Asynchronous Overlapping Weighted Multi-Subdomain Decomposition for Elliptic Problem

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Abstract

Our aim in this paper is to give a new approach for overlapping multi-subdomain decomposition and there linked multisplitting methods called weighted overlapping multi-subdomain decomposition which can be viewed as a generalization of asynchronous multi-subdomain decomposition and its corresponding parallel optimized Schwarz methods. As applications, we consider a Dirichlet problem with elliptic operator which is the most impressive results when theory domain decomposition was applied and we give some comparison of our generalization.

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Keywords: Asynchronous iterations, additive and multiplicative Schwarz alternating method, multisplitting method, subdomain decomposition methods and fixed point iteration
1. Introduction

In recent years, we see more and more large problem whose resolution calls on parallel computing environment. The domain decomposition method has more studied in several directions and considered as the most promising ones for parallel computer systems. We start by presenting the general context of our work and then give some results of analysis of the corresponding asynchronous iterations for the solution of second order elliptic partial differential equations which extend the work in [8]. Last and recent work in this area was give more details of a general framework for fixed point methods in general or specific space domain and other on product space. We will give in this paper a general formulation which includes generalized asynchronous multi-subdomain decomposition, generalized Schwarz alternating procedure and their Jacobi, Gauss-Seidel variants. We give a definition of asynchronous iterations which describe more involved synchronous or asynchronous parallel algorithms. For the asynchronous methods for overlapping subdomain decomposition we refer to [11] for more details. And for recent developments for the multi-subdomain multiplicative Schwarz (Gauss-Seidel) method we refer to [2, 1] and for optimized schwarz method to [10]. We introduce a weighted $L^\infty$ norms, which allows us to obtain a stronger convergence property than the usual $L^\infty$ one, even for the standard algorithms (see [6] V.2, p. 294), provided that the problem and its decomposition are sufficiently regular. After the introduction and the given notations and definitions, we present in the next section the problem formulation and state our basic assumptions. For an elliptic problem, we define the collection of the linear fixed point mapping $T^k$ which are defined as a composition of the linear mapping with a suitable restriction operator. We define then the weighted multi-subdomain decomposition formulation and there asynchronous variants. We also study the contraction property of the extended fixed point mapping combined with weighted matrices called multisplittings. Our results treat comparison between constants of contractions associated to the various fixed points mapping and the link with the study of overlap for the elliptic problem and there pseudo linear and linear problem. We finally give a comparison for different operators and a remark that shows some situations encountered in the literature.

2. Notations, definitions and asynchronous iterations:

2.1. Notations and definitions

Let us introduce:
Let $\Omega$ be an open bounded domain of $\mathbb{R}^n$ with boundary $\partial \Omega$.

A finite number of disjoint subsets of indexes $\tilde{\Omega}_k$, $k = 1, ..., m$ ($m \ll n$) of integer interval $\{1, ..., n\}$, satisfying:

$$\begin{align*}
\tilde{\Omega}_i \cap \tilde{\Omega}_j &= \emptyset, \ i \neq j, \ \bigcup_{i=1}^m \tilde{\Omega}_i = \tilde{\Omega} \\
\tilde{\Gamma}_i &= \partial \tilde{\Omega}_i \cap \Omega \\
\tilde{\gamma}_i &= \partial \tilde{\Omega}_i \cap \partial \Omega
\end{align*}$$

(2.1)

$E = E_\Omega$ (resp $E_{\Omega_k}$) a Banach space associated to $\Omega$ (resp to $\Omega_k$) by:

$$\begin{align*}
E &= E_\Omega = \prod_{i \in \Omega} E_i, \ E_{\Omega_k} = \prod_{i \in \Omega_k} E_i \\
\text{for simplicity we take } E_i &= \mathbb{R} \quad \forall i \in \Omega_k
\end{align*}$$

$A \in \mathcal{L}(\mathbb{R}^n)$ (resp $A_{\Omega \in \mathcal{L}(E_\Omega)}$) and $A_{\Omega, \Omega'} \in \mathcal{L}(\mathbb{R}^n, E_\Omega)$) denotes the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp from $E_\Omega$ to $E_\Omega$ and from $\mathbb{R}^n$ to $E_\Omega$).

We enlarge $\tilde{\Omega}_i, i \in \{1, ..., m\}$ to construct the desired overlapping decomposition which will be used by our formulation to obtain the overlapping multi-subdomain decomposition such that:

$$\tilde{\Omega}_i \subset \Omega_i \subset \Omega \ \forall i \in \{1, ..., m\}$$

(2.2)

and we define the corresponding boundaries of the overlapping subdomain decomposition by

$$\Gamma_i = \partial \Omega_i \cap \Omega, \ \gamma_i = \partial \Omega_i \cap \partial \Omega$$

(2.3)

and for the strict overlap, we write:

$$\tilde{\Omega}_i \cap \Gamma_i = \emptyset, \ i = 1, ..., m$$

(2.4)

For the exchange of information between subdomains and there overlap, we will also employ the index notation

$$\begin{align*}
\Gamma_i^j &= \Gamma_i \cap \tilde{\Omega}_j, \ j \in J(i) \\
\Omega_{i,j} &= \Omega_i \cap \Omega_j, \ j \in I(i)
\end{align*}$$

(2.5)

where the index set $I, J$ is defined by

$$\begin{align*}
J(i) &= \{j/\Gamma_i \cap \tilde{\Omega}_j \neq \emptyset, \ j \neq i\} \\
I(i) &= \{j/\Omega_i \cap \Omega_j \neq \emptyset, \ j \neq i\}
\end{align*}$$

(2.6)
We employed for $z \in E_\Omega$ the notation $[z]_\Omega$ (resp $z|_\Omega$) a restriction of $z$ to $E_\Omega$.

