Generalized Hyers-Ulam Stability of a 3-Dimensional Quadratic Functional Equation in Modular Spaces

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Abstract
In this paper, we prove the stability problem for a 3-dimensional quadratic functional equation

$$9f\left(\frac{x+y+z}{3}\right) - f(x+y) - f(y+z) - f(x+z) + f(x) + f(y) + f(z) = 0$$

in modular spaces by applying the direct method.

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1 Introduction

In 1940, Ulam [14] proposed the problem concerning the stability of group homomorphisms. In the following year, Hyers [4] gave an affirmative answer to this problem for additive mappings between Banach spaces. Thereafter, many mathematicians came to deal with this problem (cf. [1, 3, 8, 12]).

A solution of the functional equation

\[ f(x + y) - f(x - y) - 2f(x) - 2f(y) = 0 \]  

is called a quadratic mapping ([2, 13]). A functional equation is called a quadratic functional equation if every solution of that equation is a quadratic mapping and any quadratic mapping is a solution of the equation ([5, 6, 7]).

In 1959, Nakano [10] and Musielak and Orlicz [9] defined a modular on a vector space to construct a modular structure on the space.

Definition 1.1 Let $X$ be a real vector space.
(a) A functional $\rho : X \rightarrow [0, \infty]$ is called a modular if for arbitrary $x, y \in X$,

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scaler $\alpha$ with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta > 0$,

(b) We say that $\rho$ is a convex modular if the last condition (iii) is replaced by

(iii') $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta > 0$.

A modular $\rho$ defines a corresponding modular space, i.e., the vector space $X_\rho$ given by $X_\rho = \{ x \in X : \rho(\lambda x) \rightarrow 0$ as $\lambda \rightarrow 0 \}$.

Definition 1.2 Let $\{x_n\}$ and $x$ be in $X_\rho$.

(i) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is $\rho$-convergent to $x$ and write $x_n \rightarrow x$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$.

(ii) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is called $\rho$-Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

(iii) A subset $S$ of $X_\rho$ is called $\rho$-complete if and only if every $\rho$-Cauchy sequence is $\rho$-convergent to an element of $S$. 

In this paper, we consider the following 3-dimensional functional equation
\[ 9f\left(\frac{x+y+z}{3}\right) - f(x+y) - f(y+z) - f(x+z) + f(x) + f(y) + f(z) = 0 \quad (2) \]
where \( f \) is a mapping from a real vector space to a \( \rho \)-complete modular space.

Firstly, we show that the functional equation (2) is a quadratic functional equation, and, by applying the direct method in [4], we further prove the stability of that equation. More precisely, starting from the given mapping \( f \) that approximately satisfies the functional equation (2), we explicitly construct an exact solution \( F \) of that equation, which approximates the mapping \( f \), given by
\[ F(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \quad \text{or} \quad F(x) = \lim_{n \to \infty} \frac{f(3^n x)}{3^n}. \]

2 Main results

Throughout this section, let \( V \) and \( W \) be real vector spaces. And let \( \rho \) be a convex modular on a real vector space \( Y \). For a given mapping \( f : V \to W \), we use the following abbreviations
\[
\begin{align*}
Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y), \\
Df(x,y,z) &:= 9f\left(\frac{x+y+z}{3}\right) - f(x+y) - f(y+z) - f(x+z) \\
&\quad + f(x) + f(y) + f(z)
\end{align*}
\]
for all \( x, y, z \in V \). Notice that the solution of the functional equation \( Qf \equiv 0 \) is called a quadratic mapping.

In the next theorem we will show that the functional equation \( Df \equiv 0 \) is a quadratic functional equation.

**Theorem 2.1** A mapping \( f : V \to W \) satisfies \( Df(x,y,z) = 0 \) for all \( x,y,z \in V \) if and only if \( f \) is a quadratic mapping.

**Proof.** If \( f : V \to W \) satisfies \( Df(x,y,z) = 0 \) for all \( x,y,z \in V \), then, since \( Df(0,0,0) = 9f(0) = 0 \), we have \( Df(x,x,x) = 12f(x) - 3f(2x) = 0 \) and \( Df(3x,0,0) = 9f(x) - f(3x) + f(0) = 0 \) for all \( x \in V \). It follows that
\[
Df(-x,-x,2x) = f(2x) + 2f(-x) - f(-2x) - 2f(x) = 2f(x) - 2f(-x) = 0
\]
which implies that \( f(x) = f(-x) \) for all \( x \in V \). Hence, we conclude that \( Qf(x,y) = -Df(x,y,-y) \) for all \( x, y \in V \), and so \( f \) is a quadratic mapping.
Conversely, let \( f : V \rightarrow W \) be a quadratic mapping, i.e., \( Qf(x, y) = 0 \) for all \( x, y \in V \). Then we have the equality \( f(\frac{x}{n}) = \frac{f(x)}{n^2} \) for all \( x \in V \) and \( n \in \mathbb{N} \). Hence we obtain that

