Subjective Probability Theory on Ordered Normed Linear Spaces

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Abstract

In this paper we establish the first steps of a subjective distribution theory on ordered normed linear spaces, relying on the geometric elements of the partial ordering. The existence either of order-units (interior-points), or quasi-interior points, either of a bounded base, or the fact that the cone is well-based, implies the definition of the analog of all the well-known probability distributions in these spaces.

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1 Introduction

In mathematical economics, the problem of defining a probability measure, which is compatible with a likelihood function \( v \) which is assigned on some family of sets \( \mathcal{E} \), being a sub-class of some \( \sigma \)-algebra \( \mathcal{A} \) is a typical problem in Decision Making, see for example the papers [2] and [10]. The Ellsberg paradox in Decision Making, is exactly related to this topic, [3]. The basic idea is that people always choose a known probability of winning over an unknown
probability of winning even if the known probability is low and the unknown probability could be a guarantee of winning. Due to this paradox, there is a need to define subjective probability measures on state -spaces, by using "known" likelihoods. The "known" probability may be one of the well-known cumulative probability distributions. \( \mathcal{E} \) contains both the set \( \Omega \) of the states of the world and the empty set \( \emptyset \). These appropriate likelihood functions are the capacities. A capacity is a map \( v : \mathcal{E} \rightarrow [0, 1] \), such that \( v(\emptyset) = 0, v(\Omega) = 1 \), while if \( A \subseteq B, A, B \in \mathcal{E}, v(A) \leq v(B) \). More specifically, elements of a partially ordered linear space \( E \) may be considered to be the set of states of the world \( \Omega \), if the function space \( E \subseteq E \) is a path space of some class of stochastic processes. Then, if there is a capacity \( v \) being defined on a class \( E \subseteq \mathcal{E} \subseteq E \) of paths, the question of the definition of a probability measure on the measurable space \( (E, \sigma(\mathcal{E})) \) arises.

In the present paper we show that if the \( \sigma \)-algebra is the one of the Borel sets \( \mathcal{B}_E \) generated by the class of open sets with respect to the norm topology of some partially ordered normed linear space \( E \), there is a variety of capacities which may be ordered on appropriate classes \( E \subseteq \mathcal{B}_E \), according to the geometry of the partial ordering of the space \( E \). These capacities may be extended to a Borel probability measure, while the monotonicity properties of them are compatible to any of the usual, real-valued, cumulative distribution functions. We recall that by the notation \( [h, g) \subseteq E \), we denote the set \( \{x \in E | g \geq x \geq h, g \neq x\} \), if \( E \) is an ordered linear space and its partial ordering \( \geq \) is defined by a cone \( E_+ \). We have the following relevant

**Theorem 1.** If \( E \) is a partially ordered normed space, and

(i) \( \mathcal{E} = \{k(B - B), k \in \mathbb{R}_+\} \), where \( B \) is a closed, bounded base of \( E_+ \)

(ii) \( \mathcal{E} = \{k[-e, e], k \in \mathbb{R}_+\} \), where \( e \in E_+ \) is an order-unit of \( E_+ \)

(iii) \( \mathcal{E} = \{[f(s)u, f(t)u] \}, \) where \( u \in E_+ \) is a quasi-interior point of \( E \) \( f(t) > f(s), t, s \in E \) and \( f \) is a strictly positive functional of \( E \),

then a capacity \( v_F \) may be defined on \( \mathcal{E} \), compatible with some cumulative distribution function \( F \).

**Proof:**

(i) If \( k_1 \leq k_2 \), then \( F(k_1) \leq F(k_2) \), which implies that \( k_1(B - B) \subseteq k_2(B - B) \) and the map \( v_F(k(B - B)) = F(k) \), is a capacity.

(ii) If \( k_1 \leq k_2 \), then \( F(k_1) \leq F(k_2) \), which implies that \( k_1[-e, e] \subseteq k_2[-e, e] \) and the map \( v_F(k[-e, e]) = F(k) \), is a capacity.
(iii) If \( f(t_2) \geq f(t_1), f(s_1) < f(s_2) \), then \([f(t_1)u, f(s_1)u] \subseteq [f(t_2)u, f(s_2)u]\) and the map \( \nu_F([f(s)u, f(t)u]) = F(f(t)) - F(f(s)) \) is a capacity, because \( F(f(s_1)) - F(f(t_1)) \leq F(f(s_2)) - F(f(t_2)) \), under the above inclusion.

