Note on the Long Wave Stability of Shear Flows with Variable Bottom Topography

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Abstract

We consider extended Taylor-Goldstein problem of hydrodynamic stability dealing with incompressible, inviscid, stratified shear flows of arbitrary cross section. For this problem we obtained an instability region for an unstable mode which depends on breadth function, basic velocity profile, vorticity variation, wave number, Richardson number and stratification parameter. Furthermore, long wave stability result, namely, if $k \leq k_c > 0$ (for some critical wave number $k_c$) implies stability of the mode.

Mathematics Subject Classification: 76E05

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1 Introduction

The stability of shear flow of an incompressible, inviscid but density stratified fluids to infinitesimal normal mode disturbance in sea straits of arbitrary cross section was initiated in [3]. Mathematical approach of the above problem was found in [1]. For this problem [1] proved that a sufficient condition for stability is that minimum Richardson number $J_0$ greater than or equal to one-quarter. Also [1] proved a semicircle instability region in the upper half of $c_r - c_i$ plane. A semi-elliptical instability region lying inside the semicircle of [1] has been proved in [5]. Howard’s conjecture namely $kc_i$ tends to zero as the wave number $k$ tends to infinity for class of basic flows has been derived in [2]. Long wave instability of some examples of shear flows with piecewise linear velocity profiles has been shown [4].

In our present work, we derived an instability region for the extended Taylor-Goldstein problem of hydrodynamic stability. Our new instability region depends on minimum and maximum of basic velocity profile $U_0(z)$, breadth function $b(z)$, wave number $k$, minimum and maximum of stratification parameter $N^2(z)$, Richardson number $J_0$ and vorticity function $\left(\frac{U'}{b}\right)'$. Furthermore, a long wave stability result, namely, if $k \leq k_c > 0$, where $k_c$ is some critical value of wave number $k$ then the flow is stable is proved for flows with smooth basic flow variables. The long wave stability is proved only for stratified shear flows. The result has been illustrated with examples.

2 Extended Taylor-Goldstein Problem

The extended Taylor-Goldstein problem (cf. [1]) is given by the second order ordinary differential equation

$$\left[\frac{(bW)''}{b}\right] + \left[\frac{N^2}{(U_0 - c)^2} - \frac{b \left[\frac{U''}{b}\right]'}{U_0 - c} - k^2\right] W = 0,$$

with boundary conditions

$$W(0) = 0 = W(D).$$

(1)

(2)

Here, the real part of $W(z)e^{ik(z-ct)}$ is the vertical velocity of a normal mode disturbance, $c = c_r + ic_i$ is the complex phase velocity, $b(z)$ is the breadth function, $k > 0$ is the wave number, $U_0(z)$ is the basic velocity profile.
3 Instability Region:

**Theorem 3.1** For the existence of unstable mode, the following integral relations are true

\[
\int \left[ \frac{|(bW')|^2}{b} + k^2 b |W|^2 \right] dz \leq \int \frac{N^2 [(U_0 - c_r)^2 - c^2_i]}{|U_0 - c|^4} b |W|^2 dz \\
+ \int \frac{b \left( \frac{U_0'}{b} \right)'}{|U_0 - c|^2} b |W|^2 dz.
\]

**Proof:**

Multiplying (1) by \((bW^*)\), integrating over \([0, D]\) and applying (2), we get

\[
\int \left[ \frac{|(bW')|^2}{b} + k^2 b |W|^2 \right] dz + \int \frac{b \left( \frac{U_0'}{b} \right)'}{U_0 - c} b |W|^2 dz \\
- \int \frac{N^2}{(U_0 - c)^2} b |W|^2 dz = 0.
\]