**Definition 1.** We say that a vector $z$ is nonnegative (positive), denoted $z \geq 0$ ($z > 0$), if all its entries are nonnegative (positive). A matrix $B$ is said to be nonnegative, denoted $B \geq 0$, if all its entries are nonnegative. We compare two matrices $A \geq B$, when $A - B \geq 0$, and two vectors $x \geq y$ ($x > y$) when $x - y \geq 0$ ($x - y > 0$).

**Definition 2.** Let $A \in \mathcal{L}(\mathbb{R}^n)$. The representation $A = M - N$ is called a splitting if $M$ is nonsingular.
It is called a convergent splitting if $\rho(M^{-1}N) < 1$.
A splitting $A = M - N$ is called:
(a) regular if $M^{-1} \geq 0$ and $N \geq 0$
(b) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

### 2.2. Asynchronous iteration

The asynchronous iteration which are used in the mathematical modelling of the parallel treatment of problems taking into account interaction processes is described as follow:
Let $E_i, i \in \{1, \ldots, m\}$ be Banach spaces equipped with the norms $| \cdot |_i$.
Consider $E = \prod_{i=1}^{m} E_i$ equipped with the norm

$$ | \cdot |_{e, \infty} = \max_{1 \leq k \leq m} \max_{i \in \Omega} | \cdot |_i $$

Define

$J = \{ J(p) \}_{p \in \mathbb{N}}$ a sequence of nonempty subsets of $\{1, \ldots, m\}$
$S = \{ \bar{\rho}_1(p), \ldots, \bar{\rho}_m(p) \}_{p \in \mathbb{N}}$ a sequence of $N^m$ such that:

(h1) $\forall i \in \{1, \ldots, m\}$, the subset $\{ p \in \mathbb{N}, i \in J(p) \}$ is infinite
(h2) $\forall i \in \{1, \ldots, m\}, \forall p \in \mathbb{N}, \bar{\rho}_i(p) \leq p$
(h3) $\forall i \in \{1, \ldots, m\}$, $\lim_{p \to \infty} \bar{\rho}_i(p) = \infty$

The asynchronous iteration associated to the operator $T$ and denoted $(T, u^0, J, S)$ is defined by,

$$ \begin{cases} 
    \text{Given } u^0 = (u^{0,1}, \ldots, u^{0,m}) \\
    \text{for } p = 1, 2, \ldots \\
    \text{for } l = 1, \ldots, m \\
    u^{t,p+1} = \begin{cases} 
        T_l(u^{1,\bar{\rho}_1(p)}, \ldots, u^{m,\bar{\rho}_m(p)}) & \text{if } l \in J(p) \\
        u^{t,p} & \text{if } l \notin J(p) 
    \end{cases}
\end{cases} $$

(2.7)

where $J(p)$ is the set of components to be updated at step $p$, $p - \bar{\rho}_i(p)$ is the delay due to $l^{th}$ processor when it computes the $l^{th}$ block at the $p^{th}$ iteration.
If we take \( \tilde{\rho}_i(p) = p \) then (2.7) describes the synchronous algorithms.
If we take
\[
\begin{align*}
\tilde{\rho}_i(p) &= p \quad \text{for } p = 1, 2, \ldots \text{ and } i = 1, 2, \ldots \\
J(p) &= \{1, 2, \ldots, m\} \quad \text{for } p = 1, 2, \ldots
\end{align*}
\]
then (2.7) describes the algorithm of Jacobi.
If we take
\[
\begin{align*}
\tilde{\rho}_i(p) &= p \quad \text{for } p = 1, 2, \ldots \text{ and } i = 1, 2, \ldots \\
J(p) &= 1 + p \pmod{m} \quad \text{for } p = 1, 2, \ldots
\end{align*}
\]
then (2.7) describes the algorithm of Gauss Seidel.

**Theorem 3.** Let \( T \) be a mapping from \( D(T) \subset E \) in \( E \) and suppose that:
(a) \( D(T) = \prod_{i=1}^{a} D_i(T) \) where \( D_i(T) \) are closed and convexes.
(b) \( T(D(T)) \subset D(T) \)
(c) \( \exists u^* \in D(T) \), such that \( u^* = T(u^*) \)
(d) \( \forall u \in D(T), |T(u) - u^*|_{e,\infty} \leq \mu|u - u^*|_{e,\infty} \) with \( 0 < \mu < 1 \)
then every asynchronous algorithm \( (T, u^0, J, S) \) associated to \( T \) with a starting point \( u^0 \in D(T) \), converges to the fixed point \( u^* \) of \( T \).

*Proof. See [14] \qed*

**Definition 4.** A mapping \( T \) which satisfies condition (d) of Theorem 3 will be said \( | \cdot |_{e,\infty} \) contractive with respect to \( u^* \).

If \( T \) is contractive in the norm \( | \cdot |_{e,\infty} \) on \( D(T) \) we say that it is \( | \cdot |_{e,\infty} \) contractive on \( D(T) \).

### 3. Weighted overlapping multi-subdomain decomposition

#### 3.1. Problem definition

We consider the Dirichlet problem model
\[
\begin{align*}
Au &= f \text{ in } \Omega \\
u &= g \text{ on } \partial \Omega
\end{align*}
\]
with given \( f \in C(\Omega) \), \( g \in C(\partial \Omega) \) and \( A \) an elliptic operator.
Let decompose (3.1), on multi-subdomain decomposition as :
\[
\begin{align*}
k &= 1, \ldots, m \\
Au_k &= f_k \text{ in } \Omega_k \\
u_k &= g_k \text{ on } \partial \Omega_k
\end{align*}
\]

We assume that the problem (3.2) have a unique solution in a suitable space denoted $u^* \in C = C(\Omega) \cap H^1(\Omega)$ and $g_k \in C(\partial \Omega_k)$ where $g_k = [g]_{\partial \Omega_k}$.