The above Theorem and the rest of the paper is an extension of the partial probabilistic information model by Lehrer [7], mentioned in [10]. The assumption that \( \Omega \) is a finite set is removed in the present paper, an assumption which is present in [7] and [2]. If for example \( \Omega = E \) is a path space of some stochastic process, then \( \Omega \) is an infinite-dimensional space, as we mentioned above.

## 2 Partially ordered linear spaces and probability distributions

Consider some partially ordered linear space \( E \), whose positive cone \( E_+ \) has a bounded base \( B \), such that \( 0 \notin \overline{B} \) and \( E_+ \) gives an open decomposition, according to [6, Th.3.8.11]. For the notion of open decomposition, see [6, Ch.3.3]-for each neighborhood \( U \) of zero, the set \( K \cap U - K \cap U \), where \( K \) is the ordering cone of the space, is a neighborhood of zero. A relative Proposition is [6, Pr.3.3.1], which denotes that a cone with non-empty interior gives an open decomposition. As it is well-known by [5, Th.4], the property of well-based cone is equivalent to the fact that 0 is a point of continuity of \( C \), both with the existence of quasi-interior points in \( E^\ast \). Let us denote by \( C \) the set \( \cup_{t \in [0,1]} t \cdot B \). Then, \( V = C - C \) is a bounded, convex neighborhood of 0. We may also consider a cumulative distribution function \( F : \mathbb{R}_+ \rightarrow [0,1] \).

**Theorem 2.** The map \( \mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+ \), induces a Borel probability measure on \( E \).

**Proof:** In this case, \( B - B \) is a closed, convex neighborhood of zero, where \( B \) is the bounded base for which \( 0 \notin \overline{B} \). We may denote again by \( B \) the bounded base defined by a functional \( f \in E^\ast \) in this case. Hence, since the Borel \( \sigma \)-algebra \( \mathcal{B}_E \) for the norm topology contains the class of the sets \( K = \{k(B - B), k \in \mathbb{R}_+\}, \sigma(K) = \mathcal{B}_E \).

**Example 3.** A Bishop-Phelps cone is a cone in some normed linear space \( E \), such that \( K(f, a) = \{ x \in E | f(x) \geq a \|x\| \} \), where \( f \in E^\ast \) with \( \|f\| = 1 \) and \( a \in (0,1) \). According to what is mentioned in [6, p.127] has interior points, hence \( E_+ \) gives an open decomposition.

We recall some definitions regarding Schauder bases of Banach spaces - see in [9].

**Definition 4.** If a Banach space has a basis \( x_n, n = 1, 2, ... \), this basis is:
(i) of type $P^*$, if the coefficient functionals $f_n, n = 1, 2, \ldots$ of $x_n, n = 1, 2, \ldots$ are a basis of type $P$ in $E^*$.

(ii) of type $\ell_+$, if there is some $d > 0$, such that $\| \sum_{i=1}^{n} a_i x_i \| \geq d \cdot \text{sum}_{i=1}^{n} a_i$.

(iii) of type $P$, if $x_n, n = 1, 2, \ldots$ is bounded and $\sup_{n \geq 1} \| \sum_{i=1}^{n} x_i \| < \infty$.

(iv) of type $(\ell_+)^*$, if the coefficient functionals $f_n, n = 1, 2, \ldots$ of $x_n, n = 1, 2, \ldots$ are a basis of type $\ell_+^*$ in $E^*$.

(v) of type $a\ell_+$, if there exists a sequence of scalars $\varepsilon_n, n = 1, 2, \ldots$ with $|\varepsilon_n| = 1$, such that the basis $\varepsilon_n x_n, n = 1, 2, \ldots$ of $E$ is of type $\ell_+$.

As a consequence of the last Theorem, we take the following Corollaries, related to the above Definition.

**Corollary 5.** The map $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any Banach space having a basis of type $P^*$.

**Proof:** According to [9, Th.10.1], the existence of a basis of type $P^*$ in $E$, is equivalent to the existence of a basis of type $\ell_+$. If a basis is of type $\ell_+$, then according to [9, Th.10.2] the cone $K = \{ \sum_{i=1}^{\infty} a_i x_i | a_i \geq 0, i = 1, 2, \ldots, \sum_{i=1}^{\infty} a_i < \infty \}$ has a bounded base $B = B_f$, defined by a functional $f \in E^*$.

**Corollary 6.** The map $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any Banach space having a basis of type $a\ell_+$.

**Proof:** According to [9, Th.10.3], the existence of a basis of type $a\ell_+$ in $E$, is equivalent to the existence of a basis of type $\ell_+$. If a basis is of type $\ell_+$, then according to [9, Th.10.2] the cone $K = \{ \sum_{i=1}^{\infty} a_i x_i | a_i \geq 0, i = 1, 2, \ldots, \sum_{i=1}^{\infty} a_i < \infty \}$ has a bounded base $B = B_f$, defined by a functional $f \in E^*$.

