On Fuzzy Almost $r$-$M$ $\beta$-Continuous Mappings on Fuzzy $r$-Minimal Structures

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Abstract

We introduce the concept of fuzzy almost $r$-$M$ $\beta$-continuous mappings on fuzzy $r$-minimal structures, and investigate its properties and the relationships among fuzzy $r$-$M$ $\beta$-continuity, fuzzy weakly $r$-$M$ $\beta$-continuity and fuzzy almost $r$-$M$ $\beta$-continuity. In particular, we investigate characterizations for the continuity in terms of fuzzy $r$-minimal semiopen sets, fuzzy $r$-minimal preopen sets and fuzzy $r$-minimal regular-open sets in fuzzy $r$-minimal spaces.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [10]. Chang [1] defined fuzzy topological spaces by using fuzzy sets. In [7], Ramadan introduced the
concept of smooth topological space, which is a generalization of fuzzy topological space. We introduced the concept of fuzzy $r$-minimal space [8] which is an extension of the smooth fuzzy topological space. The concepts of fuzzy $r$-minimal open sets and fuzzy $r$-$M$ continuous mappings are also introduced and studied. In [5,6], we introduced the concepts of fuzzy $r$-minimal semiopen ($\beta$-open) sets and fuzzy $r$-$M$ semicontinuous ($\beta$-continuous) mappings, and investigate properties of such concepts. In [3], we introduced the concept of fuzzy $r$-minimal preopen set and fuzzy $r$-$M$ precontinuous mapping on fuzzy $r$-minimal spaces. In this paper, we introduce the concept of fuzzy almost $r$-$M$ $\beta$-continuous mapping, which is an extended fuzzy $r$-$M$ $\beta$-continuous mapping. We also investigate characterizations of such mappings in terms of generalized fuzzy $r$-minimal open sets-fuzzy $r$-minimal semiopen sets, fuzzy $r$-minimal pre-open sets and fuzzy $r$-minimal regular-open sets- in a fuzzy $r$-minimal space. And we study the relationships among fuzzy $r$-$M$ $\beta$-continuity, fuzzy weakly $r$-$M$ $\beta$-continuity and fuzzy almost $r$-$M$ $\beta$-continuity.

2. Preliminaries

Let $I$ be the unit interval $[0,1]$ of the real line. A member $A$ of $I^X$ is called a fuzzy set [10] of $X$. By $\tilde{0}$ and $\tilde{1}$, we denote constant maps on $X$ with value 0 and 1, respectively. For any $A \in I^X$, $A^c$ denotes the complement $\tilde{1} - A$. All other notations are standard notations of fuzzy set theory. An fuzzy point $x_\alpha$ in $X$ is a fuzzy set $x_\alpha$ defined as follows

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

A smooth topology [7] on $X$ is a map $T : I^X \rightarrow I$ which satisfies the following properties:

1. $T(\tilde{0}) = T(\tilde{1}) = 1$.
2. $T(A_1 \cap A_2) \geq T(A_1) \land T(A_2)$.
3. $T(\cup A_i) \geq \land T(A_i)$.

The pair $(X,T)$ is called a smooth topological space [7].

Let $X$ be a nonempty set and $r \in (0,1] = I_0$. A fuzzy family $M : I^X \rightarrow I$ on $X$ is said to have a fuzzy $r$-minimal structure [8] if the family $M_r = \{ A \in I^X \mid M(A) \geq r \}$ contains $\tilde{0}$ and $\tilde{1}$.

Then the $(X,M)$ is called a fuzzy $r$-minimal space [8] (simply $r$-FMS). Every member of $M_r$ is called a fuzzy $r$-minimal open set. A fuzzy set $A$ is called a fuzzy $r$-minimal closed set if the complement of $A$ (simply, $A^c$) is a fuzzy $r$-minimal open set. The fuzzy $r$-minimal closure of $A$, denoted by $mC(A,r)$, is defined as $mC(A,r) = \cap\{ B \in I^X : B^c \in M_r \text{ and } A \subseteq B \}$. The fuzzy $r$-minimal interior of $A$, denoted by $mI(A,r)$, is defined as $mI(A,r) = \cup\{ B \in I^X : B \in M_r \text{ and } B \subseteq A \}$.