We define a non-negative peace-wise mapping $\Theta^l_k$ as follow:

$$
\Theta^l_k = \begin{cases}
I \text{ on } \Omega_k \\
\Theta^l_k \forall k \in I(k) \\
\sum_{k=1}^m \Theta^l_k = I
\end{cases}
$$

where $I$ (resp $O$) denote the identity (resp null function).

In order to define our general algorithm, we associate to (3.3) the extend problem defined by :

$$
\begin{cases}
l = 1, \ldots, m - 1 \\
k = 1, \ldots, m \\
A v^l_k = f_k \text{ in } \Omega_k, \quad v^l_k = g_k \text{ on } \partial \Omega_k \\
v^l = \sum_{i=1}^m \Theta^l_k v^l_k
\end{cases}
$$

Definition 5. We call the problem (3.4) the weighted multi-subdomain decomposition.

3.2. Weighted multi-subdomain decomposition fixed point mapping definition

3.2.1. Multi-subdomain decomposition fixed point mapping definition

For each $l = 1, \ldots, m$, we associate to (3.4’) the elliptic problem as :

$$
\begin{cases}
k = 1, \ldots, m \\
A v^l_k = f_k \text{ in } \Omega_k \\
v^l_k = g_k \text{ on } \Gamma_k \\
v^l_k|_{\Gamma^l_k} = v^l_j|_{\Gamma^l_k} \forall j \in J(k)
\end{cases}
$$

In order to resolve this problem, let us introduce :

- Hypothesis :

  - (H1): The elliptic operator $A$ satisfies both the standard maximum principle with respect to $\Omega_i$ and $\partial \Omega_i$ and the Hopf maximum principle with respect to $\gamma_i$. 

(H2): Moreover to the Greens function
\[ G_i(x, y) \succ 0, \quad x, y \in \Omega_i, \quad x \neq y \]
corresponds a positive Poisson Kernel \( \partial G_i/\partial v_i(x, y) \) continuously differentiable on \( \Omega_i \times \partial \Omega_i, \ x \neq y \) where \( \partial/\partial v_i \) denotes the unit outward normal derivative with respect to \( \partial \Omega_i \).

(H3): There exists a positive eigenfunction \( e \in C(\bar{\Omega}) \) associated with the smallest eigenvalue \( \lambda \) of \( A \).

\[ Ae = \lambda e, \quad \lambda \in \mathbb{R}^+ \]

For each \( l \in \{1, \ldots, m\} \), in order to resolve (3.5), we associate the decoupled system as:
\[
\begin{cases}
A v^l_k = f_k & \text{in } \Omega_k \\
v^l_k = g_k & \text{on } \Gamma_k \\
v^l_k = 0 & \text{on } \Gamma^j_k
\end{cases}
\]
(3.6)
denote \( u^l_k = v^l_k|_{\Gamma^j_k} \) and let us consider an element \( w^l \) such that
\[
\begin{cases}
k = 1, \ldots, m \\
A v^l_k = f_k & \text{in } \Omega_k \\
v^l_k = g_k & \text{on } \Gamma_k \\
v^l_k|_{\Gamma^j_k} = w^l_{k,j}
\end{cases}
\]
(3.7)
where
\[
\begin{cases}
w^l_k = \{\ldots, w^l_{k,j}, \ldots\} \\
w^l_{k,j} = v^l_{j}|_{\Gamma^j_k} & \forall j \in J(k)
\end{cases}
\]
(3.8)
Then, we define the fixed point mapping \( T^l \) by:
\[ v^l = T^l(w^l) + u^l \]
(3.9)
and there corresponding weighted norm by:
\[ ||e||_{e,\infty} = \max_{1 \leq k \leq m} \max_{x \in \Omega_k} \frac{|e(x)|}{e(x)} \]

**Proposition 6.** Under hypothesis and (3.7)-(3.8), the fixed point mapping \( F^l(w^l) = T^l(w^l) + u^l \) is \( ||e||_{e,\infty} \) contractive with constant of contraction \( \mu_l < 1 \).

**Proof.** See [11, 8] and there related references.

**Lemma 7.** Any asynchronous algorithm \( (F^l, w^0, J, S) \) corresponding to \( F^l \) and starting with \( w^0 \) converge to the solution \( u^* \) of (3.1).
3.2.2. Weighted multi-subdomain decomposition fixed point mapping definition

Define the fixed point mapping:

\[ T^{WM} : \mathcal{C}^m \to \mathcal{C}^m \]

\[ W = (\bar{w}^1, \ldots, \bar{w}^m) \mapsto W = (w^1, \ldots, w^{m-1}) \]

such that for \( 1 \leq l \leq m \)

\[
\begin{cases}
  w^l = F^l(z^l) \\
  z^l = \sum_{k=1}^{m} \Theta^l_k \bar{w}^k
\end{cases}
\]  

(3.10)

**Proposition 8.** Denote \( U^* = (u^*, \ldots, u^*) \) where \( u^* \) is the solution of (3.1), then the fixed point mapping \( T^{WM} \) is \( \Pi_{e,\infty} \) contractive with respect to \( U^* \), it’s constant of contraction is

\[ \mu = \max_{1 \leq l \leq m} \mu_l \]

and \( U^* \) is the fixed point of \( T^{WD} \).