**Corollary 7.** The map $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any Banach space having a basis of type $\ell_+$.

**Proof:** The same proof with the one above.

**Example 8.** The usual basis $e_n, n = 1, 2, \ldots$ of the Banach space $\ell^1$ is a basis of type $\ell_+$. A bounded base is defined by $1 = (1, 1, \ldots)$.

**Corollary 9.** The map $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on the Banach space $E^*$ if the coefficient functionals of this basis in $E^*$ are a basic sequence of type $P$.

**Proof:** If this is the case, $E^*$ has a basic sequence which is equivalent to a basic sequence of type $\ell_+$ and there is a bounded base $B$ in $E^*$, defined by a functional in $E^{**}$, such that the map $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$ induces a Borel probability measure on the Banach space $E^*$. 


Example 10. A basis of type $P$ in $L^1[0,1]$ exists, see [4, p.522-23].

Corollary 11. The map $\mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on the Banach space $E$ if the coefficient functionals of this basis in $E^*$ are a basis of type $(\ell_+)^*$.

Proof: According to [9, Def.10.1], any basis of type $(\ell_+)^*$ is a basic sequence of type $\ell_+$ in $E^*$, hence the previous proof may be repeated.

Corollary 12. The map $\mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+$, where $B$ is a bounded base of $E_+$, induces a Borel probability measure on any AL-space $E$.

Proof: Every AL-space is lattice -isomorphic to some $L^1$ space, according to the Theorem of Kakutani. Hence, we may identify $E$ with some $L^1$ space. Any functional $f \in \text{int}(L_1^\infty)$ defines a bounded base on $L_1^\infty = E_+$. The map $\mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on the Banach space $E$.

Corollary 13. The map $\mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any normed space $E$, ordered by a Bishop-Phelps cone.

Proof: In this case, $B - B$ is a closed, convex neighborhood of zero, where $B = B_f = \{x \in K(f,a) | f(x) = 1\}$. Hence, since the Borel $\sigma$-algebra $B_E$ for the norm topology contains the class of the sets $K = \{k[-e,e], k \in \mathbb{R}_+\}$, $\sigma(K) = B_E$.

An analogous definition of probability measures is taken, in case where $E_+$ has an order -unit $e$. In this case, $[-e,e]$ is a bounded, convex neighborhood of 0.

Theorem 14. The map $\mu_F(k[-e,e]) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on $E$.

Proof: In this case, $[-e,e]$ is a closed, convex neighborhood of zero. Hence, since the Borel $\sigma$-algebra $B_E$ for the norm topology contains the class of the sets $K = \{k[-e,e], k \in \mathbb{R}_+\}$, $\sigma(K) = B_E$.

Example 15. Pick $e \in \ell_\infty$, where $\ell_\infty$ is endowed with the $\|\cdot\|_\infty$- topology, and $e_n \geq r \geq 0$. Then, $e$ is an order-unit of $\ell_\infty$.

Corollary 16. The map $\mu_F(k[-e,e]) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any AM-space.

Example 17. Pick $e = 1 \in L^\infty_+(\Omega, \mathcal{F}, \mu)$, where $L^\infty(\Omega, \mathcal{F}, \mu)$ over a probability space $(\Omega, \mathcal{F}, \mu)$ is an AM-space, endowed with the $\|\cdot\|_\infty$-norm, while $e = 1$ is an order-unit.
Corollary 18. The map $\mu_F(k[-e,e]) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on any Banach space $E$ ordered by a Bishop-Phelps cone defined as above, if $e$ is an interior point of it.

Proof: If we consider some $\epsilon > 0$, such that $a(1 + 2\epsilon) < 1$, where $a$ is the one which appears in the definition of the Bishop-Phelps cone $K(f,a)$, there exists some $x_0 \in E$ with $\|x_0\| = 1$ and $f(x) > a(1 + 2\epsilon)$. For all the elements of the closed ball $\|y\| \leq a\epsilon$, we have $\|x_0 + y\| \leq 1 + \epsilon$, and $f(x + y) \geq a(1 + \epsilon)$, hence $x_0 + y \in K(f,a)$, and we put $x_0 = e$.

Another definition of probability measures is possible, if $F$ is defined all over the set of real numbers and $E_+$ is a closed cone, which has an order -unit $u$. Notice that by $(su,tu)$, we denote the set $\{x \in E|su \leq x \leq tu, x \neq su\}$.