**Theorem 2.1** ([8]). Let $(X,M)$ be an $r$-FMS and $A, B \in I^X$.

1. $mI(A,r) \subseteq A$ and if $A$ is a fuzzy $r$-minimal open set, then $mI(A,r) = A$. 
Fuzzy almost $r$-$M$ $\beta$-continuous mappings

(2) $A \subseteq mC(A, r)$ and if $A$ is a fuzzy $r$-minimal closed set, then $mC(A, r) = A$.

(3) If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.

(4) $mI(A, r) \cap mI(B, r) \supseteq mI(A \cap B, r)$ and $mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r)$.

(5) $mI(mI(A, r), r) = mI(A, r)$ and $mC(mC(A, r), r) = mC(A, r)$.

(6) $\bar{I} - mC(A, r) = mI(\bar{I} - A, r)$ and $\bar{I} - mI(A, r) = mC(\bar{I} - A, r)$.

Let $(X, \mathcal{M})$ be an $r$-FMS and $A \in I^X$. Then a fuzzy set $A$ is called a fuzzy $r$-minimal $\beta$-open set [6] in $X$ if

$$A \subseteq mC(mI(mC(A, r), r), r).$$

A fuzzy set $A$ is called a fuzzy $r$-minimal $\beta$-closed set if the complement of $A$ is fuzzy $r$-minimal $\beta$-open.

Then any union of fuzzy $r$-minimal $\beta$-open sets is fuzzy $r$-minimal $\beta$-open and in general, the intersection of two fuzzy $r$-minimal $\beta$-open sets is not fuzzy $r$-minimal $\beta$-open (see [6]). For $A \in I^X$, $m\beta C(A, r)$ and $m\beta I(A, r)$, respectively, are defined as the following:

- $m\beta C(A, r) = \cap \{ F \in I^X : A \subseteq F, F \text{ is fuzzy } r\text{-minimal } \beta\text{-closed} \}$;
- $m\beta I(A, r) = \cup \{ U \in I^X : U \subseteq A, U \text{ is fuzzy } r\text{-minimal } \beta\text{-open} \}$.

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be $r$-FMS’s. Then $f : X \rightarrow Y$ is said to be

- (1) fuzzy $r$-$M$ continuous mapping [8] if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{M}_r$.
- (2) fuzzy $r$-$M$ $\beta$-continuous [6] if for each point $x_\alpha$ and each fuzzy $r$-minimal open set $V$ containing $f(x_\alpha)$, there exists a fuzzy $r$-minimal $\beta$-open set $U$ containing $x_\alpha$ such that $f(U) \subseteq V$.
- (3) fuzzy weakly $r$-$M$ $\beta$-continuous [2] if for each fuzzy point $x_\alpha$ and each fuzzy $r$-minimal open set $V$ containing $f(x_\alpha)$, there exists a fuzzy $r$-minimal $\beta$-open set $U$ containing $x_\alpha$ such that $f(U) \subseteq mC(V, r)$.

Obviously every fuzzy $r$-$M$ $\beta$-continuous mapping is fuzzy weakly $r$-$M$ $\beta$-continuous but the converse need not be true (See [2]).

3. Fuzzy Almost $r$-$M$ $\beta$-Continuous Mappings

**Definition 3.1.** Let $f : X \rightarrow Y$ be a mapping between $r$-FMS’s $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. Then $f$ is said to be fuzzy almost $r$-$M$ $\beta$-continuous if for fuzzy point $x_\alpha$ in $X$ and each fuzzy $r$-minimal open set $V$ containing $f(x_\alpha)$, there is a fuzzy $r$-minimal $\beta$-open set $U$ containing $x_\alpha$ such that $f(U) \subseteq mI(mC(V, r), r)$.

Obviously, we have the following relationships:

fuzzy $r$-$M$ continuity $\Rightarrow$ fuzzy $r$-$M$ $\beta$-continuity $\Rightarrow$ fuzzy almost $r$-$M$ $\beta$-continuity $\Rightarrow$ fuzzy weakly $r$-$M$ $\beta$-continuity

The converses are not always true as shown in the example below:
Example 3.2. Let $X = I = [0, 1]$, and let $A$, $B$ and $C$ be fuzzy sets defined as follows

$A(x) = \begin{cases} \frac{1}{4}x + \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \frac{1}{4}x + \frac{3}{4}, & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$