Equating real parts, we get

\[
\int \left[ \frac{|(bW')|^2}{b} + k^2 b |W|^2 \right] dz + \int \frac{b \left( \frac{U_0'}{b} \right)'}{U_0 - c} b |W|^2 dz \\
- \int \frac{N^2}{|U_0 - c|^4} b |W|^2 dz = 0;
\]

\[
\int \left[ \frac{|(bW')|^2}{b} + k^2 b |W|^2 \right] dz \\
\leq \int \frac{N^2 [(U_0 - c_r)^2 - c^2_i]}{|U_0 - c|^4} b |W|^2 dz - \int \frac{b \left( \frac{U_0'}{b} \right) (U_0 - c_r)}{|U_0 - c|^2} b |W|^2 dz;
\]

\[
\int \left[ \frac{|(bW')|^2}{b} + k^2 b |W|^2 \right] dz \\
\leq \int \frac{N^2 [(U_0 - c_r)^2 - c^2_i]}{|U_0 - c|^4} b |W|^2 dz + \int \frac{b \left( \frac{U_0'}{b} \right) (U_0 - c_r)}{|U_0 - c|^2} b |W|^2 dz.
\]  

(3)

**Theorem 3.2** The instability region is given by
\[ [c_r - \left( \frac{U_{\text{max}} + U_{\text{min}}}{2} \right)]^2 + c_i^2 + J_0 \left( 1 + \frac{A_1}{c_i^2} \right)^2 c_i^2 \leq \left[ \frac{U_{\text{max}} - U_{\text{min}}}{2} \right]^2, \]

where

\[ A_1 = \frac{(U_{\text{max}} - U_{\text{min}})^2 \left[ 2 N_{\text{max}}^2 - N_{\text{min}}^2 + b \left( \frac{U_0'}{b} \right) \right] \} \max \left[ \frac{U_{\text{max}} - U_{\text{min}}}{4} \right] }{ \frac{\pi^2 b_{\text{min}}}{D_{\text{bmax}}^2} + k}. \]

Proof: Using the transformation \( W = (U_0 - c)F \), we get

\[ (bW)' = (U_0 - c)(bF)' + U_0'(bF). \]  

This implies that

\[ \frac{||bW'||^2}{b} \geq |U_0 - c|^2 \frac{||(bF)'||^2}{b} - 2 |U_0 - c| ||(bF)'|| |U_0'|| |F| + |U_0'|^2 b |F|^2. \]  

The use of Cauchy-Schwartz inequality gives

\[ \int |U_0 - c| |U_0'||((bF)')| |F| dz \leq \left[ \int |U_0'|^2 b |F|^2 dz \right]^{\frac{1}{2}} \left[ \int |U_0 - c|^2 \frac{||(bF)'||^2}{b} dz \right]^{\frac{1}{2}}. \]

\[ \int |U_0 - c| |U_0'||((bF)')| |F| dz \leq BE, \]

where

\[ B^2 = \int |U_0'|^2 b |F|^2 dz; \]  

\[ E^2 = \int |U_0 - c|^2 Q dz; \]

and

\[ Q = \frac{||(bF)'||^2}{b} + k^2 b |F|^2. \]  

Using (7), (8), (9) in (5), we get

\[ \int \left[ \frac{||(bW)'||^2}{b} + k^2 b |W|^2 \right] dz \geq E^2 - 2BE + B^2; \]

i.e.,

\[ \int \left[ \frac{||(bW)'||^2}{b} + k^2 b |W|^2 \right] dz \geq [B - E]^2. \]

We know that

\[ (U_0 - c_r)^2 - c_i^2 \leq 2(U_0 - c_r)^2 - |U_0 - c|^2, \]
Long wave stability of shear flows

substituting (11) in (3), we get

$$\int \left[ \frac{|(bW)'|^2}{b} + k^2 b|W|^2 \right] dz \leq 2 \int \frac{N^2(U_0 - c_r)^2}{|U_0 - c|^4} b|W|^2 dz$$

$$- \int \frac{N^2}{|(U_0 - c)|^2} b|W|^2 dz + \int \left( \frac{U_0'}{b} \right)' (U_0 - c_r) \left[ \frac{U_0 - c}{(U_0 - c_r)} \right] b|W|^2 dz = 0.$$  