**Proof.** Let \( W = T^{WD}(\bar{W}) \), by (3.10) and Proposition 6, we have:

\[ |F^l(\sum_{k=1}^{m} \Theta^l_k \bar{w}^k) - u^*|_{e,\infty} \leq \mu_l |\sum_{k=1}^{m} \Theta^l_k \bar{w}^k - u^*|_{e,\infty} = \mu_l |\sum_{k=1}^{m} \Theta^l_k (\bar{w}^k - u^k)|_{e,\infty} \]

and

\[
\max_{1 \leq k \leq m, x \in \Omega_k} \max_{1 \leq l \leq m} \frac{|F^l(\sum_{k=1}^{m} \Theta^l_k \bar{w}^k) - u^*|}{e(x)} \leq \mu_l \max_{1 \leq k \leq m, x \in \Omega_k} \frac{|(\sum_{k=1}^{m} \Theta^l_k (\bar{w}^k - u^k))(x)|}{e(x)}
\]

\[ \leq \mu_k \max_{1 \leq k \leq m, x \in \Omega_k} \frac{|\sum_{k=1}^{m} \Theta^l_k(x)(\bar{w}^k(x) - u^k(x))|}{e(x)} \]

\[ \leq \mu_k \max_{1 \leq k \leq m, x \in \Omega_k} \frac{|\sum_{k=1}^{m} \Theta^l_k(x)||\bar{w}^k(x) - u^k(x)|}{e(x)} \]

By the definition of \( \Theta \), we have

\[
\max_{1 \leq k \leq m, x \in \Omega_k} \frac{|F^l(\sum_{k=1}^{m} \Theta^l_k \bar{w}^k) - u^*|}{e(x)} \leq \mu_l \max_{1 \leq k \leq m, x \in \Omega_k} \frac{||\bar{w}^k(x) - u^k(x)|}{e(x)}
\]

\[ \leq \max_{1 \leq k \leq m} \mu_l \max_{1 \leq k \leq m, x \in \Omega_k} \frac{|\bar{w}^k(x) - u^k(x)|}{e(x)} \]

Consequently

\[ |W - U^*|_{e,\infty} \leq \mu|\bar{W} - U^*|_{e,\infty} \]

\[ \square \]
**Corollary 9.** Under the assumption of proposition 8, any asynchronous algorithm \((T^{WD}, U^0, J, S)\) corresponding to \(T^{WD}\) and starting with \(U^0\) converge to the solution of (3.1).

### 3.2.3. Limited studies and particular situation

In this section, we use the weighted multi-subdomain decomposition which can generalize some recent work in this direction. By introducing the non-negative peacewise mapping \(\Theta^l_k\), our formulation method allows in order to give our framework to include:

1. Discrete cases
   
   (a) to take \(m = 2\) and \(\Theta^l_k = I\) for \(k = 1, 2\) we found the classical Schwarz method and there additive and multiplicative variants (see [4, 3]).
   
   (b) to take \(\Theta^l_k = \theta I\), \(\theta \in \mathbb{R}^+\) we found the dumbed Schwarz method and there additive and multiplicative variants (see [5]).
   
   (c) to take \(\Theta^l_k = \theta^l_k I\), \(\theta^l_k \in \mathbb{R}^+\) depending on the index \(l\) in order to give a presentation of either the Schwarz alternating method or general Schwarz multisplitting variant methods.

2. Continuous cases
   
   (a) to take \(l = 1\) and \(\Theta^l_k(x) = \text{dist}(x, \Omega_k)\), we found some resolution of classical Schwarz method (see [12])
   
   (b) to take \(\Theta^l_k(x)\) depending on both the index \(l\) and on the element \(x\) of \(\mathbb{R}^n\).

### 3.3. Discrete weighted multi-subdomain decomposition iteration and some convergence comparison results

#### 3.3.1. Discrete weighted multi-subdomain iteration

In order to give an overview of our analogous algebraic formulation, let us consider the following problem: for \(l = 1, ..., m\)

\[
\begin{align*}
    Au^l + \Lambda(u^l) &= f \text{ in } \Omega \\
    u^l &= g \text{ on } \partial\Omega \\
\end{align*}
\]  

(3.11)

where \(\Lambda\) a is a continuous diagonal (possibly multivalued) maximal monotone operator and there discretized problem

\[
Ax^l + \Lambda(x^l) = b
\]  

(3.12)
where \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) and \( b \in \mathbb{R}^n \) and note that \( Ax^* + \Lambda(x^*) = b \) where \( x^* \) was the unique solution of (3.11).

**Lemma 10.** Under (H1) and (3.11) notation, for \( l = 1, ..., m \) \( A \) and \( A_\Omega \) are M-matrix.

**Proof.** See [4, 11] and [13]. \( \square \)

Let us define for \( k = 1, ..., m \) a collection of the fixed mapping \( T^k \):

\[
x^k \in \mathbb{R}^n \mapsto y^k \in \mathbb{R}^n
\]

where

\[
\begin{cases}
[b]_\Omega_k - A_\Omega_k c_{\Omega_k} x^k - A_\Omega_k y^k \in \Lambda_k (y^k) \\
[b]_\Omega_k - A_\Omega_k c_{\Omega_k} x^k - A_\Omega_k c_{\Omega_k} y^k \in \Lambda_k (y^k)
\end{cases}
\]  

(3.13)

In (3.13), we consider this decomposition because we never have to use any component of \( y^k \) in \( \mathbb{C}_{\Omega_k} \) in the evaluation of \( T^K \), for more details see [4].

