Generalized Hyers-Ulam Stability of a 3-Dimensional Quadratic Functional Equation

Sun Sook Jin

Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Republic of Korea

Yang-Hi Lee

Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Republic of Korea

Copyright © 2016 Sun-Sook Jin and Yang-Hi Lee. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract
In this paper, we investigate the stability of a functional equation

\[ f(ax + ay + az) - a^2f(x + y) - a^2f(y + z) - a^2f(x + z) + a^2f(x) + a^2f(y) + a^2f(z) = 0 \]

by applying the direct method in the sense of Hyers.

Mathematics Subject Classification: 39B82, 39B52

Keywords: stability, generalized Hyers-Ulam stability, direct method

1 Introduction
In 1940, Ulam [8] proposed the stability problem of group homomorphisms. A year later, Hyers [4] gave a partial result for the first time to this problem of
the functional equation
\[ f(x + y) - f(x) - f(y) = 0. \] (1)

Thereafter, many mathematicians came to deal with this problem (cf. [1, 3, 6]) and specially Rassias obtained a generalized theorem of Hyers’. On the other hand, we call the functional equation
\[ f(x + y) - f(x - y) - 2f(x) - 2f(y) = 0 \] (2)

In this article, we consider the 3-dimensional functional equation
\[ f(ax + ay + az) - a^2 f(x + y) - a^2 f(y + z) - a^2 f(x + z) + a^2 f(x) + a^2 f(y) + a^2 f(z) = 0 \] (3)
where \( a \) be a fixed rational number with \( a \not\in \{-1, 0, 1\} \). Firstly, we will show that the functional equation (3) is equivalent to the above quadratic functional equation (2). For this reason we call the functional equation (3) the 3-dimensional quadratic functional equation. And then, we will obtain the stability of this functional equation (3) using the Hyers’ method presented in [4]. Namely, starting from the given mapping \( f \) that approximately satisfies the functional equation (3), a solution \( F \) of the functional equation (3) is explicitly constructed by the formula
\[ F(x) := \lim_{n \to \infty} \frac{f(a^n x)}{a^{2n}} \text{ or } F(x) := \lim_{n \to \infty} a^{2n} f\left( \frac{x}{a^n} \right) \]
which approximates the mapping \( f \).

2 Main results

Throughout this section, let \( V \) and \( W \) be real vector spaces. And let \( X \) and \( Y \) be a real normed space and a real Banach space, respectively.

For given mapping \( f : V \to W \) and a fixed rational number \( a \), we denote the following notation
\[ Df(x, y, z) := f(ax + ay + az) - a^2 f(x + y) - a^2 f(y + z) - a^2 f(x + z) + a^2 f(x) + a^2 f(y) + a^2 f(z) \]
for all \( x, y, z \in V \).
Lemma 2.1 If \( f : V \to Y \) is a mapping such that \( Df(x,y,z) = 0 \) for all \( x, y, z \in V \setminus \{0\} \), then 
\[
Df(x,y,z) = 0
\]
for all \( x, y, z \in V \).

Proof. Notice that the equality 
\[
f(0) = \frac{Df(2x,-x,-x) + Df(-2x,x,x)}{2}
\]
for any \( x \in V \setminus \{0\} \) implies that \( f(0) = 0 \). As well as, we show that \( f(ax) = a^2f(x) \) from the equality
\[
f(ax) - a^2f(x) = Df(3x,-x,-x) + Df(2x,-2x,x) - Df(x,x,-x)
\]
for all \( x \in V \setminus \{0\} \). So we easily prove that
\[
Df(x,y,0) = f(ax + ay) - a^2f(x + y) = 0
\]
as well as \( Df(x,0,z) = 0, Df(0,y,z) = 0, Df(x,0,0) = 0, Df(0,0,z) = 0, Df(0,y,0) = 0 \), and \( Df(0,0,0) = 0 \) for all \( x, y, z \in V \setminus \{0\} \) as desired.

