On the Exact Sequence and Variety of Polydules

Darmajid
Algebra Research Division
Institut Teknologi Bandung, Indonesia

Intan Muchtadi-Alamsyah
Algebra Research Division
Institut Teknologi Bandung, Indonesia

Dellavitha Nasution
Algebra Research Division
Institut Teknologi Bandung, Indonesia

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Abstract

Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field $k$. We study variety $\mathcal{W}_d^\Lambda(k)$ parameterizing $\Lambda$-polydules. This variety carries an action of an algebraic group such that orbits correspond to quasi-isomorphism classes of complexes in the derived category. We investigate some relation between exact triangle of polydules and variety of polydules.

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1 Introduction

Throughout this paper, let \( \Lambda \) be a finite-dimensional algebra over an algebraically closed field \( k \). Keller has observed in [2] that by fixing dimensions in homology \( d = (d_a, \ldots, d_b) \) we obtain a variety, denoted by \( \mathcal{M}_d(\mathcal{k}) \), which parameterizes polydules with homology dimensions \( d \). There is an algebraic group \( \mathfrak{S}_d^{\Lambda}(\mathcal{k}) \) acting on \( \mathcal{M}_d(\mathcal{k}) \) such that the orbits correspond to the quasi-isomorphism classes of polydules. Jensen, Madsen, and Su [1] use variety \( \mathcal{M}_d(\mathcal{k}) \) to study about degeneration for polydules with finite dimension in homology. In this paper we investigate some relation between exact triangle of polydules and varieties of polydules.

2 Preliminary Notes

In this section we recall some definitions on polydules over algebra and the variety that parameterize it. For details, we refer to [1].

Let \( A \) be an algebra over commutative ring \( R \). A polydule over \( A \), or \( A \)-polydule is a \( \mathbb{Z} \)-graded \( R \)-module \( M = \bigoplus M_i \) with \( R \)-linear \( \mathbb{Z} \)-graded maps \( m_i^M : A^{(\otimes_R)t-1} \otimes_R M \to M, \ t \geq 1 \) of degree \( 2 - t \) satisfying the polydule structure: \( m_1^M m_1^M = 0; \ m_1^M m_2^M = m_2^M (1 \otimes m_1^M) \); and for \( t \geq 3 \),

\[
\sum_{j=1}^{t} (-1)^{(t-1)} m_{t-j+1}^M (1^{\otimes t-j} \otimes m_j^M) = \sum_{j=1}^{t-2} (-1)^{j-1} m_{t-1}^M (1^{\otimes t-j-2} \otimes \mu \otimes 1^{\otimes j}),
\]

where \( 1 \) is the identity map and \( \mu \) is the multiplication in \( A \).

A morphism \( f : M \to N \) between two \( A \)-polydules \( M \) and \( N \) is a family of maps \( f_t : A^{(\otimes_R)t-1} \otimes_R M \to N, \ t \geq 1 \) of degree \( 1 - t \) satisfying the rules:

\[
f_t m_i^M = m_i^N f_t; \ f_t m_1^M = m_1^N (1 \otimes f_t) + m_2^N f_t; \ \text{and for } t \geq 3,
\]

\[
\sum_{j=1}^{t} (-1)^{(t-1)} f_{t-j+1} (1^{\otimes t-j} \otimes m_j^M) + \sum_{j=1}^{t-2} (-1)^{j} f_{t-1} (1^{\otimes t-j-2} \otimes \mu \otimes 1^{\otimes j}) = \sum_{r=1}^{t} (-1)^{(t-r+1)} m_{t-r+1}^N (1^{\otimes t-r} \otimes f_t).
\]

Let \( \mathcal{D} (\text{Mod } \Lambda) \) and \( \mathcal{D}_\infty (\Lambda) \) denote the derived category of category \( \Lambda \)-modules and \( \Lambda \)-polydules, respectively. Let \( \mathcal{D} (\text{mod } \Lambda) \) denote the full subcategory of \( \mathcal{D} (\text{Mod } \Lambda) \). We recall Proposition 2.5 in [1].