$B(x) = -\frac{1}{2}(x - 1), \quad \text{if } x \in I;$

$C(x) = \frac{1}{2}x, \quad \text{if } x \in I.$

(1) Let us consider two fuzzy $\frac{2}{3}$-minimal structures

$\mathcal{M}(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \bar{0}, \bar{1}, A, \\ 0, & \text{otherwise.} \end{cases}$

$\mathcal{N}(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \bar{0}, B, C, B \cup C \\ 0, & \text{otherwise.} \end{cases}$

Note that $A \subseteq mC(B, \frac{2}{3}), mC(C, \frac{2}{3})$ and $mC(B \cup C, \frac{2}{3})$. The identity mapping $\bar{f} : (X, \mathcal{M}) \to (X, \mathcal{N})$ is fuzzy weakly $\frac{2}{3}$-M $\beta$-continuous. Note that $mImC(B, \frac{2}{3}) = mImC(C, \frac{2}{3}) = mImC(B \cup C, \frac{2}{3}) = B \cup C$ and $B \cup C \subseteq A$.

So the identity mapping $\bar{f}$ is not fuzzy almost $\frac{2}{3}$-M $\beta$-continuous.

(2) Let us consider two fuzzy $\frac{2}{3}$-minimal structures

$\mathcal{M}(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \bar{0}, \bar{1}, B, \\ 0, & \text{otherwise.} \end{cases}$

$\mathcal{N}(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \bar{0}, B, C, B \cup C \\ 0, & \text{otherwise.} \end{cases}$

Note that $B \subseteq mImC(B, \frac{2}{3}) = mImC(C, \frac{2}{3}) = mImC(B \cup C, \frac{2}{3}) = B \cup C$.

Obviously the identity mapping $\bar{f} : (X, \mathcal{M}) \to (X, \mathcal{N})$ is fuzzy almost $\frac{2}{3}$-M $\beta$-continuous but not fuzzy $\frac{2}{3}$-M $\beta$-continuous.

Let $(X, \mathcal{M})$ be an $r$-FMS and $A \in I^X$. Then in [6], we showed that:

(1) $m\beta I(A, r) \subseteq mC(mI(mC(m\beta I(A, r), r), r))$ \subseteq $mC(mI(mC(A, r), r), r)$.

(2) $m\beta I(A, r) = A \cap mC(mI(mC(A, r), r), r)$.

Theorem 3.3. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on $r$-FMS’s $(X, \mathcal{M})$ and $(Y, \mathcal{N})$. Then the following statements are equivalent:

(1) $f$ is fuzzy almost $r$-M $\beta$-continuous.

(2) $f^{-1}(V) \subseteq mC(mI(mC(f^{-1}(mI(mC(V, r), r)), r), r), r)$ for each fuzzy $r$-minimal open $V$ in $Y$. 
Proof. (1) ⇒ (2) Let \( V \) be a fuzzy \( r \)-minimal open set and \( x_\alpha \in f^{-1}(V) \). Then \( f(x_\alpha) \in V \), and from fuzzy weakly \( r-M \) \( \beta \)-continuity, there exists a fuzzy \( r \)-minimal \( \beta \)-open set \( U_{x_\alpha} \) containing \( x_\alpha \) such that \( f(U_{x_\alpha}) \subseteq mI(mC(V, r), r) \).

Thus \( x_\alpha \in U_{x_\alpha} \subseteq f^{-1}(mI(mC(V, r), r)) \). Since \( U_{x_\alpha} \) is a fuzzy \( r \)-minimal \( \beta \)-open set, \( x_\alpha \in U_{x_\alpha} \subseteq mC(mI(mC(U_{x_\alpha}, r), r), r) \subseteq mC(mI(mC(f^{-1}(mI(mC(V, r), r)), r), r), r) \).

Hence we have the following: \( f^{-1}(V) \subseteq mC(mI(mC(f^{-1}(mI(mC(V, r)), r)), r), r) \).

(2) ⇒ (1) Let \( x_\alpha \) be a fuzzy point in \( X \) and \( V \) a fuzzy \( r \)-minimal open set containing \( f(x_\alpha) \). Then by hypothesis, \( f(x_\alpha) \in f^{-1}(V) \subseteq f^{-1}(mI(mC(V, r), r)) \cap mC(mI(mC(f^{-1}(mI(mC(V, r)), r), r), r), r) \). Put \( U = f^{-1}(mI(mC(V, r))) \cap mC(mI(mC(f^{-1}(mI(mC(V, r)), r), r), r), r) \). Then \( U \) is a fuzzy \( r \)-minimal \( \beta \)-open set containing \( x_\alpha \) such that \( U \subseteq f^{-1}(mI(mC(V, r))) \). Hence \( f \) is fuzzy weakly \( r-M \) \( \beta \)-continuous.