To prove the stability of the equation \( Df(x,y,z) = 0 \), we also use the following abbreviations
\[
Af(x,y) := f(x+y) - f(x) - f(y),
Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y),
\]
for all \( x, y, z \in V \). Recall each solution of \( Qf = 0 \) is called a quadratic mapping.

Now we will show that the functional equation \( Df = 0 \) is equivalent to the quadratic functional equation \( Qf = 0 \) with the assumption that \( a \) is a fixed rational number.

Lemma 2.2 Let \( a \) be a fixed rational number such that \( a \notin \{0,1,-1\} \). A mapping \( f : V \to W \) satisfies the functional equation \( Df(x,y,z) = 0 \) if and only if \( f \) is a quadratic mapping.

Proof. Suppose that \( f : V \to W \) is a solution of the functional equation \( Df(x,y,z) = 0 \). Then we have that \( f(0) = Df(0,0,0) = 0 \) and
\[
Af_o(x,y) = \frac{1}{a^2}Df_o\left(\frac{x+y}{2},\frac{y-x}{2},\frac{x-y}{2}\right) - \frac{1}{a^2}Df_o\left(\frac{x+y}{2},\frac{x+y}{2},\frac{-x-y}{2}\right)
\]
for all $x, y \in V$, where $f_o(x) := \frac{f(x) - f(-x)}{2}$. Thus $f_o$ is an additive mapping and $f_o(ax) = af_o(x)$ for all $x \in V$. On the other hand, we can show that $f(ax) = Df(x, 0, 0) + a^2f(x) = a^2f(x)$ for all $x \in V$. And then

$$f_o(ax) = \frac{f(ax) - f(-ax)}{2} = \frac{a^2f(x) - a^2f(-x)}{2} = a^2f_o(x)$$

for all $x \in V$. It follows that $f_o(ax) = af_o(x) = a^2f_o(x)$, and so we get the equality $f_o(x) = 0$ for all $x \in V$. Now, together with the previous results $f(0) = 0$, $f(ax) = a^2f(x)$, and $f(-x) = f(x)$ for all $x \in V$, we get the desired equality

$$Qf(x, y) = -\frac{1}{a^2}Df(x, y, -y) = 0$$

for all $x, y \in V$.

Conversely, let $f : V \to W$ be a quadratic mapping. Then we get the equalities $f(ax) = a^2f(x)$ and $f(-x) = f(x)$ for all $x \in V$. Notice that

$$Df(x, y, z) = a^2 \left( f(x + y + z) - f(x + y) - f(y + z) - f(x + z) + f(x) + f(y) + f(z) \right)$$

$$= a^2 \left[ f(x + y + z) + f(x - y) - 2f \left( x + \frac{z}{2} \right) - 2f \left( y + \frac{z}{2} \right) \right]$$

$$- \left( f(x + y) + f(x - y) - 2f(x) - 2f(y) \right)$$

$$- \left( f(x + z) + f(x) - 2f \left( x + \frac{z}{2} \right) - 2f \left( \frac{z}{2} \right) \right)$$

$$- \left( f(y + z) + f(y) - 2f \left( y + \frac{z}{2} \right) - 2f \left( \frac{z}{2} \right) \right) \right]$$

$$= a^2 \left[ Qf \left( x + \frac{z}{2}, y + \frac{z}{2} \right) - Qf(x, y) \right]$$

$$-Qf \left( x + \frac{z}{2}, y + \frac{z}{2} \right) - Qf(x, y) \right)$$

$$= 0$$

for all $x, y, z \in V$. It completes the proof of the lemma.