Also, we know that 

$$b\left| U_0 - c_r \right| \leq \frac{U_{0\text{max}} - U_{0\text{min}}}{|U_0 - c|^2}$$

and 

$$\frac{1}{|U_0 - c|^2} \leq \frac{1}{c_i^2}.$$ 

Substituting this in the above equation, we get

$$\int \left[ \frac{|(bW)'|^2}{b} + k^2 b|W|^2 \right] dz \leq 2 \int \frac{N^2_{\text{max}}(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^4} \frac{b|W|^2}{|(U_0 - c)|^2} dz$$

$$- \int \frac{N^2_{\text{min}}}{|(U_0 - c)|^2} b|W|^2 dz + \left| \frac{U_0'}{b} \right|_{\text{max}} [U_{0\text{max}} - U_{0\text{min}}] \int \frac{b|W|^2}{|(U_0 - c)|^2} dz.$$ 

Using (4) in the above equation, we have

$$\int \left[ \frac{|(bW)'|^2}{b} + k^2 b|W|^2 \right] dz \leq$$

$$\left[ \frac{2N^2_{\text{max}}(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^4} - N^2_{\text{min}} \right] b \left( \frac{U_0'}{b} \right)_{\text{max}} [U_{0\text{max}} - U_{0\text{min}}] \int b|F|^2 dz.$$  

(12)

Using (12) in (10), we get

$$(B - E)^2$$

$$\leq \left[ \frac{2N^2_{\text{max}}(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^4} - N^2_{\text{min}} \right] b \left( \frac{U_0'}{b} \right)_{\text{max}} [U_{0\text{max}} - U_{0\text{min}}] \int b|F|^2 dz.$$  

(13)

$$E^2 = \int |U_0 - c|^2 Q dz.$$ 

We know that 

$$|U_0 - c|^2 \geq c_i^2.$$ 

Therefore, 

$$E^2 \geq c_i^2 \int \left[ \frac{|(bW)'|^2}{b} + k^2 b|F|^2 \right] dz.$$  

(14)

Using Rayleigh-Ritz inequality, we get

$$E^2 \geq c_i^2 \left[ \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right] \int b|F|^2 dz;$$

i.e.,

$$\int b|F|^2 dz \leq \frac{E^2}{c_i^2 \left[ \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right]}.$$  

(15)
Substituting (15) in (13), we get

\[ (B - E)^2 \leq \left[ \frac{2N_2(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^2} - N_{\text{min}}^2 \right] \left[ U_{0\text{max}} - U_{0\text{min}} \right] \frac{E^2}{c_i^2 \left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \]

\[ E^2 \left( \frac{B}{E} - 1 \right)^2 \leq \left[ \frac{2N_{\text{max}}^2(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^2} - N_{\text{min}}^2 \right] \left[ U_{0\text{max}} - U_{0\text{min}} \right] \frac{E^2}{c_i^2 \left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \]

Also we know that \( c_i \leq \frac{|U_{0\text{max}} - U_{0\text{min}}|}{2}, \)

\[ (B - E)^2 \leq \left[ \frac{2N_{\text{max}}^2(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^2} - N_{\text{min}}^2 \right] \left[ U_{0\text{max}} - U_{0\text{min}} \right] \frac{E^2}{c_i^2 \left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \]

\[ (B - E)^2 \leq A_1^2 \frac{2N_{\text{max}}^2(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^2} \left[ U_{0\text{max}} - U_{0\text{min}} \right] \frac{E^2}{c_i^2 \left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \]

where \( A_1^2 = \frac{(U_{0\text{max}} - U_{0\text{min}})^2}{\left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \)

\[ (\frac{B}{E} - 1) \leq A_1 \frac{2N_{\text{max}}^2(U_{0\text{max}} - U_{0\text{min}})^2}{c_i^2} \left[ U_{0\text{max}} - U_{0\text{min}} \right] \frac{E^2}{c_i^2 \left( \frac{\pi^2 b_{\text{min}}}{D^2 b_{\text{max}}} + k^2 \right)}; \]

\[ \frac{B}{E} \leq \left( 1 + \frac{A_1}{c_i^2} \right); \]

\[ B^2 \leq \left( 1 + \frac{A_1}{c_i^2} \right) E^2. \] (16)

\[ \int N^2 b|F|^2 dz \geq \left[ \frac{N_{\text{min}}^2}{(U_{0\text{min}})^2} \right] \int |U_0'|^2 b|F|^2 dz; \]

\[ i.e., \]

\[ \int N^2 b|F|^2 dz \geq J_0 B^2. \] (17)

