Viscosity Approximations for Nonexpansive Mappings with Meir-Keeler Contractions in CAT(0) Spaces

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Abstract

In this paper, we consider the Moudafi’s viscosity approximations with generalized contractions for nonexpansive mappings in a complete CAT(0) space. The results presented in this paper mainly extended the corresponding results in literature.

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1 Introduction

It is well-known that the construction of fixed point of nonexpansive mappings is an important subject in nonlinear mappings theory and its application. The existence and approximation of the fixed point of nonexpansive mappings in different situations have been studied by many papers, for example [24, 14, 3, 11, 30, 31].

In 2000, Moudafi [20] introduced the viscosity approximation method for a nonexpansive mapping $T$ in Hilbert spaces, which formally generates the sequence $\{x_n\}$ by the recursive formula:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,$$  \hspace{1cm} (1)
where \( f \) is a contraction and \( \alpha_n \subset (0, 1) \) is a real sequence.

There are very important because they are applied to convex optimization, linear programing and etc. His results have been extended in several direction see [32, 27] and reference therein. Subsequently, Suzuki [26] extended Moudafi’s results [20] and proved the strong convergence of the Moudafi’s viscosity approximations with Meir-Keeler contractions in a uniformly smooth Banach space.

In 2012, Shi and Chen [25], studied the convergence theorems of the viscosity iterations for a nonexpansive mapping \( T \):

\[
x_t = tf(x_t) \oplus (1 - t)Tx_t,
\]

and \( x_0 \in C \) is arbitrary chosen and

\[
x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)Tx_n, \quad n \geq 0,
\]

where \( \{\alpha_n\} \subset (0, 1) \). They proved \( \{x_t\} \) and \( \{x_n\} \) defined by (2) and (3) converge strongly as \( t \to 0 \) to \( \bar{x} \in Fix(T) \) such that \( \bar{x} = P_{Fix(T)}f(\bar{x}) \) in the framework of \( \text{CAT}(0) \) space satisfying property \( \mathcal{P} \), i.e., if for \( x, u, y_1, y_2 \in X \),

\[
d(x, P_{[x,y_1]}u)d(x, y_1) \leq d(x, P_{[x,y_2]}u)d(x, y_2) + d(x, u)d(y_1, y_2).
\]

By using the concept of quasilinearization, Wangkeeree and Preechasilp [28] obtained the same strong convergence results in [25] without the assumed property \( \mathcal{P} \). For more related result (see [29, 18]).

All of the above bring us the following conjectures.

**Question** Could we obtain the strong convergence for the viscosity approximation method for generalized contraction in the framework of \( \text{CAT}(0) \) space?

The purpose of this paper is to study the strong convergence theorems of the iterative schemes (2) and (3) with the Meir-Keeler contraction mapping \( f \) in a complete \( \text{CAT}(0) \) space. We prove the iterative schemes (2) and (3) converges strongly to \( \bar{x} \) such that \( \bar{x} = P_{Fix(T)}f(\bar{x}) \) which is the unique solution of the variational inequality

\[
\langle \bar{x}f(\bar{x}), \bar{x}x \rangle \geq 0, \quad x \in Fix(T).
\]

### 2 Preliminaries

Let \( X \) be a metric space \( C \) a nonempty subset of \( X \). Let \( T : C \to C \) be a self-mapping of \( C \). \( T \) is said to be nonexpansive if \( d(Tx, Ty) \leq d(x, y) \) for all \( x, y \in C \). The fixed point set of \( T \) is denoted by \( \text{Fix}(T) := \{ x \in C : Tx = x \} \).
Viscosity approximation method with Mier-Keeler Contraction

A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an $L$-function if $\psi(0) = 0, \psi(t) > 0$, for each $t > 0$ and for every $t > 0$ and for every $s > 0$ there exists $u > s$ such that $\psi(t) \leq s$, for all $t \in [s, u]$. As a consequence, every $L$-function $\psi$ satisfies $\psi(t) < t$, for each $t > 0$.