Let us consider now the extended fixed mapping \( T^{W M} \) by:

\[
(x^1, ..., x^m) \in (\mathbb{R}^n)^m \mapsto (y^1, ..., y^m) \in (\mathbb{R}^n)^m
\]

where

\[
\begin{cases}
[b]_\Omega_k - A_\Omega_k c_{\Omega_k} z^k - A_\Omega_k y^k \in \Lambda_k (y^k) \\
z^k = \sum_{l=1}^m \Theta_k^l x^l
\end{cases}
\]  

(3.14)

**Proposition 11.** Under (H3) and Lemma 10, \( T^{W M} \) is \( |e, \infty \) contractive with respect to \( U^* \).

**Proof.** Let us introduce a vector \( h \in \mathbb{R}^n, h \succ 0 (h_i \succ 0, i = 1, ..., n) \) and by (H3) there exist an \( e \) such that \( Ae = h \).

We consider also \( \varphi^{k,e} \in E_{\Omega_k} \) a solution of

\[
A_{\Omega_k} \varphi^{k,e} = -A_{\Omega_k} c_{\Omega_k} [e]_{\mathbb{C}_{\Omega_k}}
\]  

(3.15)

We have \( A_{\Omega_k} [e]_{\Omega_k} = -A_{\Omega_k} c_{\Omega_k} [e]_{\mathbb{C}_{\Omega_k}} + [h]_{\Omega_k} \), as \( A \) is an M-matrix, so \( A_{\Omega_k} \) enjoys the same property and with the strict positivity of the components of \( h \) then we have:

\[
\varphi^{k,e} \succ [e]_{\Omega_k}
\]

then

\[
\mu_k = \max_{i \in \Omega_k} \left| \frac{\varphi^{k,e}_i}{e_i} \right| < 1
\]  

(3.16)

Let

\[
\begin{cases}
b_i - \sum_{j \in \Omega_k} a_{ij} y^k - \sum_{j \in \mathbb{C}_{\Omega_k}} a_{ij} z^k \in \Lambda_i(y^k) \forall i \in \Omega_k, k = 1, ..., m \\
z^k = \sum_{l=1}^m \Theta_k^l x^l
\end{cases}
\]
Wich can be written:

\[
\begin{align*}
\chi^k_i &= b_i - \sum_{j \in \Omega_k} a_{ij} y^k_j - \sum_{j \in \Omega_k} a_{ij} z^k_j \forall i \in \Omega_k, \ k = 1, \ldots, m \quad \text{with } \chi^k_i \in \Lambda_i(y^k_i) \\
\chi^*_i &= b_i - \sum_{j \in \Omega_k} a_{ij} u^*_j - \sum_{j \in \Omega_k} a_{ij} u^*_j \forall i \in \Omega_k, \ k = 1, \ldots, m \\
z^k &= \sum_{l=1}^m \Theta_l^k x^l
\end{align*}
\]

let us write:

\[
\begin{align*}
\delta y^k_j &= y^k_j,1 - u^*_j \delta z^k_j = z^k_j - u^*_j \quad \text{and } \delta x^k_j = x^k_j - u^*_j
\end{align*}
\]

by substracting membre to membre the relation, we obtain:

\[
\chi^k_i - \chi^*_i = - \sum_{j \in \Omega_k} a_{ij} \delta y^k_j - \sum_{j \in \Omega_k} a_{ij} \delta z^k_j \forall i \in \Omega_k, \ k = 1, \ldots, m
\] (3.17)

Let us consider the multi-valued mapping:

\[
x \in \mathbb{R} \to \text{sign}(x) = \begin{cases} 
+1 & \text{if } x > 0 \\
[-1, +1] & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

Let us define \( \text{sgn} (\delta y^k) \in E_{\Omega_k} \) by:

\[
\text{sgn} (\delta y^k) = \{ \ldots, \text{sgn} (\delta y^k_i), \ldots \} \forall i \in \Omega_k
\]

Then, let us multiply the two membre of (3.17) by

\[
\text{sgn} (\delta y^k_i) \in \text{sign} (\delta y^k_i) \forall i \in \Omega_k
\]

which gives rise to:

\[
\begin{align*}
\forall i \in \Omega_k, \ k = 1, \ldots, m \\
\left( \chi^{k,1}_i - \chi^*_i \right) \text{sgn} (\delta y^k_i) + \left( \sum_{j \in \Omega_k} a_{ij} \delta y^k_j \right) \text{sgn} (\delta y^k_i) = - \left( \sum_{j \in \Omega_k} a_{ij} \delta z^k_j \right) \text{sgn} (\delta y^k_i)
\end{align*}
\] (3.18)

by the monotonity property of \( \Lambda \),

\[
\left( \chi^{k,1}_i - \chi^*_i \right) \text{sgn} (\delta y^k_i) \geq 0
\]

then:

\[
\left( \sum_{j \in \Omega_k} a_{ij} \delta y^k_j \right) \text{sgn} (\delta y^k_i) \leq - \left( \sum_{j \in \Omega_k} a_{ij} \delta z^k_j \right) \text{sgn} (\delta y^k_i) \forall i \in \Omega_k, \ k = 1, \ldots, m
\]
As $A$ is $M$-matrix, then:

$$\sum_{j \in \Omega_k} a_{ij} |\delta y_j^k| \leq \left( \sum_{j \in \Omega_k} a_{ij} \delta y_j^k \right) \operatorname{sgn} (\delta y_i^k) \forall i \in \Omega_k, \, k = 1, \ldots, m$$

and,

$$- \left( \sum_{j \in \emptyset_{\Omega_k}} a_{ij} \delta z_j^k \right) \operatorname{sgn} (\delta y_i^k) \leq - \sum_{j \in \emptyset_{\Omega_k}} a_{ij} |\delta z_j^k| \forall i \in \Omega_k, \, k = 1, \ldots, m$$

so the previous inequations imply:

$$\sum_{j \in \Omega_k} a_{ij} |\delta y_j^k| \leq - \sum_{j \in \emptyset_{\Omega_k}} a_{ij} |\delta z_j^k| \forall i \in \Omega_k, \, k = 1, \ldots, m$$