**Lemma 2.1.** Let \( \Lambda \) be a finite dimensional \( \mathcal{k} \)-algebra. A \( \Lambda \)-polydule \( M \) is in the essential image of the composed functor \( \mathcal{D}^b (\text{mod } \Lambda) \hookrightarrow \mathcal{D} (\text{Mod } \Lambda) \xrightarrow{\sim} \mathcal{D}_\infty (\Lambda) \) if and only if it is quasi-isomorphic to a \( \Lambda \)-polydule \( N \) with \( m_i^N = 0 \) and \( \text{dim}_k (M) \) finite.

We denote by \( \mathcal{D}_\infty^{\text{fin}} (\Lambda) \) the full subcategory of \( \mathcal{D}_\infty (\Lambda) \) consisting of objects in this image of that composed functor.
Let $\mathfrak{B}_\Lambda = \{v_1 = 1, v_2, \ldots, v_n\}$ be a $k$-basis for $\Lambda$. Define $\ell (\mathfrak{B}_\Lambda)$ as a collection of all finite non-empty sequences of elements in $\mathfrak{B}_\Lambda$, that is

$$\ell (\mathfrak{B}_\Lambda) := \{S = (\lambda_{S1}, \lambda_2, \lambda_1) \mid \lambda_i \in \mathfrak{B}_\Lambda, \forall i \in \{1, 2, \ldots, |S|\}\}.$$ 

We also define $\ell^0 (\mathfrak{B}_\Lambda) = \ell (\mathfrak{B}_\Lambda) \cup \{\emptyset\}$. Following [1], we fix two integers $a < b$ and a finite dimensional $k$-vector space $W = \bigoplus_{i=a}^{b} \mathcal{W}^i$ with $\dim_k (\mathcal{W}^i) = d_i$ for $i = a, \ldots, b$. Let $d$ denote the vector $d = (d_a, \ldots, d_a)$. We denote the affine variety $\mathcal{M}_d^\Lambda (k)$ which parameterize the possible $\Lambda$-polydules structures on $W$. A point $w = (W(S, i))$ in the variety $\mathcal{M}_d^\Lambda (k)$ is a collection of matrices $W(S, i) \in M_{d_i-|S|+1 \times d_i} (k)$, one for each sequence $S \in \ell (\mathfrak{B}_\Lambda)$ and integer $i$ which satisfy $a + |S| - 1 \leq i \leq b$. We also define the algebraic group $\mathfrak{G}_d^\Lambda (k)$ which act on $\mathcal{M}_d^\Lambda (k)$. An element $f = (F(S, i))$ in $\mathfrak{G}_d^\Lambda (k)$ is a collection of matrices $F(S, i) \in M_{d_i-|S| \times d_i} (k)$, one for each sequence $S \in \ell^0 (\mathfrak{B}_\Lambda)$ and integer $i$ which satisfy $a + |S| \leq i \leq b$. The $\mathfrak{G}_d^\Lambda (k)$-orbits in $\mathcal{M}_d^\Lambda (k)$ correspond to isomorphism classes of $\Lambda$-polydules with given dimension in homology.

### 3 Main Results

We have the following result about the point in variety of polydules.

**Theorem 3.1.** Let $d = (d_a, d_{a+1}, \ldots, d_b), d' = (d'_a, d'_{a+1}, \ldots, d'_b), d'' = (d''_a, d''_{a+1}, \ldots, d''_b)$ with $d = d' + d''$. Then, $u \in \mathcal{M}_d^\Lambda (k)$ and $v \in \mathcal{M}_{d'}^\Lambda (k)$ if and only if $w \in \mathcal{M}_{d''}^\Lambda (k)$ where for each pairs $(S, i), W(S, i) = \begin{bmatrix} U(S, i) & Z(S, i) \\ 0 & V(S, i) \end{bmatrix}$ for some $Z(S, i) \in M_{d_i-|S|+1 \times d_i'} (k)$ and $(Z(S, i))$ satisfying the relation

$$\sum_{S = [S_1, S_2], S_1, S_2 \neq \emptyset} (-1)^{|S_1|(|S_2|+1)} (U(S_1, i - |S_2| + 1) Z(S_2, i) + Z(S_1, i - |S_2| + 1) V(S_2, i))$$

$$= \sum_{l=1}^{n} \sum_{r=1}^{\ell - 1} (-1)^{l-1} c^{S,l}_r Z(S_{l/r}, i).$$

(1)