\[
Df(x, y, z) = f(x + y + z) - f(x + y) - f(y + z) - f(x + z) + f(x) + f(y) + f(z)
\]

\[
= f(x + y + z) + f(x - y) - 2f\left(x + \frac{z}{2}\right) - 2f\left(y + \frac{z}{2}\right) - f(x + z) - f(x)
\]

\[
+ 2f\left(x + \frac{z}{2}\right) + 2f\left(\frac{z}{2}\right) - f(y + z) - f(y) + 2f\left(y + \frac{z}{2}\right) + 2f\left(\frac{z}{2}\right)
\]

\[
- f(x + y) - f(x - y) + 2f(x) + 2f(y)
\]

\[
= Qf\left(x + \frac{z}{2}, y + \frac{z}{2}\right) - Qf\left(x + \frac{z}{2}, \frac{z}{2}\right) - Qf\left(y + \frac{z}{2}, \frac{z}{2}\right) - Qf(x, y)
\]

\[
= 0
\]

for all \( x, y, z \in V \), as we desired.

In the following lemma, we will show that \( f \) is a quadratic mapping even if \( Df(x, y, z) = 0 \) for all \( x, y, z \in V \setminus \{0\} \).

**Lemma 2.2** If a mapping \( f : V \rightarrow W \) satisfies \( Df(x, y, z) = 0 \) for all \( x, y, z \in V \setminus \{0\} \), then \( f \) satisfies \( Df(x, y, z) = 0 \) for all \( x, y, z \in V \).

**Proof.** Notice that \( f(0) = 0 \) since \( 18f(0) = Df(2x, -x, -x) + Df(-2x, x, x) \) for \( x \in V \setminus \{0\} \). And then we have

\[
6f(x) - 6f(-x) = 3Df(2x, -x, -x) + Df(x, x, x) - Df(-x, -x, -x),
\]

\[
9f(x) - f(3x) = Df(-9x, 3x, 3x) + Df(6x, -6x, 3x) - Df(3x, -3x, -3x)
\]

which follow that \( f(3x) = 9f(x) \) and \( f(x) = f(-x) \) for all \( x \in V \setminus \{0\} \). So, we can say that

\[
Df(x, y, 0) = 9f\left(\frac{x + y}{3}\right) - f(x + y) - f(y) - f(x) + f(x) + f(y) + f(0) = 0,
\]

\[
Df(x, 0, 0) = 9f\left(\frac{x}{3}\right) - f(x) = 0
\]

for all \( x, y \in V \setminus \{0\} \). Similarly we easily get the equalities \( Df(x, 0, z) = 0, Df(0, y, z) = 0, Df(0, y, 0) = 0, Df(0, 0, z) = 0 \) for all \( x, y, z \in V \setminus \{0\} \). Therefore, we have proved \( Df(x, y, z) = 0 \) for all \( x, y, z \in V \) as we desired.

**Remark** Note that \( \rho \) is an increasing function since it is a convex modular on a real vector space \( Y \). Suppose \( 0 < \alpha < \beta \), then property (iii) of Definition 1.1 with \( y = 0 \) shows that \( \rho(\alpha x) = \rho((\alpha/\beta)\beta x) \leq \rho(\beta x) \) for all \( x \in X \).
Moreover, if $\rho$ is a convex modular on $X$, then property (iii') of Definition 1.1 shows that
\[
\rho \left( \sum_{i=1}^{n} \alpha_i x_i \right) = \rho \left( \alpha_1 x_1 + (1 - \alpha_1) \left( \frac{\alpha_2}{1 - \alpha_1} x_2 + \ldots + \frac{\alpha_n}{1 - \alpha_1} x_n \right) \right)
\leq \alpha_1 \rho(x_1) + (1 - \alpha_1) \rho \left( \frac{\alpha_2}{1 - \alpha_1} x_2 \right)
+ \left( 1 - \frac{\alpha_2}{1 - \alpha_1} \right) \rho \left( \frac{\alpha_3}{1 - \alpha_1 - \alpha_2} x_3 + \ldots + \frac{\alpha_n}{1 - \alpha_1 - \alpha_2} x_n \right)
\leq \alpha_1 \rho(x_1) + \alpha_2 \rho(x_2) + (1 - \alpha_1 - \alpha_2) \rho \left( \frac{\alpha_3}{1 - \alpha_1 - \alpha_2} x_3 + \ldots + \frac{\alpha_n}{1 - \alpha_1 - \alpha_2} x_n \right)
\leq \ldots \leq \sum_{i=1}^{n} \alpha_i \rho(x_i)
\tag{3}
\]
for all $x_1, \ldots, x_n \in X$ and all $\alpha_i > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$.