With vectorial form notation, we have respectively:

$$(A_{\Omega_k} \delta y^k) . \operatorname{sgn} (\delta y^k) \leq (-A_{\Omega_k} \epsilon_{\Omega_k} \delta z^k) . \operatorname{sgn} (\delta y^k)$$

and

$$A_{\Omega_k} |\delta y^k| \leq (A_{\Omega_k} \delta y^k) . \operatorname{sgn} (\delta y^k)$$

and

$$(-A_{\Omega_k} \epsilon_{\Omega_k} \delta z^k) . \operatorname{sgn} (\delta y^k) \leq -A_{\Omega_k} \epsilon_{\Omega_k} |\delta z^k|$$

then, we obtain in the vectorial form

$$A_{\Omega_k} |\delta y^k| \leq -A_{\Omega_k} \epsilon_{\Omega_k} |\delta z^k|$$

$A_{\Omega_k}$ is, as $A$, an $M$-matrix, so:

$$|\delta y^k| \leq -A_{\Omega_k}^{-1} A_{\Omega_k} \epsilon_{\Omega_k} |\delta z^k|$$

with:

$$B_k = -A_{\Omega_k}^{-1} A_{\Omega_k} \epsilon_{\Omega_k} \in \mathcal{L}(\mathbb{R}^n, E_{\Omega_k})$$

which allows to write:

$$|\delta y_i^k| \leq \sum_j |b_{ij}^k| e_j \frac{|\delta z_j^k|}{e_j} \leq \left( \sum_j |b_{ij}^k| e_j \right) \max_j \frac{|\delta z_j^k|}{e_j}$$

The relation (3.15) gives rise to:

$$\varphi_{k,e} = A_{\Omega_k}^{-1} \left( A_{\Omega_k} \epsilon_{\Omega_k} e \right) = B_k e$$
Then, we have:

\[ |\delta y^k_i| \leq \varphi^e_i \max_{j \in \Omega_k} \frac{|\delta z^k_j|}{e_j}, \quad i \in \Omega_k \]

that is to say:

\[ \frac{|\delta y^k_i|}{e_i} \leq \frac{\varphi^e_i}{e_i} \max_{j \in \Omega_k} \frac{|\delta z^k_j|}{e_j}, \quad i \in \Omega_k \]

\[ \max_{i \in \Omega_k} \frac{|\delta y^k_i|}{e_i} \leq \max_{i \in \Omega_k} \frac{\varphi^e_i}{e_i} \max_{j \in \Omega_k} \frac{|\delta z^k_j|}{e_j} = \mu_k \max_{j \in \Omega_k} \frac{|\delta z^k_j|}{e_j} \]

with

\[ \delta z^k = \sum_{l=1}^m \Theta^k_l x^l - u^* = \sum_{l=1}^m \Theta^k_l (x^l - u^*) \]

Since the weighting matrices \( \Theta^k_l \) are diagonal, we have

\[ \frac{|\delta z^k_j|}{e_j} = \sum_{l=1}^m (\Theta^k_l)_{jj} \frac{(x^l - u^*)_j}{e_j} = \sum_{l=1}^m (\Theta^k_l)_{jj} \frac{|\delta x^k_j|}{e_j} \]

then

\[ \max_{j \in \Omega_k} \frac{|\delta z^k_j|}{e_j} \leq \max_{j \in \Omega_k} \frac{|\delta x^k_j|}{e_j} \]

and

\[ \max_{1 \leq k \leq m} \max_{i \in \Omega_k} \frac{|\delta y^k_i|}{e_i} \leq \max_{1 \leq k \leq m} \mu_k \left( \max_{1 \leq k \leq m} \max_{j \in \Omega_k} \frac{|\delta x^k_j|}{e_j} \right) \leq \mu |\delta X|_{\infty,e} \]

which proves that:

\[ |T^{WM}(X) - U^*|_{\infty,e} \leq \mu |X - U^*|_{\infty,e} \] where \( \mu = \max_{1 \leq k \leq m} \mu_k < 1 \)

\[ \square \]

Corollary 12. Any asynchronous algorithm \((T^{WD}, U^0, J, S)\) corresponding to \(T^{WD}\) and starting with \(U^0\) converge to the solution of (3.12).

Proof. Direct application of Theorem 3. \( \square \)

4. Comparison results

In this section, we will give some comparison results about the constants of contraction of the fixed point mapping corresponding to different overlapping domain decomposition and there weighted multi-subdomain decomposition.
Without lose a generality, we restrict our comparison to algebraic linear problem in the form:

\[
\begin{aligned}
x^k \in \mathbb{R}^n & \quad T^k \rightarrow y^k \in \mathbb{R}^n \\
M_k y^k - N_k x^k &= b
\end{aligned}
\]

where \( M_k = \begin{pmatrix} A_{\Omega} & 0 \\ 0 & A_{\Omega^c \cup \Omega^c} \end{pmatrix} \) and \( N_k = \begin{pmatrix} 0 & A_{\Omega^c \cup \Omega^c} \\ A_{\Omega^c \cup \Omega^c} & 0 \end{pmatrix} \)

### 4.1. General case

For \( k = 1, \ldots, m \), let us consider two subsets of indexes \( \Omega^1_k \) and \( \Omega^2_k \) satisfying:

\( \tilde{\Omega}_k \subseteq \Omega^1_k \subseteq \Omega^2_k \)