Theorem 19. The map $\mu_F((se,te)) = F(t) - F(s), t, s \in \mathbb{R}$, induces a Borel probability measure on $E$.

Proof: Since the Borel $\sigma$-algebra $\mathcal{B}_E$ for the norm topology contains the class of the sets $K = \{(-su,tu), s, t \in \mathbb{R}\}$, $\sigma(K) = \mathcal{B}_E$.

Corollary 20. The map $\mu_F((se,te)) = F(t) - F(s), t, s \in \mathbb{R}$, induces a Borel probability measure on any AM-space.

Corollary 21. The map $\mu_F((se,te)) = F(t) - F(s), t, s \in \mathbb{R}$, induces a Borel probability measure on any Banach space ordered by a Bishop-Phelps cone defined as above, if $e$ is an interior point of it.

Definition 22. A basis $x_n, n = 1, 2, \ldots$ in a Banach space is a Hilbertian basis if $\|\sum_{i=1}^{n}a_ix_i\| \leq c\sqrt{\sum_{i=1}^{n}|a_i|^2}$, where $c > 0$.

Lemma 23. If a Banach space $E$ has a Hilbertian basis, then a Bishop-Phelps cone may be defined on $E$.

Proof: If $c > 1$ and $f$ is the functional such that $f_n = f(x_n) = a_n$, while it is well known that in this case $\sum_{i=1}^{\infty}|a_i|^2 = |f(x)|^2$, if also $\|f\| = 1$, then $K(f,\frac{1}{c})$ is a Bishop-Phelps cone.

Example 24. According to the [9, Rem.11.1], every Banach space with a basis, has a Hilbertian basis. Indeed, if $x_n, n = 1, 2, \ldots$ is a normalized basis of $E$, then $\{\frac{1}{n}x_n\}$ is a Hilbertian basis of $E$.

Theorem 25. If a Hilbertian base $x_n, n = 1, 2, \ldots$ defines a Bishop-Phelps cone $K$ on a Banach space $E$, a Borel probability measure may be defined by a continuous linear functional $f \in E^*$ such that $f_n = a_n = f(x_n)$ and a cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$, as follows: If $B$ is the base that the functional $f$ defines on the cone $K$, then $\mu_F(k(B - B)) = F(k), k \in \mathbb{R}_+$, induces a Borel probability measure on $E$. 
Proof: The base is \( B = B_f = \{ x \in K(f, \frac{1}{c}) | f(x) = 1 \} \) and the map 
\( \mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+ \), induces a Borel probability measure on \( E \).

**Theorem 26.** If a Hilbertian base \( x_n, n = 1, 2, \ldots \) defines a Bishop-Phelps cone \( K(f, \frac{1}{c}) \) on a Banach space \( E \), a Borel probability measure may be defined by a continuous linear functional \( f \in E^* \) such that \( f_n = a_n = f(x_n) \) and a cumulative distribution function \( F : \mathbb{R}_+ \to [0,1] \), as follows: If \( e \) is an interior point of the cone \( K \), then \( \mu_F(k[-e,e]) = F(k), k \in \mathbb{R}_+ \) induces a Borel probability measure on \( E \).

Proof: If we consider some \( \epsilon > 0 \), such that \( a(1 + 2\epsilon) < 1 \), where \( a \) is the one which appears in the definition of the Bishop-Phelps cone \( K(f,a) \), where \( a = \frac{1}{c} \). there exists some \( x_0 \in E \) with \( \|x_0\| = 1 \) and \( f(x) > a(1 + 2\epsilon) \). For all the elements of the closed ball \( \|y\| \leq \epsilon \), we have \( \|x_0 + y\| \leq 1 + \epsilon \), and \( f(x + y) \geq a(1 + \epsilon) \), hence \( x_0 + y \in K(f,a) \), and we put \( x_0 = e \).

**Theorem 27.** If a Hilbertian base \( x_n, n = 1, 2, \ldots \) defines a Bishop-Phelps cone \( K(f, \frac{1}{c}) \) on a Banach space \( E \), a Borel probability measure may be defined by a continuous linear functional \( f \in E^* \) such that \( f_n = a_n = f(x_n) \) and a cumulative distribution function \( F : \mathbb{R}_+ \to [0,1] \), as follows: If \( e \) is an interior point of the cone \( K(f, \frac{1}{c}) \), then \( \mu_F((se,te]) = F(t) - F(s) \) induces a Borel probability measure on \( E \).