**Theorem 2.3** Let $a$ be a rational constant with $a \not\in \{0, 1, -1\}$ and let $\varphi : (V \setminus \{0\})^3 \to [0, \infty)$ be a function satisfying either

$$\sum_{i=0}^{\infty} \varphi(a^i x, a^i y, a^i z) a^{2i} < \infty \quad (5)$$

or

$$\sum_{i=0}^{\infty} a^{2i} \varphi \left( \frac{x}{a^i}, \frac{y}{a^i}, \frac{z}{a^i} \right) < \infty \quad (6)$$
for all \(x, y, z \in V \setminus \{0\}\). If a mapping \(f : V \to Y\) satisfies \(f(0) = 0\) and
\[
\|Df(x, y, z)\| \leq \varphi(x, y, z)
\]
for all \(x, y, z \in V \setminus \{0\}\), then there exists a unique quadratic mapping \(F : V \to Y\) such that
\[
\|f(x) - F(x)\| \leq \begin{cases} 
\sum_{i=0}^{\infty} \frac{\varphi(a^i x)}{a^{2i+2}} & \text{if } \varphi \text{ satisfies (5)}, \\
\sum_{i=0}^{\infty} a^{2i} \Phi\left(\frac{x}{a^{i+1}}\right) & \text{if } \varphi \text{ satisfies (6)}
\end{cases}
\]
holds for all \(x \in V \setminus \{0\}\), where
\[
\Phi(x) := \varphi(3x, -x, -x) + \varphi(2x, -2x, x) + \varphi(x, x, -x).
\]

**Proof.** We prove in two cases, either \(\varphi\) satisfies (5) or \(\varphi\) satisfies (6).

**Case 1.** Let \(\varphi\) satisfy (5). It follows from (4) and (7) that
\[
\frac{f(a^nx)}{a^{2n}} - \frac{f(a^{n+m}x)}{a^{2n+2m}} \leq \sum_{i=n}^{n+m-1} \frac{f(a^i x)}{a^{2i}} - \frac{f(a^{i+1} x)}{a^{2i+2}}
\]
\[
= \sum_{i=n}^{n+m-1} \frac{\|Df(a^i x, a^i x, -a^i x) - Df(3a^i x, -a^i x, -a^i x) - Df(2a^i x, -2a^i x, a^i x)\|}{a^{2i+2}}
\]
\[
\leq \sum_{i=n}^{n+m-1} \frac{\varphi(3a^i x, -a^i x, -a^i x) + \varphi(2a^i x, -2a^i x, a^i x) + \varphi(a^i x, a^i x, -a^i x)}{a^{2i+2}}
\]
for all \(x \in V \setminus \{0\}\). So, it is easy to show that the sequence \(\{f(a^nx)/a^{2n}\}\) is a Cauchy sequence for all \(x \in V \setminus \{0\}\). Since \(Y\) is complete and \(f(0) = 0\), the sequence \(\{f(a^nx)/a^{2n}\}\) converges for all \(x \in V\). Hence, we can define a mapping \(F : V \to Y\) by
\[
F(x) := \lim_{n \to \infty} \frac{f(a^nx)}{a^{2n}}
\]
for all \(x \in V\). Moreover, if we put \(n = 0\) and let \(m \to \infty\) in (9), we obtain the first inequality in (8). From the definition of \(F\), we get
\[
\|DF(x, y, z)\| = \lim_{n \to \infty} \frac{\|Df(a^nx, a^ny, a^nz)\|}{a^{2n}} \leq \lim_{n \to \infty} \frac{\varphi(a^n x, a^n y, a^n z)}{a^{2n}} = 0
\]
for all \(x, y, z \in V \setminus \{0\}\) i.e., \(DF(x, y, z) = 0\) for all \(x, y, z \in V \setminus \{0\}\). By Lemma 2.1, \(F\) is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping \(F' : V \to Y\) which satisfies the first
inequality in (8). Notice that \( F'(x) = \frac{F'(a^nx)}{a^{2n}} \) holds for all \( x \in V \). Using (5) and (8), we obtain
\[
\lim_{n \to \infty} \left\| f(a^n x) - F'(x) \right\| = \lim_{n \to \infty} \left\| \frac{f(a^n x) - F'(a^n x)}{a^{2n}} \right\|
\leq \lim_{n \to \infty} \sum_{i=0}^{\infty} \frac{\Phi(a^{i+n} x)}{a^{2n+2i+2}}
= \lim_{n \to \infty} \sum_{i=n}^{\infty} \frac{\Phi(a^i x)}{a^{2i+2}}
= 0
\]
for all \( x \in V \setminus \{0\} \), i.e., \( F'(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^{2n}} = F(x) \) for all \( x \in V \).