Moreover, if $u$ and $v$ correspond to a $\Lambda$-polydules $U$ and $V$, respectively with $m^U_1 = 0 = m^V_1$ then $w$ correspond to a $\Lambda$-polydule $W$ where $W^i = U^i \oplus V^i$ for any $i \in \mathbb{Z}$.

**Proof.** (⇒) Let $u \in \mathcal{M}_d^\Lambda (k)$ and $v \in \mathcal{M}_{d'}^\Lambda (k)$. We construct a collection of matrices $w = (W(S, i))$ as follow. For each pair $(S, i)$, let $W(S, i) = \begin{bmatrix} U(S, i) & Z(S, i) \\ 0 & V(S, i) \end{bmatrix}$ for some $Z(S, i) \in M_{d_i-|S|+1 \times d_i'} (k)$ where $(Z(S, i))$ satisfying the relation 1. The collection of matrices $(Z(S, i))$ is not empty since the collection of zeroes matrices is satisfying that condition. We will
show that $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$. Since $(Z(S, i))$ satisfying the relation 1, then by the properties of matrices we obtain

$$\sum_{S=[S_1,S_2]}(-1)^{S_1(|S_2|+1)}W(S_1,i-|S_2|+1)W(S_2,i)=\sum_{l=1}^{[S]-1}\sum_{r=1}^{n}(-1)^{l-1}c_r^{S,i}W(S_{l/r},i).$$

Hence, $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$.

$(\Leftarrow)$ Let $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$ where for each pair $(S, i), W(S, i) = \begin{bmatrix} U(S, i) & Z(S, i) \\ 0 & V(S, i) \end{bmatrix}$ for some $Z(S, i) \in \mathcal{M}_{d_1-|S|+1 \times d_2'}(k)$. By calculation, we obtain that both $U(S, i)$ and $V(S, i)$ satisfy the same equation as equation in $W(S, i)$ for each $(S, i)$. Therefore, $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$ and $\mathbf{v} \in \mathcal{M}_{d_2'}^A(k)$. Now assume that $\mathbf{u}$ and $\mathbf{v}$ correspond to a $\Lambda$-polydual $U$ and $V$, respectively with respect to the $k$-basis $\mathfrak{B}_U^i = \{e_{1U}^i, e_{2U}^i, \ldots, e_{d_1U}^i\}$ for $U^i$ and $\mathfrak{B}_V^i = \{e_{1V}^i, e_{2V}^i, \ldots, e_{d_2V}^i\}$ for $V^i$, $i \in \{a, \ldots, b\}$. Define $W^i = U^i \oplus V^i$. Let $\mathfrak{B}_V^i = \{e_{1W}^i, e_{2W}^i, \ldots, e_{d_W}^i\}$ where $e_{jW}^i = \begin{bmatrix} e_{jU}^i \\ 0 \end{bmatrix}$ for $j \in \{1, 2, \ldots, d_1^i\}$ and $e_{jW}^i = \begin{bmatrix} 0 \\ e_{j-d_1^i}^i \end{bmatrix}$ for $j \in \{d_1^i+1, d_1^i+2, \ldots, d^i\}$ be a $k$-basis for $W^i$. Define $\Lambda$-polydual structure in $W$ by the formula $m_W^{|S|-1+i}(S, -) : W^i \rightarrow W^{|S|+1}$ given by sending $\sum_{j=1}^{d^i} \alpha_j e_{jW}^i$ to $\begin{bmatrix} e_{1W}^i \\ \vdots \\ e_{d^i-1-|S|+1W}^i \end{bmatrix} W(S, i) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{d^i} \end{bmatrix}$. Then, $W(S, i)$ is representative matrix of linear map $m_W^{|S|-1+i}(S, -)$. Hence, $\mathbf{w}$ correspond to a $\Lambda$-polydual $W$ where $W^i = U^i \oplus V^i$ for any $i \in \mathbb{Z}$. \hfill $\Box$

Now, we have the relation between exact sequence and variety of $\Lambda$-polydules.

