A Short Note on a Simple Proof of Schauder’s Fixed Point Theorem and its Generalization without Continuity and Compactness Assumptions

Wei-Shih Du

Department of Mathematics
National Kaohsiung Normal University
Kaohsiung 82444, Taiwan

Copyright © 2016 Wei-Shih Du. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this note, we give a simple proof of Schauder’s fixed point theorem by applying Park’s fixed point theorem. A generalized Fan’s minimax inequality and a generalized Schauder’s fixed point theorem are also established. In our new generalized Schauder’s fixed point theorem, the compactness assumption is replaced by a finite open (resp., closed) cover and the continuity assumption is removed.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Convex space, Park’s fixed point theorem, Fan’s minimax inequality, Schauder’s fixed point theorem

1. Introduction and preliminaries

Throughout this paper we denote the set of real numbers by $\mathbb{R}$. Let $A$ and $B$ be nonempty sets. A multivalued map $T : A \rightrightarrows B$ is a function from $A$ to the power set $2^B$ of $B$. We denote

$$T(A) = \bigcup\{T(x) : x \in A\}$$
and let $T^+ : B \to A$ be defined by the condition that $x \in T^+(y)$ if and only if $y \in T(x)$. Let $\langle A \rangle$ denote the set of nonempty finite subsets of $A$. Let $X$ be a vector space and $D$ be a nonempty subset of $X$. A function $f : X \to \mathbb{R}$ is called \textit{quasiconcave} on $X$ if for any $x, y \in X$,

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\} \quad \text{for all } \lambda \in [0, 1].$$

The convex hull of $D$ is denoted by $coD$. We call $(X, D)$ a \textit{convex space} if $coD \subset X$ and $X$ has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Lassonde [1] and Park [2]. If $X=D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [3]. It is obvious that any normed linear space is a convex space.

The famous Schauder fixed point theorem proved in 1930 (see [4]) and has been generalized in various directions by using different methods; see, for instance, [5-13] and references therein. The famous Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem established in 1929 [14] is a powerful tool in the application of nonlinear analysis. Since then a number of generalizations in various different directions of KKM theorem have been investigated by several authors; see [15], [16] and references therein. In 2004, Park [17] establish the following fixed point theorem in convex spaces which is actually equivalent to the version of the KKM theorem for convex spaces:

\textbf{Theorem 1.1. (Park [17])} Let $(X, D)$ be a convex space and $P : X \to D$ a multimap. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open (resp., closed) subsets $G_i \subset P^-(z_i)$ for each $i = 1, 2, \cdots, n$ such that

$$co\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^{n} G_i,$$

then the map $coP : X \to X$ has a fixed point $x_0 \in X$ (i.e., $x_0 \in coP(x_0)$).

In the present paper, a simple proof of Schauder’s fixed point theorem is given by applying Park’s fixed point theorem. Moreover, we also establish a generalized Fan’s minimax inequality and a generalized Schauder’s fixed point theorem. In our new generalized Schauder’s fixed point theorem, the compactness assumption is replaced by a finite open (resp., closed) cover and the continuity assumption is removed.

\textbf{2. Main results}

Applying Park’s fixed point theorem, we first establish the following generalized Fan’s minimax inequality.
Theorem 2.1. (Generalized Fan’s minimax inequality) Let \((X, D)\) be a convex space and \(r \in \mathbb{R}\). Suppose that \(f : X \times X \to \mathbb{R}\) is a real-valued function such that

(i) \(f(x, x) \leq r\) for all \(x \in X\);

(ii) for each \(y \in X\), \(x \mapsto f(x, y)\) is quasiconcave on \(X\);

(iii) there exist \(z_1, z_2, \cdots, z_n \in D\) and nonempty open (resp., closed) subsets \(G_i \subset \{y \in X : f(z_i, y) > r\}\) for each \(i = 1, 2, \cdots, n\) such that \(\text{co}\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^{n} G_i\).

Then there exists \(y_0 \in X\) such that \(f(x, y_0) \leq r\) for all \(x \in X\).

Proof. Suppose that for each \(y \in X\), there exists \(x \in X\) such that \(f(x, y) > r\). Let \(P : X \to X\) be a set-valued map defined by

\[ P(y) = \{x \in X : f(x, y) > r\} \]

and

\[ P^-(x) = \{y \in X : f(x, y) > r\}. \]

Then \(P(y) \neq \emptyset\) for all \(y \in X\). By (ii), \(P(y)\) is convex for all \(y \in X\). By (iii), there exist \(z_1, z_2, \cdots, z_n \in D\) and nonempty open (resp., closed) subsets \(G_i \subset P^-(z_i)\) for each \(i = 1, 2, \cdots, n\) such that \(\text{co}\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^{n} G_i\).

Applying Theorem 1.1, there exists \(v \in X\) such that \(v \in \text{co}P(v) = P(v)\) or \(f(v, v) > r\), which contradict with (i). Hence there exists \(y_0 \in X\) such that \(f(x, y_0) \leq r\) for all \(x \in X\). \(\square\)

A simple proof of Schauder’s fixed point theorem can be given by using Theorem 2.1.