**Proposition 13.** With our previous notations and assumptions, the collection of fixed point mapping \( T^{k,1} \) and \( T^{k,2} \) associated respectively to \( \Omega^1_k \) and \( \Omega^2_k \) are contractives with constants of contraction:

\( \mu^2_k \leq \mu^1_k \)

**Proof.** With our previous notations and assumptions, for \( k = 1, \ldots, m \), \( \forall i \in \Omega^1_k \) assume that \( \varphi^{k,e,1} \) and \( \varphi^{k,e,2} \) are solutions of

\[
\begin{aligned}
\sum_{j \in \Omega^1_k} a_{ij} \varphi^{k,e,1}_j &= - \sum_{j \in \Omega^2_k} a_{ij} \varphi^{k,e,1}_j \\
\sum_{j \in \Omega^2_k} a_{ij} \varphi^{k,e,2}_j &= - \sum_{j \in \Omega^2_k} a_{ij} \varphi^{k,e,2}_j
\end{aligned}
\]

where:

\( \mathcal{C}_{\Omega^1_k} = \Omega^2_k \cap \Omega^1_k \cup \mathcal{C}_{\Omega^2_k} \) and \( \Omega^2_k \cap \Omega^1_k \cap \mathcal{C}_{\Omega^2_k} = \emptyset \)

so:

\[- \sum_{j \in \mathcal{C}_{\Omega^1_k}} a_{ij} e_j = - \sum_{j \in \Omega^2_k \cap \Omega^1_k} a_{ij} e_j - \sum_{j \in \mathcal{C}_{\Omega^2_k}} a_{ij} e_j\]

then:

\[\sum_{j \in \Omega^1_k} a_{ij} \varphi^{k,e,1}_j = - \sum_{j \in \Omega^2_k \cap \Omega^1_k} a_{ij} e_j - \sum_{j \in \mathcal{C}_{\Omega^2_k}} a_{ij} e_j\]

we have also:

\[\sum_{j \in \Omega^1_k} a_{ij} \varphi^{k,e,2}_j = - \sum_{j \in \Omega^2_k \cap \Omega^1_k} a_{ij} \varphi^{k,e,2}_j - \sum_{j \in \mathcal{C}_{\Omega^2_k}} a_{ij} e_j\]

with:

\( \varphi^{k,e,2}_j \prec e_j \quad \forall j \in \Omega^2_k \)
then, we have:

\[ \sum_{j \in \Omega_k^1} a_{ij} \left( \varphi_j^{k,e,2} - \varphi_j^{k,e,2} \right) = \sum_{j \in \Omega_k^2 \cap \Omega_k^1} a_{ij} \left( e_j - \varphi_j^{k,e,2} \right) \geq 0 \quad (4.1) \]

finally, we obtain:

\[ \varphi_j^{k,e,1} - \varphi_j^{k,e,2} \geq 0 \ \forall j \in \Omega_k^1 \]

then

\[ \frac{\varphi_j^{k,e,1}}{e_j} \geq \frac{\varphi_j^{k,e,2}}{e_j} \ \forall j \in \Omega_k^1 \]

\[ \max_{j \in \Omega_k^2} \left| \frac{\varphi_j^{k,e,2}}{e_j} \right| = \mu_k^2 \leq \max_{j \in \Omega_k^1} \left| \frac{\varphi_j^{k,e,1}}{e_j} \right| \ \forall j \in \Omega_k^1 \]

which proves:

\[ \mu_k^2 \leq \max_{j \in \Omega_k^1} \left| \frac{\varphi_j^{k,e,1}}{e_j} \right| = \mu_k^1 < 1 \]

\[ \square \]

**Corollary 14.** The weighted multi subdomain decomposition \( T^{WM,1}_k \) and \( T^{WM,2}_k \) corresponding to the collection \( T^{k,1} \) and \( T^{k,2} \), are contractives with constants of contraction:

\[ \mu^2 \leq \mu^1 \]

**Proof.** From the previous proposition and the proof of proposition 11, we have for \( k = 1, \ldots, m \) : \( \mu^2 \leq \mu^1 \). Then,

\[ \mu^2 = \max_{1 \leq k \leq m} \mu_k^2 \leq \max_{1 \leq k \leq m} \mu_k = \mu^1 \]

\[ \square \]

### 4.2. Strict inequalities

Let us assume now that:

\[ \left\{ A \ and \ A_{\Omega_k^1} \ \forall k = 1, \ldots, m \right\} \]

are irreductible matrices

Let us introduce:

\[ V_{\Omega_k^1} = \left\{ j \in C_{\Omega_k^1} / \exists i \in \Omega_k^1 \text{ with } a_{ij} \neq 0 \right\} \]
then by the irreducibility of $A$, we have:

$$V_{\Omega_k^1} \neq \emptyset$$

Let us assume that:

$$\tilde{\Omega}_k \subseteq \Omega_k^1 \subset \Omega_k^2 and V_{\Omega_k^1} \cap \Omega_k^2 \neq \emptyset$$

(4.2)