**Case 2.** Let \( \varphi \) satisfy (6). It follows from (7) and (4) that
\[
\left\| a^{2n} f\left( \frac{x}{a^n} \right) - a^{2n+2m} f\left( \frac{x}{a^{n+m}} \right) \right\|
\leq \sum_{i=n}^{n+m-1} \left\| a^{2i} f\left( \frac{x}{a^i} \right) - a^{2i+2} f\left( \frac{x}{a^{i+1}} \right) \right\|
= \sum_{i=n}^{n+m-1} a^{2i} \left\| Df\left( \frac{3x}{a^{i+1}}, \frac{x}{a^{i+1}}, -x \right) + Df\left( \frac{2x}{a^{i+1}}, \frac{-2x}{a^{i+1}}, \frac{x}{a^{i+1}} \right) - Df\left( \frac{x}{a^{i+1}}, \frac{x}{a^{i+1}}, \frac{-x}{a^{i+1}} \right) \right\|
\leq \sum_{i=n}^{n+m-1} a^{2i} \Phi\left( \frac{x}{a^{i+1}} \right)
\]
for all \( x \in V \setminus \{0\} \). So, it is easy to show that the sequence \( \left\{ a^{2n} f\left( \frac{x}{a^n} \right) \right\} \) is a Cauchy sequence for all \( x \in V \setminus \{0\} \). Since \( Y \) is complete and \( f(0) = 0 \), the sequence \( \left\{ a^{2n} f\left( \frac{x}{a^n} \right) \right\} \) converges for all \( x \in V \). Hence, we can define a mapping \( F : V \to Y \) by
\[
F(x) := \lim_{n \to \infty} a^{2n} f\left( \frac{x}{a^n} \right)
\]
for all \( x \in V \). Moreover, if we put \( n = 0 \) and let \( m \to \infty \) in (10), we obtain the second inequality in (8). From the definition of \( F \), we get
\[
\left\| DF(x, y, z) \right\| = \lim_{n \to \infty} \left\| a^{2n} Df\left( \frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n} \right) \right\| \leq \lim_{n \to \infty} a^{2n} \varphi\left( \frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n} \right) = 0
\]
for all \( x, y, z \in V \setminus \{0\} \) i.e., \( DF(x, y, z) = 0 \) for all \( x, y, z \in V \setminus \{0\} \). By Lemma 2.1, \( F \) is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping \( F' : V \to Y \) which satisfies the second inequality in (8). Notice that \( F'(x) = a^{2n} F'(\frac{x}{a^n}) \) for all \( x \in V \). Using (6) and
(8), we get
\[
\lim_{n \to \infty} \left\| a^{2n} f \left( \frac{x}{a^n} \right) - F'(x) \right\| = \lim_{n \to \infty} \left\| a^{2n} f \left( \frac{x}{a^n} \right) - a^{2n} F' \left( \frac{x}{a^n} \right) \right\|
\leq \lim_{n \to \infty} \sum_{i=0}^{\infty} a^{2n+2i} \phi \left( \frac{x}{a^{n+i}} \right)
= \lim_{n \to \infty} \sum_{i=n}^{\infty} a^{2i} \phi \left( \frac{x}{a^i} \right)
= 0
\]
for all \( x \in V \setminus \{0\} \), i.e, \( F'(x) = \lim_{n \to \infty} a^{2n} f \left( \frac{x}{a^n} \right) = F(x) \) for all \( x \in V \).

Now we will establish another proof for the stability of the functional equation \( Df(x, y, z) = 0 \).