Substituting (16) in (17), we get

\[ \int N^2 b|F|^2 dz \geq J_0 E^2 \left( 1 + \frac{A_1}{c_i^2} \right)^2. \] (18)

Substituting (18) in (14), we have

\[ \int N^2 b|F|^2 dz \geq J_0 \left( 1 + \frac{A_1}{c_i^2} \right)^2 c_i^2 \int Q dz. \] (19)
From [1], we have
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 \int Q \, dz + \int N^2 b |F|^2 \, dz \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2.
\] (20)

Substituting (19) in (20), we get
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 \int Q \, dz + J_0 \left( 1 + \frac{A_1}{c_i^2} \right)^2 c_i^2 \int Q \, dz
\leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2,
\]
i.e.,
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 + J_0 \left( 1 + \frac{A_1}{c_i^2} \right)^2 c_i^2 \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2. \quad (21)
\]

**Theorem 3.3** The instability region is given by
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 + J_0 \left( 1 + \frac{A_2}{c_i^2} \right)^2 c_i^2 \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2,
\]
where
\[
A_2^2 = \frac{(U_{0\text{max}} - U_{0\text{min}})^2}{k^2} \left[ 2N_{\text{max}}^2 - \frac{N_{\text{min}}^2}{4} + b \left( \frac{U_0}{v_s} \right) \right] \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{4} \right].
\]

**Proof:**
In (14), if the first term of right hand side is dropped being positive, we get
\[
E^2 \geq c_i^2 \int k^2 b |F|^2 \, dz.
\]

Proceeding in the same way as Theorem 3.2, we get the result.
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 + J_0 \left( 1 + \frac{A_2}{c_i^2} \right)^2 c_i^2 \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2. \quad (22)
\]

### 4 Stability Results:

**Theorem 4.1** If \( k \leq k_c > 0 \), where \( k_c \) is some critical value of wave number \( k \) then the flow is stable.

**Proof:**
Equation (22) can be written as
\[
\left( c_r - \left( \frac{U_{0\text{max}} + U_{0\text{min}}}{2} \right) \right)^2 + c_i^2 + J_0 \left( 1 + \frac{2A_2}{c_i^2} + \frac{A_2^2}{c_i^4} \right) c_i^2 \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2.
\]
i.e.,
\[ 
[c_r - \frac{(U_{0\text{max}} + U_{0\text{min}})}{2}]^2 + c_i^2(1 + J_0) + 2A_2 J_0 + \frac{J_0 A_2^2}{c_i^4} \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2;
\]

i.e.,
\[ 
[c_r - \frac{(U_{0\text{max}} + U_{0\text{min}})}{2}]^2 + c_i^2(1 + J_0) + \frac{J_0 A_2^2}{c_i^4} \leq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2 - 2A_2 J_0.
\]

From the above equation \( 2A_2 J_0 \geq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2 \) implies stability

i.e.,
\[ 
\frac{2J_0 (U_{0\text{max}} - U_{0\text{min}}) \left[ 2N_{\text{max}}^2 - \frac{N_{\text{min}}^2}{4} + \left| b \left( \frac{U'_0}{b} \right) \right|_{\text{max}} \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{4} \right] \right]^\frac{1}{2}}{k} \geq \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2
\]

implies stability

i.e.,
\[ 
2J_0 (U_{0\text{max}} - U_{0\text{min}}) \left[ 2N_{\text{max}}^2 - \frac{N_{\text{min}}^2}{4} + \left| b \left( \frac{U'_0}{b} \right) \right|_{\text{max}} \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{4} \right] \right]^\frac{1}{2} \leq k \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2
\]

implies stability

i.e., \( k \leq k_c > 0 \),

where \( k_c = \frac{2J_0 (U_{0\text{max}} - U_{0\text{min}}) \left[ 2N_{\text{max}}^2 - \frac{N_{\text{min}}^2}{4} + \left| b \left( \frac{U'_0}{b} \right) \right|_{\text{max}} \left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{4} \right] \right]^\frac{1}{2}}{\left[ \frac{U_{0\text{max}} - U_{0\text{min}}}{2} \right]^2} \)

then the flow is stable. For homogeneous shear flows \( J_0 = 0 \). \( k \leq 0 \) implies stability, which is impossible. Therefore for \( J_0 > 0 \), \( k \leq k_c > 0 \), implies the flow is stable. This long wave stability is proved only for stratified shear flows.