(2) ⇔ (3) Obvious.

\( \Box \)

Lemma 3.4. ([6]) Let \((X, M)\) be an \( r \)-FMS and \( A \in I^X \). Then

(1) \( x_\alpha \in m\beta C(A, r) \) if and only if \( A \cap V \neq \emptyset \) for every \( r \)-minimal \( \beta \)-open set \( V \) containing \( x_\alpha \).

(2) \( x_\alpha \in m\beta I(A, r) \) if and only if there exists a fuzzy \( r \)-minimal \( \beta \)-open set \( G \) such that \( G \subseteq A \).

Theorem 3.5. Let \( f : X \rightarrow Y \) be a mapping between \( r \)-FMS’s \((X, M_X)\) and \((Y, M_Y)\). Then the following statements are equivalent:

(1) \( f \) is fuzzy almost \( r-M \) \( \beta \)-continuous.

(2) \( f^{-1}(B) \subseteq m\beta I(f^{-1}(mI(mC(B, r), r)), r) \) for each fuzzy \( r \)-minimal open set \( B \) of \( Y \).

(3) \( m\beta C(f^{-1}(mI(F, r)), r) \subseteq f^{-1}(F) \) for each fuzzy \( r \)-minimal closed set \( F \) in \( Y \).

Proof. (1) ⇒ (2) Let \( B \) be a fuzzy \( r \)-minimal open set in \( Y \). Then for each \( x_\alpha \in f^{-1}(V) \), there exists a fuzzy \( r \)-minimal \( \beta \)-open set \( U \) of \( x_\alpha \) such that \( f(U) \subseteq mI(mC(B, r), r) \).

It implies that \( x_\alpha \in m\beta I(f^{-1}(mI(mC(B, r)), r)), r) \). Consequently, \( f^{-1}(B) \subseteq m\beta I(f^{-1}(mI(mC(B, r)), r), r) \).

(2) ⇒ (1) Let \( x_\alpha \) be a fuzzy point in \( X \) and \( V \) a fuzzy \( r \)-minimal open set containing \( f(x_\alpha) \). By hypothesis, \( x_\alpha \in m\beta I(f^{-1}(mI(mC(V, r), r)), r) \).

From definition of the operator \( m\beta I \), there exists a fuzzy \( r \)-minimal \( \beta \)-open set \( U \) containing \( x_\alpha \) such that \( U \subseteq f^{-1}(mI(mC(V, r))) \). So, we have \( f(U) \subseteq f(f^{-1}(mI(mC(V, r))) \subseteq mI(mC(V, r), r) \). Hence \( f \) is fuzzy almost \( r-M \) \( \beta \)-continuous.

(2) ⇔ (3) Obvious.

\( \Box \)
Let $X$ be a nonempty set and $\mathcal{M} : I^X \rightarrow I$ a fuzzy family on $X$. The fuzzy family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ \cite{9} if for $A_i \in \mathcal{M}$ ($i \in J$),

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

**Theorem 3.6.** \cite{9} Let $(X, \mathcal{M})$ be an $r$-FMS with the property $(\mathcal{U})$. Then

(1) For $A \in I^X$, $mI(A, r) = A$ if and only if $A$ is fuzzy $r$-minimal open.

(2) For $F \in I^X$, $mC(F, r) = F$ if and only if $F$ is fuzzy $r$-minimal closed.

**Theorem 3.7.** Let $f : X \rightarrow Y$ be a mapping between $r$-FMS’s $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ have property $(\mathcal{U})$, then the following statements are equivalent:

(1) $f$ is fuzzy almost $r$-$M$ $\beta$-continuous.

(2) $f^{-1}(B) \subseteq m\beta I(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy $r$-minimal open set $B$ of $Y$.

(3) $f^{-1}(mI(B, r)) \subseteq m\beta I(f^{-1}(mI(mC(mI(B, r), r), r)), r)$ for each $B \in I^Y$.

(4) $m\beta C(f^{-1}(mI(mC(B, r), r)), r)) \subseteq f^{-1}(mC(B, r))$ for each $B \in I^Y$.