**Theorem 2.4** Let \( \varphi : V^3 \to [0, \infty) \) be a function satisfying either (5) or (6) for all \( x, y, z \in V \). If a mapping \( f : V \to Y \) satisfies \( f(0) = 0 \) and (7) for all \( x, y, z \in V \), then there exists a unique quadratic mapping \( F : V \to Y \) such that
\[
\| f(x) - F(x) \| \leq \begin{cases} 
\sum_{i=0}^{\infty} \frac{\varphi(a^i x, 0, 0)}{a^{2i+2}} & \text{if } \varphi \text{ satisifes (5)}, \\
\sum_{i=0}^{\infty} a^{2i} \varphi \left( \frac{x}{a^i}, 0, 0 \right) & \text{if } \varphi \text{ satisifes (6)}
\end{cases}
\]
for all \( x \in V \).

**Proof.** We prove in two cases, either \( \varphi \) satisfies (5) or \( \varphi \) satisfies (6).

**Case 1.** Let \( \varphi \) satisfy (5). It follows from (7) that
\[
\left\| \frac{f(a^n x)}{a^{2n}} - \frac{f(a^{n+m} x)}{a^{2n+2m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{f(a^i x)}{a^{2i}} - \frac{f(a^{i+1} x)}{a^{2i+2}} \right\|
= \sum_{i=n}^{n+m-1} \left\| - Df(a^i x, 0, 0) \right\|
\leq \sum_{i=n}^{n+m-1} \frac{\varphi(a^i x, 0, 0)}{a^{2i+2}}
\]
for all \( x \in V \). The rest of the proof is similar to the proof of the case 1 in Theorem 2.3.

**Case 2.** Let \( \varphi \) satisfy (6). It follows from (7) that
\[
\left\| a^{2n} f \left( \frac{x}{a^n} \right) - a^{2n+2m} f \left( \frac{x}{a^{n+m}} \right) \right\| \leq \sum_{i=n}^{n+m-1} \left\| a^{2i} f \left( \frac{x}{a^i} \right) - a^{2i+2} f \left( \frac{x}{a^{i+1}} \right) \right\|
\]
\[= \sum_{i=n}^{n+m-1} a^{2i} \left\| Df \left( \frac{x}{a^{i+1}}, 0, 0 \right) \right\| \]
\[\leq \sum_{i=n}^{n+m-1} a^{2i} \varphi \left( \frac{x}{a^{i+1}}, 0, 0 \right) \]
for all \( x \in V \). The rest of the proof is similar to the proof of the case 2 in Theorem 2.3.

Now, using Theorem 2.4, we can obtain Hyers-Ulam-Rassias stability of the equation (1.3).

**Corollary 2.5** Let \( X \) be a normed space, let \( a \) be a fixed rational number such that \( a \notin \{0, 1, -1\} \), and let \( p, \theta \) be positive real constants with \( p \neq 2 \). If a mapping \( f : X \to Y \) satisfies the inequality
\[ \| Df(x, y, z) \| \leq \theta (\| x \|^p + \| y \|^p + \| z \|^p) \] (12)
for all \( x, y, z \in X \), then there exists a unique quadratic mapping \( F : X \to Y \) such that
\[ \| f(x) - F(x) \| \leq \frac{\theta \| x \|^p}{|a^2 - |a|^p|} \] (13)
for all \( x \in X \).

**Proof.** If we put \( \varphi(x, y, z) := \theta (\| x \|^p + \| y \|^p + \| z \|^p) \) for all \( x, y, z \in X \), then \( \varphi \) satisfies (5) when \( (|a| - 1)(2 - p) > 0 \) and \( \varphi \) satisfies (6) when \( (|a| - 1)(2 - p) < 0 \). From the inequality \( \| f(0) \| = \| Df(0,0,0) \| \leq 0 \), we get \( f(0) = 0 \). Therefore, by Theorem 2.4, there exists a unique quadratic mapping satisfying the inequality (13) for all \( x \in X \).

In the case of \( p < 0 \) in (12), using Theorem 2.3, we establish the superstability of the functional equation (3).