Now we will prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y, z) = 0$.

**Theorem 2.3** Let $V$ be a real vector space and let $Y_\rho$ be a $\rho$-complete modular space. Suppose $f : V \to Y_\rho$ satisfies the condition $f(0) = 0$ and an inequality of the form
\[
\rho(Df(x, y, z)) \leq \varphi(x, y, z)
\tag{4}
\]
for all $x, y, z \in V \setminus \{0\}$, where $\varphi : (V \setminus \{0\})^3 \to [0, \infty)$ be a function such that
\[
\sum_{i=0}^{\infty} \varphi(2^i x, 2^i y, 2^i z) \frac{4^i}{3^{i+1}} < \infty
\tag{5}
\]
for all $x, y, z \in V \setminus \{0\}$. Then there exists a unique quadratic mapping such that
\[
\rho(f(x) - F(x)) \leq \sum_{i=0}^{\infty} \tilde{\varphi}(2^i x) \frac{4^i}{3^{i+1}}
\tag{6}
\]
for all $x \in V$, where $\tilde{\varphi} : V \to [0, \infty)$ is a function defined by $\tilde{\varphi}(x) = \varphi(x, x, x)$ if $x \in V \setminus \{0\}$ and $\tilde{\varphi}(0) = 0$.

**Proof.** It follows from (3) and (4) that
\[
\rho \left( \frac{f(2^n x)}{4^n} - \frac{f(2^{n+m} x)}{4^{n+m}} \right) = \rho \left( \sum_{i=n}^{n+m-1} \left( \frac{f(2^i x)}{4^i} - \frac{f(2^{i+1} x)}{4^{i+1}} \right) \right)
\]
can define mappings $J, F$ and obtain the inequality (6). From the definition of $x$ for all $x \in V \setminus \{0\}$ and $f(0) = 0$, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ converges for all $x \in V$. Hence, we can define mappings $J_n f, F : V \to Y_\rho$ by $J_n f(x) := \frac{f(2^n x)}{4^n}$ and $F(x) := \lim_{n \to \infty} \frac{f(2^n x)}{4^n}$ for all $x \in V$. Also we obtain the inequality

$$\rho(J_n f(x) - F(x)) \leq \sum_{i=n}^{\infty} \frac{\varphi(2^i x, 2^i x, 2^i x)}{3 \cdot 4^{i+1}}$$

(7)

for all $x \in V \setminus \{0\}$ as $m \to \infty$ in (9). Moreover, if we put $n = 0$ in (8), we obtain the inequality (6). From the definition of $F$ and (3), we get

$$\rho\left( \frac{1}{16} DF(x, y, z) \right)$$

$$= \rho\left( \frac{9}{16} (F - J_n f) \left( \frac{x + y + z}{3} \right) + \frac{1}{16} (J_n f - F)(x + y) \right.$$

$$+ \frac{1}{16} (J_n f - F)(x + z) + \frac{1}{16} (J_n f - F)(y + z) - \frac{1}{16} (J_n f - F)(x)$$

$$- \frac{1}{16} (J_n f - F)(y) - \frac{1}{16} (J_n f - F)(z) + \frac{1}{16} DJ_n f(x, y, z) \right)$$

$$\leq \frac{9}{16} \rho\left( \left( F - J_n f \right) \left( \frac{x + y + z}{3} \right) \right) + \frac{1}{16} \rho((J_n f - F)(x + y))$$

$$+ \frac{1}{16} \rho((J_n f - F)(x + z)) + \frac{1}{16} \rho((J_n f - F)(y + z)) + \frac{1}{16} \rho((J_n f - F)(x))$$

$$+ \frac{1}{16} \rho((J_n f - F)(y)) + \frac{1}{16} \rho((J_n f - F)(z)) + \frac{1}{16} \rho(DJ_n f(x, y, z))$$

$$\leq \frac{9}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i(x + y + z)/3)}{3 \cdot 4^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i(x + y))}{3 \cdot 4^{i+1}}$$

$$+ \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i(x + z))}{3 \cdot 4^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i(y + z))}{3 \cdot 4^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i x)}{3 \cdot 4^{i+1}}$$

$$+ \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i y)}{3 \cdot 4^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(2^i z)}{3 \cdot 4^{i+1}} + \frac{1}{16} \frac{\varphi(2^n x, 2^n y, 2^n z)}{4^n}$$

