** Graceful labeling of $D(C_4 \cup K_{1,6})$

## 3 Discussion

Theorem 2.2 illustrates that the graph $D(G)$, the graph obtained by duplication of all the vertices of a graph $G$ preserves $\alpha$-labeling, the stronger version...
of graceful labeling, whenever $G$ has an $\alpha$-labeling. It is interesting to ask the question,

What are the other graph operations, when they are applied on graceful graphs or $\alpha$-labeled graphs, the resultant graphs are also graceful or would admit $\alpha$-labeling?

Theorems 2.3 and 2.4 illustrate that duplication operation can bring the gracefulness in the graphs obtained by the duplication of all the vertices of certain non-graceful graphs. It is interesting to explore, what are the other non-graceful graphs $G$’s for which $D(G)$’s are graceful?

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