**Proposition 15.** Under our previous assumptions and notations, we have:

$$\mu_k^2 < \mu_k^1 and \mu^2 < \mu^1$$

**Proof.** Let us consider again (4.1) which reads:

$$A_{\Omega_k^1} \left( \varphi_{k,e}^{k,e,1} - [\varphi_{k,e}^{k,e,2}]_{\Omega_k^1} \right) = -A_{\Omega_k^1,\Omega_k^2 \ominus \Omega_k^1} (e - \hat{\varphi}_{k,e}^{k,e,2})$$

where $\hat{\varphi}_{k,e}^{k,e,2} \in \mathbb{R}^n$ is an extension of $\varphi_{k,e}^{k,e,2}$ from $E_{\Omega_k^2}$ to $\mathbb{R}^n$, that is to say $\hat{\varphi}_{k,e}^{k,e,2} \in \mathbb{R}^n$ and:

$$\hat{\varphi}_{j}^{k,e,2} = \varphi_{j}^{k,e,2} \forall j \in \Omega_k^2$$

moreover:

$$[e]_{\Omega_k^2} - \varphi_{k,e}^{k,e,2} \succ 0$$

Then by assumption (3.15):

$$A_{\Omega_k^1} \left( \varphi_{k,e}^{k,e,1} - [\varphi_{k,e}^{k,e,2}]_{\Omega_k^1} \right) = -A_{\Omega_k^1,\Omega_k^2 \ominus \Omega_k^1} (e - \hat{\varphi}_{k,e}^{k,e,2}) \npreceq 0$$

By the irreducibility assumption of any matrix $A_{\Omega_k^1} \forall k = 1, ..., m$, we know that:

all the coefficients of $A_{\Omega_k^1}^{-1}$ are strictly positive

and (4.2) implies that:

$$\varphi_{k,e}^{k,e,1} - [\varphi_{k,e}^{k,e,2}]_{\Omega_k^1} \succ 0$$

so we obtain:

$$\mu_k^2 < \mu_k^1 and \mu^2 < \mu^1$$

4.3. **Comparison between $A = -\Delta$ and $A = -\varepsilon \Delta + I$**

By (H2), we have $Ae = \lambda e$ and let decompose $e$ us follow: For $k = 1, ..., m$,

$$\begin{cases}
  e_k = [e]_{\Omega_k} \\
  e_k = \varphi_{k,e} + \psi_{k,e}
\end{cases}$$
such that

\[
\begin{align*}
A\varphi^{k,e} &= 0 \text{ in } \Omega_k \\
\varphi^{k,e} &= e_k \text{ on } \partial\Omega_k
\end{align*}
\]

and

\[
\begin{align*}
A\psi^{k,e} &= \lambda e_k \text{ in } \Omega_k \\
\psi^{k,e} &= e_k \text{ on } \partial\Omega_k
\end{align*}
\]

\[\varphi^{k,e} \succ 0, \psi^{k,e} \succ 0 \text{ in } \Omega_k\]

If we replace \(A\) by \(A = -\varepsilon \Delta + I\), then

\[
\begin{align*}
-\varepsilon \Delta e + e &= (\varepsilon \lambda + 1)e \text{ in } \Omega \\
e &= 0 \text{ on } \partial\Omega
\end{align*}
\]

(4.3)

and \(A\) by \(A = -\Delta\), then

\[
\begin{align*}
-\Delta e &= \lambda e \text{ in } \Omega \\
e &= 0 \text{ on } \partial\Omega
\end{align*}
\]

For \(k = 1, \ldots, m\),

We decompose (4.3) us :

\[
\begin{align*}
-\varepsilon \Delta \varphi^{k,e,1} + \varphi^{k,e,1} &= 0 \text{ in } \Omega_k \\
\varphi^{k,e,1} &= e_k \text{ on } \partial\Omega_k
\end{align*}
\]

(4.4)

and

\[
\begin{align*}
-\Delta \varphi^{k,e,2} &= 0 \text{ in } \Omega_k \\
\varphi^{k,e,2} &= e_k \text{ on } \partial\Omega_k
\end{align*}
\]

(4.5)

\[\varphi^{k,e,1} \succ 0, \varphi^{k,e,2} \succ 0 \text{ in } \Omega_k\]

**Proposition 16.** Under previous notations, we have \(\mu^1 \prec \mu^2 \prec 1\)

*Proof.* The (4.5) can be write us :

\[
\begin{align*}
-\varepsilon \Delta \varphi^{k,e,2} &= 0 \text{ in } \Omega_k \\
\varphi^{k,e,2} &= e_k \text{ on } \partial\Omega_k
\end{align*}
\]

(4.6)

By subtracting (4.4) from (4.6), we have :

\[
\begin{align*}
-\varepsilon \Delta (\varphi^{k,e,1} - \varphi^{k,e,2}) &= -\varphi^{k,e,1} \text{ in } \Omega_k \\
\varphi^{k,e,1} - \varphi^{k,e,2} &= 0 \text{ on } \partial\Omega_k
\end{align*}
\]

By (H1), we obtain

\[\varphi^{k,e,1} - \varphi^{k,e,2} \prec 0 \text{ in } \Omega_k\]
consequently

\[ \frac{\varphi^{k,e,1}}{e(x)} \prec \frac{\varphi^{k,e,2}}{e(x)} \text{ in } \Omega_k \]

then

\[ \mu^1_k = \max_{x \in \Omega_k} \frac{\varphi^{k,e,1}}{e(x)} < \max_{x \in \Omega_k} \frac{\varphi^{k,e,2}}{e(x)} = \mu^2_k \]

so

\[ \mu^1 = \max_{1 \leq k \leq m} \mu^1_k < \mu^2 = \max_{1 \leq k \leq m} \mu^2_k \]

Remark 17. In order to study some situation as additive (resp multiplicative) Schwarz method, we need to applied block Jacobi iteration and there asynchronous iteration to the fixed point mapping \( T^{W,M} \) defined in the previous section (resp we need to applied block Gauss-Seidel iteration and there asynchronous iteration to the fixed point mapping \( T^{W,M} \) defined in the previous section). Another important point to find the optimized Schwarz method is to take the weighted matrices \( \Theta^l_k = \Theta_k \) independently of \( l \).

References


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