Proof: Since the Borel \( \sigma \)-algebra \( B_E \) for the norm topology contains the class of the sets \( K = \{ (-se, te], s, t \in \mathbb{R} \} \), \( \sigma(K) = B_E \). By using the previous Example we deduce the following

**Theorem 28.** If a basis \( x_n, n = 1, 2, \ldots \) exists in a Banach space \( E \), a Borel probability measure may be defined by a continuous linear functional \( f \in E^* \) such that \( f_n = a_n = f(x_n) \) and a cumulative distribution function \( F : \mathbb{R}_+ \to [0,1] \), as follows: If \( B \) is the base that the functional \( f \) defines on the corresponding Hilbertian basis Bishop-Phelps cone \( K \), then \( \mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+, \) induces a Borel probability measure on \( E \).

Proof: As we mentioned above, the corresponding base is \( B = B_f = \{ x \in K(f, \frac{1}{c}) | f(x) = 1 \} \) and the map \( \mu_F(k(B-B)) = F(k), k \in \mathbb{R}_+, \) induces a Borel probability measure on \( E \).

**Theorem 29.** If a basis \( x_n, n = 1, 2, \ldots \) exists in a Banach space \( E \), a Borel probability measure may be defined by a continuous linear functional \( f \in E^* \) such that \( f_n = a_n = f(x_n) \) and a cumulative distribution function \( F : \mathbb{R}_+ \to [0,1] \), as follows: If \( e \) is an interior point of the corresponding Hilbertian basis Bishop-Phelps cone \( K \), then \( \mu_F(k[-e,e]) = F(k), k \in \mathbb{R}_+ \) induces a Borel probability measure on \( E \).
Proof: Since the Borel $\sigma$-algebra $B_E$ for the norm topology contains the class of the sets $K = \{k[-e,e], k \in \mathbb{R}_+\}$, $\sigma(K) = B_E$.

**Theorem 30.** If a basis $x_n, n = 1, 2, \ldots$ exists in a Banach space $E$, a Borel probability measure may be defined by a continuous linear functional $f \in E^*$ such that $f_n = a_n = f(x_n)$ and a cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$, as follows: If $e$ is an interior point of the corresponding Hilbertian basis Bishop-Phelps cone $K$, then $\mu_F((se,te]) = F(t) - F(s)$ induces a Borel probability measure on $E$.

Proof: Since the Borel $\sigma$-algebra $B_E$ for the norm topology contains the class of the sets $K = \{(-su, tu], s, t \in \mathbb{R}\}$, $\sigma(K) = B_E$.

**Definition 31.** An element $u \in E_+$ is called quasi-interior point, if the solid subspace $\cup_{n=1}^{\infty}[-nu, nu]$ is norm-dense in $E$.

The following result extends the previous ones for cones with quasi-interior points.

**Theorem 32.** Let $E$ be an ordered normed linear space, whose positive cone $E_+$ has quasi-interior points, while $u$ is such a point. Then a probability measure may be defined on $E$, with respect to a quasi-interior point $u \in E_+$ and a cumulative distribution function $F : \mathbb{R} \to [0,1]$, by using the following equality:

$$\mu_F((su, tu]) = F(t) - F(s).$$

Proof: Since the Borel $\sigma$-algebra $B_E$ for the norm topology contains the class of the sets $K = \{(-su, tu], s, t \in \mathbb{R}\}$, $\sigma(K) = B_E$.

The following Proposition is well-known, as a consequence of the Kantorovich Theorem:

Let $E$ be a Banach lattice, while $u$ is a quasi-interior point of $E_+$. Then $f : E_+ \to \mathbb{R}_+$ such that

$$f(x) = \inf\{y \in \mathbb{R} | x \leq y \cdot u\}$$

is additive, hence it is extended to a functional $f \in E^*$.

As a consequence of the above, we take the following:

**Theorem 33.** Let $E$ be a Banach lattice, being a subspace of $L^0(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space. Then, a probability measure on $E$, may be defined as follows, if $E_+$ has quasi-interior points:

$$\mu_F((su, tu]) = F(f(t)) - F(f(s)).$$
Proof: Since the Borel $\sigma$-algebra $\mathcal{B}_E$ for the norm topology contains the class of the sets $K = \{(-su,tu), s, t \in \mathbb{R}\}$, $\sigma(K) = \mathcal{B}_E$.

Below, we mention some examples of quasi-interior points of the corresponding cones.

Example 34. (i) Any $r \in L^1_+[0,1]$, such that $r(t) > 0$, $\lambda$-a.e. is a norm quasi-interior point, but not an interior point of $L^1_+[0,1]$.

(ii) Any $r \in \ell^p, p \geq 1$, such that $r_n > 0, n \in \mathbb{N}$ is a norm quasi-interior point, but not an interior point of $\ell^p$.

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