Corollary 2.2. (Schauder [4]) Let \(X\) be a nonempty compact convex subset of a normed linear space \(E\) and let \(T : X \to X\) be a continuous map. Then \(T\) has a fixed point in \(X\).

Proof. On the contrary, suppose that \(Tx \neq x\) for all \(x \in X\). Define \(f : X \times X \to \mathbb{R}\) by

\[ f(x, y) = \|y - Ty\| - \|x - Ty\|. \]

Let \(P : X \to X\) be a set-valued map defined by

\[ P(y) = \{x \in X : f(x, y) > 0\} \]
Then $P(y)$ is convex for all $y \in X$. For each $x \in C$, by the continuity of $T$, $y \to f(x, y)$ is l.s.c. and hence $P^-(x)$ is open for all $x \in X$. For each $y \in X$, by our hypothesis, there exists $x_y \in X$ with $x_y \neq y$ such that $Ty = x_y$ and hence

$$
\|y - Ty\| > 0 = \|x_y - Ty\|.
$$

So we deduce

$$
X = \bigcup_{x \in X} P^-(x).
$$

Since $X$ is compact, there exists $z_1, z_2, \cdots, z_n \in X$, such that $P^-(z_i) \neq \emptyset$ for all $i = 1, 2, \cdots, n$ and $X = \bigcup_{i=1}^{n} P^-(z_i)$. Let

$$
G_i = P^-(z_i) \text{ for all } i = 1, 2, \cdots, n.
$$

Then $G_i$ is nonempty and open for all $i = 1, 2, \cdots, n$. Clearly,

$$
co\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^{n} G_i.
$$

Applying Theorem 2.1, there exists $v \in X$ such that $Tv = v$. This leads a contradiction. Therefore $T$ has a fixed point in $X$. \qed

**Remark 2.3.** The classical proof of Schauder’s fixed point theorem was shown by using a partition of unity method and Brouwer fixed point theorem. So our proof in Theorem 2.1 is quite different from the well-known classical proofs of Schauder’s fixed point theorem in the literature.

Finally, from Theorem 2.1, we can establish the following new generalized Schauder’s fixed point theorem without the compactness assumption and the continuity assumption.

**Theorem 2.4.** Let $X$ be a nonempty convex subset of a normed linear space $E$ and let $T : X \to X$ be a selmap. Suppose that there exist $z_1, z_2, \cdots, z_n \in X$ and nonempty open (resp., closed) subsets $G_i \subset \{y \in X : \|y - Ty\| > \|z_i - Ty\|\}$ for each $i = 1, 2, \cdots, n$ such that

$$
co\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^{n} G_i.
$$

Then $T$ has a fixed point in $X$.

**Proof.** Define $f : X \times X \to \mathbb{R}$ by

$$
f(x, y) = \|y - Ty\| - \|x - Ty\|.
$$

Then $f$ is continuous and $f(x, y) > 0$ for all $x, y \in X$ with $x \neq y$. By the continuity of $T$, $y \to f(x, y)$ is l.s.c. for all $x \in X$. For each $x \in C$, by the continuity of $T$, $y \to f(x, y)$ is l.s.c. and hence $f^-(x)$ is open for all $x \in X$. For each $y \in X$, by our hypothesis, there exists $x_y \in X$ with $x_y \neq y$ such that $Ty = x_y$ and hence

$$
\|y - Ty\| > 0 = \|x_y - Ty\|.
$$

So we deduce

$$
X = \bigcup_{x \in X} f^-(x).
$$

Since $X$ is compact, there exists $v \in X$ such that $Tv = v$. This leads a contradiction. Therefore $T$ has a fixed point in $X$. \qed
Clearly, \( f(x,x) = 0 \) for all \( x \in X \). By our hypothesis, there exist \( z_1, z_2, \ldots, z_n \in X \) and nonempty open (resp., closed) subsets \( G_i \subset \{ y \in X : f(z_i, y) > 0 \} \) for each \( i = 1, 2, \ldots, n \) such that

\[
\text{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{i=1}^{n} G_i.
\]

It is easy to see that for each \( y \in X \), \( x \to f(x,y) \) is quasiconcave on \( X \). By Theorem 2.1, there exists \( v \in X \) such that \( f(x,v) \leq 0 \) or \( \|v - Tx\| \leq \|x - Tx\| \) for all \( x \in X \). Since \( TX \subset X \) and \( x_0 \in X \), we have

\[
0 \leq \|x_0 - Tx_0\| \leq \|Tx_0 - Tx_0\| = 0.
\]

Hence \( \|x_0 - Tx_0\| = 0 \) and \( Tx_0 = x_0 \).

\[\square\]

**Remark 2.5.** In our new generalized Schauder’s fixed point theorem, the compactness assumption is replaced by a finite open (resp., closed) cover and the continuity assumption is removed.

**Acknowledgments**

This research was supported by grant no. MOST 104-2115-M-017-002 of the Ministry of Science and Technology of the Republic of China.

**References**


Received: June 1, 2016; Published: July 5, 2016