**Proof.** It follows from Theorem 3.5 and Theorem 3.6. \hfill \Box

Let $(X, \mathcal{M})$ be an $r$-FMS and $A \in I^X$. Then a fuzzy set $A$ is said to be fuzzy $r$-minimal regular open (resp., fuzzy $r$-minimal regular closed) \cite{4} if $A = mI(mC(A, r), r)$ (resp., $A = mC(mI(A, r), r)$).

**Theorem 3.8.** Let $f : X \rightarrow Y$ be a mapping between $r$-FMS’s $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has property $(\mathcal{U})$, then the following statements are equivalent:

(1) $f$ is fuzzy almost $r$-$M$ $\beta$-continuous.

(2) $f^{-1}(F) = m\beta C(f^{-1}(F), r)$ for any fuzzy $r$-minimal regular closed set $F$ in $Y$.

(3) $f^{-1}(V) = m\beta I(f^{-1}(V), r)$ for any fuzzy $r$-minimal regular open set $V$ in $Y$.

**Proof.** (1) $\Rightarrow$ (2) Let $F$ be any fuzzy $r$-minimal regular closed set of $Y$. Then from the property $(\mathcal{U})$, $F$ is $F = mC(mI(F, r), r)$. So by Theorem 3.5, $m\beta C(f^{-1}(F), r) = m\beta C(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$. This implies that $f^{-1}(F) = m\beta C(f^{-1}(F), r)$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $V$ be a fuzzy $r$-minimal open set containing $f(x_\alpha)$. Since $mI(mC(V, r), r)$ is fuzzy $r$-minimal regular open, by hypothesis, $f^{-1}(mI(mC(V, r), r)) = m\beta I(f^{-1}(mI(mC(V, r), r)), r)$.

So $f^{-1}(mI(mC(V, r), r))$ is a fuzzy $r$-minimal $\beta$-open set. Put $U = f^{-1}(mI(mC(V, r), r))$. Then $U$ is a fuzzy $r$-minimal $\beta$-open set containing $x_\alpha$ satisfying $U \subseteq f^{-1}(mI(mC(V, r), r))$. Then this implies $f(U) \subseteq mI(mC(V, r), r)$. Hence $f$ is fuzzy almost $r$-$M$ $\beta$-continuous. \hfill \Box
Let \((X, \mathcal{M})\) be an \(r\)-FMS and \(A \in I^X\). Then a fuzzy set \(A\) is said to be
(1) fuzzy \(r\)-minimal semiopen [5] if \(A \subseteq mC(mI(A, r), r)\);
(2) fuzzy \(r\)-minimal preopen [3] if \(A \subseteq mI(mC(A, r), r)\);
(3) fuzzy \(r\)-minimal \(\beta\)-open [6] if \(A \subseteq mC(mI(mC(A, r), r), r)\).

A fuzzy set \(A\) is called a fuzzy \(r\)-minimal semiclosed (resp., fuzzy \(r\)-minimal preclosed, fuzzy \(r\)-minimal \(\beta\)-closed) set if the complement of \(A\) is a fuzzy \(r\)-minimal semiopen (resp., fuzzy \(r\)-minimal preopen, fuzzy \(r\)-minimal \(\beta\)-open) set.

**Theorem 3.9.** Let \(f : X \to Y\) be a mapping on \(r\)-FMS’s \((X, \mathcal{M}_X)\) and \((Y, \mathcal{M}_Y)\). If \(\mathcal{M}_Y\) has the property \((\mathcal{U})\), then the following statements are equivalent:

1. \(f\) is fuzzy almost \(r\)-\(M\) \(\beta\)-continuous.
2. \(m\beta C(f^{-1}(G), r) \subseteq f^{-1}(mC(G, r))\) for each fuzzy \(r\)-minimal \(\beta\)-open set \(G\) in \(Y\).
3. \(m\beta C(f^{-1}(G), r) \subseteq f^{-1}(mC(G, r))\) for each fuzzy \(r\)-minimal semi-open set \(G\) in \(Y\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(G\) be a fuzzy \(r\)-minimal \(\beta\)-open set. Then
\(G \subseteq mC(mI(mC(G, r), r), r)\) and \(mC(G, r)\) is fuzzy \(r\)-minimal regular closed.
From Theorem 3.8, it follows:

\[ m\beta C(f^{-1}(G), r) \subseteq m\beta C(f^{-1}(mC(G, r)), r) = f^{-1}(mC(G, r)). \]

(2) \(\Rightarrow\) (3) Since every fuzzy \(r\)-minimal semiopen set is fuzzy \(r\)-minimal \(\beta\)-open, it is obvious.