**Theorem 3.2.** Let $\mathbf{u} \in \mathcal{M}_{d_2}^A(k)$ and $\mathbf{v} \in \mathcal{M}_{d_2'}^A(k)$. Define $d = d_1 + d_2'$. Then, $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$ where for each pair $(S, i), W(S, i) = \begin{bmatrix} U(S, i) & Z(S, i) \\ 0 & V(S, i) \end{bmatrix}$ for some $Z(S, i) \in \mathcal{M}_{d_1-|S_i|+1 \times d_2'}(k)$ if and only if there is an exact sequence of $\Lambda$-polydules $\Sigma : 0 \rightarrow U \xrightarrow{f_1} W \xrightarrow{f_2} V \rightarrow 0$ with $W^i = U^i \oplus V^i$.

**Proof.** $(\Rightarrow)$ By Theorem 3.1, let $U$, $V$ and $W$ be a $\Lambda$-polydules which correspond to points $\mathbf{u} \in \mathcal{M}_{d_2}^A(k)$, $\mathbf{v} \in \mathcal{M}_{d_2'}^A(k)$ and $\mathbf{w} \in \mathcal{M}_{d_2}^A(k)$, respectively with $W^i = U^i \oplus V^i$ for any $i \in \mathbb{Z}$. We define an injection map $f = (f_i)$ from $U$ to $W$ recursively given by

$$f_1(\emptyset, -) : U^i \rightarrow W^i, \quad u \mapsto (u, 0), \quad \text{for} \quad i \in \{a, \ldots, b\},$$

$$f_2(1 \otimes m_U^i) = f_1(m_U^i) - m_W^i(1 \otimes f_1),$$

and for $t \geq 3$,

$$f_t(1 \otimes t-1 \otimes m_U^i) = -\sum_{j=2}^{t} (-1)^{j(t-1)} f_{t-j+1}(1 \otimes t-j \otimes m_U^j).$$
\[ - \sum_{j=1}^{t-2} (-1)^j f_{t-1} (1^{\otimes t-j-2} \otimes \mu \otimes 1^{\otimes j}) + \sum_{r=1}^{t-1} (-1)^{t+r+1} m_{t-r+1} (1^{\otimes t-r} \otimes f_r). \]

Note that \( f \) is strict monomorphism of \( \Lambda \)-polydules. Dually, we also define projection map \( g = (g_i) \) from \( W \) to \( V \) recursively analogous as \( f \) before so that we obtain strict epimorphism of \( \Lambda \)-polydules \( g \). Hence, \( \im (f) = \ker (g) \).

Therefore, \( 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0 \) is an exact sequence of \( \Lambda \)-polydules.

\((\Leftarrow) \) Suppose there exist an exact sequence of \( \Lambda \)-polydules \( \Sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0 \) with \( W^i = U^i \oplus V^i \). We apply functor \( \mathcal{D} \) to get \( \alpha \) where the morphism \( f \rightarrow \mathcal{D} \alpha \) is a sequence of zero numbers for any object \( X \).

Therefore, \( 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0 \) is an exact sequence of \( \Lambda \)-polydules.

Following Zvara in [3], we define additive function from derived category \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \) of \( \Lambda \)-polydules. Let \( \Delta : L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} L[1] \) be distinguished triangle in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \). Define additive function \( \vartheta_{\Delta} \) from any object in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \) to the set of sequence of integers given by sending each object \( X \) in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \) to \( \left( \dim_k \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (N[j] \oplus L[j], X) \right) \).