5 Examples:

Below examples will illustrate the stability results.

Example:1

Let \( U_0 = 1 - z^2 \) in \((0,1)\), \( N^2 = N_0^2 \), a constant, \( b = z \).

In this case, \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = N_{\text{min}}^2 = 1, J_0 = \frac{1}{4} \),

\[ \left| b \left( \frac{U'_0}{b} \right) \right|_{\text{max}} = 0. \]

After numerical computation, we get \( k_c = 2.64 \).

If \( k \leq 2.64 \) then the flow is stable.
Example: 2
Let \( U_0 = 1 - z^2 \) in \((0,1)\), \( N^2 = N_0^2 \), a constant, \( b = b_0 \), a constant.
In this case \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = N_{\text{min}}^2 = 1, J_0 = \frac{1}{4} \),
\[
\left| \frac{b \left( \frac{U_0'}{b} \right)'}{b} \right|_{\text{max}} = 2.
\]
After numerical computation, we get \( k_c = 3 \)
If \( k \leq 3 \) then the flow is stable.

Example: 3
Let \( U_0 = 1 - z^2 \) in \((0,1)\), \( N^2 = z \), \( b = z \), a constant.
In this case \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = 1, N_{\text{min}}^2 = 0, J_0 = \frac{1}{4} \),
\[
\left| \frac{b \left( \frac{U_0'}{b} \right)'}{b} \right|_{\text{max}} = 0.
\]
After numerical computation, we get \( k_c = 2.828 \)
If \( k \leq 2.828 \) then the flow is stable.

Example: 4
Let \( U_0 = 1 - z^2 \) in \((0,1)\), \( N^2 = z \), \( b = b_0 \), a constant.
In this case \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = 1, N_{\text{min}}^2 = 0, J_0 = \frac{1}{4} \),
\[
\left| \frac{b \left( \frac{U_0'}{b} \right)'}{b} \right|_{\text{max}} = 2.
\]
After numerical computation, we get \( k_c = 3.16 \)
If \( k \leq 3.16 \) then the flow is stable.

Example: 5
Let \( U_0 = z \) in \((0,1)\), \( N^2 = z \), \( b = b_0 \), a constant.
In this case \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = 1, N_{\text{min}}^2 = 0, J_0 = 1 \),
\[
\left| \frac{b \left( \frac{U_0'}{b} \right)'}{b} \right|_{\text{max}} = 0.
\]
After numerical computation, we get \( k_c = 11.31 \)
If \( k \leq 11.31 \) then the flow is stable.

Example: 6
Let \( U_0 = z \) in \((0,1)\), \( N^2 = N_0^2 \), a constant, \( b = b_0 \), a constant.
In this case \( U_{0\text{max}} = 1, U_{0\text{min}} = 0, N_{\text{max}}^2 = 1, N_{\text{min}}^2 = 1, J_0 = 1 \),
\[
\left| \frac{b \left( \frac{U_0'}{b} \right)'}{b} \right|_{\text{max}} = 0.
\]
After numerical computation, we get \( k_c = 10.58 \)
If \( k \leq 10.58 \) then the flow is stable.
6 Concluding Remarks

For the extended Taylor-Goldstein problem of hydrodynamic stability dealing with incompressible, inviscid, density stratified shear flows of arbitrary cross section, we derived an instability region depends on minimum and maximum of basic velocity profile $U_0(z)$, breadth function $b(z)$, wave number $k$, minimum and maximum of stratification parameter $N^2(z)$, Richardson number $J_0$ and vorticity function $\left(\frac{U_0}{b}\right)'$. Furthermore, a long wave stability result, namely, if $k \leq k_c > 0$, where $k_c$ is some critical value of wave number $k$ then the flow is stable. The long wave stability is proved only for stratified shear flows. The result has been illustrated with examples. Our long wave stability results is true for basic flows with smooth velocity and density profiles. For basic flows with non-smooth velocity and density profiles long wave instability may occur and this has been demonstrated for examples in [4].

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