(3) \(\Rightarrow\) (1) Let \(F\) be a fuzzy \(r\)-minimal regular closed set. Then \(F\) is also fuzzy \(r\)-minimal semiopen, and from hypothesis, \(m\beta C(f^{-1}(F), r) \subseteq f^{-1}(mC(F, r)) = f^{-1}(F)\). From Theorem 3.8, \(f\) is fuzzy almost \(r\)-\(M\) \(\beta\)-continuous. \(\square\)

**Theorem 3.10.** Let \(f : X \to Y\) be a mapping on \(r\)-FMS’s \((X, \mathcal{M}_X)\) and \((Y, \mathcal{M}_Y)\). If \(\mathcal{M}_Y\) has the property \((\mathcal{U})\) then \(f\) is fuzzy almost \(r\)-\(M\) \(\beta\)-continuous if and only if \(m\beta C(f^{-1}(mC(mI(mC(G, r), r), r)), r) \subseteq f^{-1}(mC(G, r))\) for each fuzzy \(r\)-minimal preopen set \(G\) in \(Y\).

**Proof.** Let \(f\) be fuzzy almost \(r\)-\(M\) \(\beta\)-continuous and \(G\) a fuzzy \(r\)-minimal pre-open set in \(Y\). Then obviously, we have \(mC(G, r) = mC(mI(mC(G, r), r), r)\) and so \(mC(G, r)\) is fuzzy \(r\)-minimal regular open. From Theorem 3.8, it follows

\[ f^{-1}(mC(G, r)) = m\beta C(f^{-1}(mC(G, r)), r) = mC\beta(f^{-1}(mC(mI(mC(G, r), r), r)), r). \]

So we have:
\[ m\beta C(f^{-1}(mC(mI(mC(G, r), r), r)), r) \subseteq f^{-1}(mC(G, r)). \]

For the converse, let \(A\) be a fuzzy \(r\)-minimal regular closed set in \(Y\). Then clearly, \(mI(A, r)\) is also fuzzy \(r\)-minimal preopen. From hypothesis and \(A = \)
$mC(mI(A, r), r)$, it follows

$$f^{-1}(A) = f^{-1}(mC(mI(A, r), r))$$

$$\geq m\beta C(f^{-1}(mC(mI(mC(mI(A, r), r)), r)), r)$$

$$= m\beta C(f^{-1}(mC(mI(A, r)), r), r)$$

$$= m\beta C(f^{-1}(A), r).$$

Hence, by Theorem 3.8, it implies that $f$ is fuzzy almost $r$-$M$ $\beta$-continuous. \hfill \Box

**Theorem 3.11.** Let $f : X \to Y$ be a mapping on $r$-FMS’s $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has the property $(U)$, then $f$ is fuzzy almost $r$-$M$ $\beta$-continuous if and only if $f^{-1}(G) \subseteq m\beta I(f^{-1}(mI(mI(G, r)), r), r)$ for each fuzzy $r$-$M$ preopen set $G$ in $Y$.

**Proof.** Let $f$ be fuzzy almost $r$-$M$ continuous and $G$ a fuzzy $r$-minimal preopen set in $Y$. Then obviously, $mI(mI(G, r), r)$ is also fuzzy $r$-minimal regular open. From Theorem 3.8, $f^{-1}(G) \subseteq f^{-1}(mI(mI(G, r), r)) = m\beta I(f^{-1}(mI(mI(G, r), r)), r)$. So $f^{-1}(G) \subseteq m\beta I(f^{-1}(mI(mI(G, r), r)), r)$.

For the converse, let $U$ be fuzzy $r$-minimal regular open. Then $U$ is fuzzy $r$-minimal preopen. From hypothesis and $A = mI(mC(A, r), r)$, $f^{-1}(U) \subseteq m\beta I(f^{-1}(mI(mI(U, r), r)), r) = m\beta I(f^{-1}(U), r)$. This implies $f^{-1}(U) = m\beta I(f^{-1}(U), r)$. So by Theorem 3.8, $f$ is fuzzy almost $r$-$M$ $\beta$-continuous. \hfill \Box

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