**Corollary 2.6** Suppose that \( a \) is a fixed rational number and \( p, \theta \) are fixed real numbers such that \( a \notin \{0, 1, -1\} \), \( \theta > 0 \), and \( p < 0 \). If a mapping \( f : X \to Y \) satisfies the inequality (12) for all \( x, y, z \in X \{0\} \), then \( f \) is itself a quadratic mapping.

**Proof.** If we put \( \varphi(x, y, z) := \theta (\| x \|^p + \| y \|^p + \| z \|^p) \) for all \( x, y, z \in X \{0\} \), then \( \varphi \) satisfies (5) when \( (|a| - 1)(2 - p) > 0 \) and \( \varphi \) satisfies (6) when \( (|a| - 1)(2 - p) < 0 \). Notice that we have the equality
\[ \| f(0) \| = \frac{\| Df(2kx, -kx, -kx) + Df(-2kx, kx, kx) \|}{2} \]
and the inequality
\[
\left\| \frac{Df(2kx, -kx, -kx) + Df(-2kx, kx, kx)}{2} \right\| \leq (2^p + 2) |k|^p \theta \|x\|^p
\]
for all \( x \in V \setminus \{0\} \) and for all real numbers \( k \), which implies the equality \( f(0) = 0 \). Therefore, by Theorem 2.3, there exists a unique quadratic mapping satisfying the inequality
\[
\|f(x) - F(x)\| \leq \frac{(2 \cdot 2^p + 3^p + 6) \theta \|x\|^p}{|a^2 - |a|^p|}
\]
for all \( x \in X \setminus \{0\} \). From (12), \( DF = 0 \), and (14), we get
\[
\|f(x) - F(x)\| \leq 2 \|Df - DF\|(2k + 1)x, -kx, -kx) + 2a^2 \| (f - F)(-2kx) \|
\leq \left( \frac{a^2(2 \cdot 2^p + 3^p + 6)(2k + 1)^p}{|a^2 - |a|^p|} + 2|k|^p \right) \theta \|x\|^p
\]
\[
\to 0, \quad \text{as} \quad k \to \infty,
\]
for all \( x \in X \setminus \{0\} \). Since \( f(0) = 0 = F(0) \), we have the equality \( f(x) = F(x) \) for all \( x \in X \) as desired.

Finally, using Theorem 2.4 again, we can obtain Hyers-Ulam stability of the equation (3).

**Corollary 2.7** Let \( X \) be a normed space. Suppose that \( \delta \) is a positive real constant. If a mapping \( f : X \to Y \) satisfies the inequality
\[
\|Df(x, y, z)\| \leq \delta
\]
for all \( x, y, z \in X \), then there exists a unique quadratic mapping \( F : X \to Y \) such that
\[
\|f(x) - F(x)\| \leq \frac{2\delta}{|a^2 - 1|} + \delta
\]
holds for all \( x \in X \).

**Proof.** Let \( g \) be the mapping defined by \( g(x) := f(x) - f(0) \) for all \( x \in X \). Then we have
\[
\|Dg(x, y, z)\| \leq \|Df(x, y, z) - Df(0, 0, 0)\| \leq 2\delta
\]
for all $x, y, z \in X$ and $g(0) = 0$. If we put $\varphi(x, y, z) := 2\delta$ for all $x, y, z \in X$, then $\varphi$ satisfies (5) when $|a| > 1$ and $\varphi$ satisfies (6) when $|a| < 1$. Therefore, by Theorem 2.4, there exists a unique quadratic mapping satisfying the inequality

$$\|f(x) - f(0) - F(x)\| = \|g(x) - F(x)\| \leq \frac{2\delta}{|a^2 - 1|}$$

for all $x \in X$. Together with $\|f(0)\| = \|Df(0, 0, 0)\| \leq \delta$, we get the desired inequality (16).

Acknowledgements. This work was supported by Gongju National University of Education Grant.

References


Received: April 5, 2016; Published: May 30, 2016