**Theorem 3.3.** If \( \Delta : L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} T[L] \) is distinguished triangle in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \) then \( \vartheta_{\Delta} (X) \) is a sequence of nonnegative integers for any object \( X \) in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \). Moreover, the distinguished triangle \( \Delta \) is split if and only if \( \vartheta_{\Delta} (X) \) is a sequence of zero numbers for any object \( X \) in \( \mathcal{D}_{\infty}^{\text{fin}} (\Lambda) \).

**Proof.** We apply functor \( \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (-, X) \) to distinguished triangle \( \Delta \),

\[ \cdots \rightarrow \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (T[L], X) \xrightarrow{h^*} \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (N, X) \xrightarrow{g^*} \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (M, X) \xrightarrow{f^*} \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (L, X) \rightarrow \cdots \]

where the morphism \( f^* \) is defined by the formula \( f^* (\alpha) = \alpha \circ f \) for every \( \alpha \in \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (M, X) \) and both of morphism \( g^* \) and \( h^* \) are defined analogous with the definition of \( f^* \). Since \( f^* : \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (M, X) \rightarrow \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (L, X) \) then \( \dim_k \left( \text{Hom}_{\mathcal{D}_{\infty}^{\text{fin}} (\Lambda)} (L, X) \right) - \text{rank} (f^*) \geq 0 \). Without loss of generality, it is enough to consider in position with index zero in the sequence \( \vartheta_{\Delta} (X) \),
that is
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(N \oplus L, X) \right) = \text{rank} (g^*) + \text{null} (g^*) + \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(L, X) \right)
\]
\[
= \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(W, X) \right) - \text{rank} (f^*) + \text{null} (g^*) + \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(L, X) \right).
\]

Then,
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(N \oplus L, X) \right) - \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(W, X) \right)
= \text{null} (g^*) + \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(L, X) \right) - \text{rank} (f^*) \geq \text{null} (g^*) \geq 0.
\]

Moreover, let $\Delta$ be a splittable sequence. Then, $h = 0$ so that we obtain $(T^j(h))^* = 0$ for any object $X$ in $D_{\infty}^{\text{fin}}(\Lambda)$ and $j \in \mathbb{Z}$. Hence, $(T^j(f))^*$ and $(T^j(g))^*$ are epimorphism and monomorphism, respectively whence we have $\text{null} (T^j(g))^* = 0$. Therefore, for each $j \in \mathbb{Z}$, we obtain
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^j(N) \oplus T^j(L), X) \right) = \dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^j(W), X) \right).
\]

Hence, $\vartheta_\Delta(X)$ is a sequence of zero numbers.

Conversely, let $\vartheta_\Delta(X)$ is a sequence of zero numbers. Then
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^j(L), X) \right) - \text{rank} (T^j(f))^* + \text{null} (T^j(g))^* = 0.
\]
Since $\text{null} (T^j(g))^* \geq 0$, and
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^j(L), X) \right) - \text{rank} (T^j(f))^* \geq 0
\]
then for each $j \in \mathbb{Z}$, we obtain
\[
\dim_k \left( \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^j(L), X) \right) = \text{rank} (T^j(f))^* \text{ and } \text{null} (T^j(g))^* = 0.
\]
Hence, for each $j \in \mathbb{Z}$, $\ker (T^j(g))^* = 0$ whence we obtain $(T^j(h))^* = 0$. In particular, if we take $X = T^{j+1}(L)$ and evaluate identity morphism $id_{T^{j+1}(L)} \in \text{Hom}_{D_{\infty}^{\text{fin}}(\Lambda)}(T^{j+1}(L), T^{j+1}(L))$, then we obtain
\[
0 = (T^j(h))^* \left( id_{T^{j+1}(L)} \right) = id_{T^{j+1}(L)} \circ T^j(h) = T^j(h).
\]

Hence, $\Delta$ is splittable sequence. \qed

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