$\to 0$, as $n \to \infty$,.
for all $x, y, z \in V \setminus \{0\}$. Hence we obtain the equality $DF(x, y, z) = 0$ for all $x, y, z \in V$ by Lemma 2.2. Moreover, according to Theorem 2.1, $f$ is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping $F' : V \to Y_\rho$ which satisfies the inequality in (6). Notice that $F'(x) = \frac{F'(2^n x)}{4^n}$ holds for all $x \in V$ and all $n \in \mathbb{N}$. From the relation

$$\rho \left( \frac{f(2^n x)}{4^n} - F'(x) \right) = \rho \left( \frac{f(2^n x) - F'(2^n x)}{4^n} \right) \leq \frac{1}{4^n} \rho(f(2^n x) - F'(2^n x)) \leq \frac{1}{4^n} \sum_{i=0}^{\infty} \tilde{\varphi}(2^{n+i} x) 3 \cdot 4^{i+1} \leq \sum_{i=n}^{\infty} \tilde{\varphi}(2^i x) 3 \cdot 4^{i+1} \to 0, \quad n \to \infty$$

for all $x \in V \setminus \{0\}$, we get the equality $F'(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} = F(x)$ for all $x \in V$, as we desired.

**Corollary 2.4** Let $X$ be a real normed space and let $p, \theta$ be real constants such that $p < 2$ and $\theta > 0$. If a mapping $f : X \to Y_\rho$ satisfies the inequality

$$\rho(df(x, y, z)) \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X \setminus \{0\}$ (with $f(0) = 0$ when $p = 0$), then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\rho(f(x) - F(x)) \leq \frac{\theta}{4 - 2p} \|x\|^p$$

for all $x \in X \setminus \{0\}$. In particular, if $p < 0$, then $f$ is a quadratic mapping.

**Proof.** Choose a $x \in V \setminus \{0\}$. Then it follows from (9) that

$$\rho(f(0)) = \rho \left( \frac{Df(2kx, -kx, -kx) + Df(-2kx, kx, kx)}{18} \right) \leq \frac{1}{18} \rho(Df(2kx, -kx, -kx)) + \frac{1}{18} \rho(Df(-2kx, kx, kx)) \leq \frac{2p + 2}{9} |k|^p \|x\|^p$$

for all nonzero real numbers $k$, i.e., $f(0) = 0$ when $p \neq 0$. If we put $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X \setminus \{0\}$, then $\varphi$ satisfies (5). Therefore, by Theorems 2.3, there exists a unique quadratic mapping $F$ satisfying the
inequality (10) for all \( x \in X \setminus \{0\} \). Moreover, if \( p < 0 \) then it follows from (3), (9), \( DF \equiv 0 \), and (10) that

\[
\rho \left( \frac{1}{14}(f(x) - F(x)) \right) \\
\leq \rho \left( \frac{1}{14}(Df - DF)((k + 1)x, -kx, -kx) + \frac{9}{14}(F - f)\left( \frac{(-k + 1)x}{3} \right) \right) \\
+ \frac{1}{14}(f - F)(-2kx) + \frac{1}{14}(F - f)((k + 1)x) + \frac{2}{14}(F - f)(-kx) \\
\leq \frac{1}{14} \rho((Df - DF)((k + 1)x, -kx, -kx)) \\
+ \frac{9}{14} \rho\left( (F - f)\left( \frac{(-k + 1)x}{3} \right) \right) + \frac{1}{14} \rho((f - F)(-2kx)) \\
+ \frac{1}{14} \rho((F - f)((k + 1)x)) + \frac{2}{14} \rho((F - f)(-kx)) \\
\leq \left( (k + 1)^p + 2 \cdot k^p + \frac{9(-k + 1)^p}{3^p} + \frac{(2k)^p + (k + 1)^p + 2 \cdot k^p}{|4 - 2^p|} \right) \frac{\theta \|x\|^p}{14} \\
\rightarrow 0, \text{ as } k \rightarrow \infty,
\]

for all \( x \in X \setminus \{0\} \). Since \( f(0) = 0 = F(0) \), the equality \( f(x) = F(x) \) holds for all \( x \in X \), as we desired.

Now we will establish another type of stability of the functional equation \( Df(x, y, z) = 0 \).

**Theorem 2.5** Let \( V \) be a real vector space and let \( Y_\rho \) be a \( \rho \)-complete modular space. Suppose \( f : V \rightarrow Y_\rho \) satisfies the condition \( f(0) = 0 \) and the inequality (4) for all \( x, y, z \in V \), where \( \varphi : V^3 \rightarrow [0, \infty) \) be a function such that

\[
\sum_{i=0}^{\infty} \varphi(3^i x, 3^i y, 3^i z) \frac{9^i}{9^i} < \infty \tag{11}
\]

for all \( x, y, z \in V \). Then there exists a unique quadratic mapping such that

\[
\rho(f(x) - F(x)) \leq \sum_{i=0}^{\infty} \varphi(3^{i+1} x) \frac{3 \cdot 9^{i+1}}{3 \cdot 9^{i+1}} \tag{12}
\]

for all \( x \in V \), where \( \varphi : V \rightarrow [0, \infty) \) is a function defined by \( \varphi(x) = \varphi(x, 0, 0) \).

**Proof.** It follows from (3) and (4) that

\[
\rho \left( \frac{f(3^n x)}{9^n} - \frac{f(3^{n+m} x)}{9^{n+m}} \right) = \rho \left( \sum_{i=n}^{n+m-1} \frac{Df(3^{i+1} x, 0, 0)}{9^{i+1}} \right)
\]
for all $x \in V$. So, it is easy to show that the sequence $\{\frac{f(3^n x)}{g^n}\}$ is a Cauchy sequence for all $x \in V$. Since $Y_\rho$ is complete, the sequence $\{\frac{f(3^n x)}{g^n}\}$ converges for all $x \in V$. Hence, we can define mappings $J_n f, F : V \to Y_\rho$ by $J_n f(x) := \frac{f(3^n x)}{g^n}$ and $F(x) := \lim_{n \to \infty} \frac{f(3^n x)}{g^n}$ for all $x \in V$. Also we obtain the inequality

$$\rho(J_n f(x) - F(x)) \leq \sum_{i=n}^{\infty} \frac{\varphi(3^{i+1} x)}{g^{i+1}}$$

(14)

for all $x \in V$ as $m \to \infty$ in (13). Moreover, if we put $n=0$ in (14), we obtain the inequality (12). From the definition of $F$ and (3), we get

$$\rho\left(\frac{1}{16} DF(x, y, z)\right) \leq \frac{9}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i (x + y + z))}{g^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i (x + y))}{g^{i+1}}$$

$$+ \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i (x + z))}{g^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i (y + z))}{g^{i+1}}$$

$$+ \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i x)}{g^{i+1}} + \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i y)}{g^{i+1}}$$

$$+ \frac{1}{16} \sum_{i=n}^{\infty} \frac{\varphi(3^i z)}{g^{i+1}} + \frac{1}{16} \varphi(3^n x, 3^n y, 3^n z)$$

$$\to 0, \text{ as } n \to \infty,$$

for all $x, y, z \in V$ i.e., $DF(x, y, z) = 0$ for all $x, y, z \in V$. By Theorem 2.1, $f$ is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping $F' : V \to Y$ which satisfies the inequality in (12). Notice that $F'(x) = \frac{F'(3^n x)}{g^n}$ for all $x \in V$. From the relations

$$\rho\left(\frac{f(3^n x)}{g^n} - F'(x)\right) \leq \frac{1}{g^n} \rho(f(3^n x) - F'(3^n x))$$

$$\leq \sum_{i=n}^{\infty} \frac{\varphi(3^{i+1} x)}{g^{i+1}} \to 0, \text{ as } n \to \infty$$

for all $x \in V$, we get the equality $F'(x) = \lim_{n \to \infty} \frac{f(3^n x)}{g^n} = F(x)$ for all $x \in V$, as we desired.

**Corollary 2.6** Let $X$ be a real normed space and let $p, \theta$ be nonnegative real constants such that $p < 2$. If a mapping $f : X \to Y_\rho$ satisfies the inequality
(9) for all $x, y, z \in X$ (with $f(0) = 0$ when $p = 0$), then there exists a unique quadratic mapping $F : X \to Y$ such that
\[
\rho(f(x) - F(x)) \leq \frac{3^p \theta}{9 - 3^p} \|x\|^p
\]
for all $x \in X$.

**Proof.** If we set $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$, then there exists a unique quadratic mapping $F$ satisfying the inequality (15) for all $x \in X$ by Theorem 2.5.

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**References**


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