**Definition 2.1.** Let $(X, d)$ be a metric space. A mapping $f : X \to X$ is said to be:

(i) A Meir-Keeler type mapping [19] if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in X$, with $\varepsilon \leq d(x, y) < \varepsilon + \delta$ we have $d(f(x), f(y)) < \varepsilon$.

(ii) A $(\psi, L)$-contraction [22] if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an $L$-function and $d(f(x), f(y)) < \psi(d(x, y))$, for all $x, y \in X$, with $x \neq y$.

**Remark 2.2.** If we take $\varepsilon = d(x, y)$, where $x \neq y$, in Definition 2.1(i), then we have that a Meir-Keeler type mapping is a nonexpansive mapping.

**Theorem 2.3.** [22] Let $(X, d)$ be a complete metric space and $f : X \to X$ be a $(\psi, L)$-contraction. Then $\text{Fix}(f) = \{x^*\}$.

**Theorem 2.4.** [19] Let $(X, d)$ be a complete metric space and $f : X \to X$ be a Meir–Keeler type mapping. Then $\text{Fix}(f) = \{x^*\}$.

**Theorem 2.5.** [17] Let $(X, d)$ be a metric space and $f : X \to X$ be a mapping. The following assertions are equivalent:

(i) $f$ is a Meir-Keeler type mapping:

(ii) there exists an $L$-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f$ is a $(\psi, L)$-contraction.

A metric space $(X, d)$ is called CAT(0) space if for each pair of points $x, y \in X$ there exists a point $w \in X$ such that for all $z \in X$

$$d^2(z, w) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y).$$

A complete CAT(0) space is sometimes called global nonpositive curvature space or Hadamard space. In the rest of paper, we denote $X$ by complete CAT(0) space. It is well-known that, in complete CAT(0) spaces, for each $x, y \in X$ there exists the unique point $z$ in the geodesic segment joining from $x$ to $y$ with

$$d(z, x) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining from $x$ to $y$, that is, $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$. A subset $C$ of $X$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

The following two lemmas use to prove the analogue of [26, Proposition 2] and [21, Proposition 3.4], respectively, in complete CAT(0) spaces.
Lemma 2.6. Let $X$ be a CAT(0) space and $C$ a convex subset of it. Let $f : C \to C$ be a Meir-Keeler type mapping. Then for each $\varepsilon > 0$ there exists $r \in (0,1)$ such that for each $x, y \in C$ with $d(x, y) \geq \varepsilon$ we have $d(f(x), f(y)) \leq rd(x, y)$.

Proof. We note that $f$ is nonexpansive. Fix $\varepsilon > 0$. Then there exists $\delta \in (0, \varepsilon)$ such that

$$d(u, v) < \varepsilon/4 + \delta \implies d(f(u), f(v)) < \varepsilon/4,$$

for $u, v \in C$. We put $r := 4\varepsilon - \delta > 0$. Fix $x, y \in C$ with $d(x, y) \geq \varepsilon$ and put $a = d(x, y)$. Then we have

$$d(f(x), f(y)) \leq \sum_{j=0}^{[a/\varepsilon]} d\left(f\left((1 - \frac{\varepsilon + \delta}{4a})(j+1)x \oplus \frac{\varepsilon + \delta}{4a}(j+1)y\right)ight)$$

$$+ d\left(f\left((1 - \frac{\varepsilon + \delta}{4a}\left[\frac{a}{\varepsilon}\right] + 1\right)x \oplus \frac{\varepsilon + \delta}{4a}\left[\frac{a}{\varepsilon}\right] + 1\right)y\right)$$

$$\leq \left(\frac{a}{\varepsilon} + 1\right)\varepsilon/4 + \left(1 - \frac{\varepsilon + \delta}{4a}\left[\frac{a}{\varepsilon}\right] + 1\right)a$$

$$= a - \delta \left(\frac{a}{\varepsilon} + 1\right) \leq rd(x, y).$$

This completes the proof.

Lemma 2.7. Let $X$ be a complete CAT(0) space and $C$ be a closed convex subset of it. Let $T : C \to C$ be a nonexpansive mapping and $f$ is a $(\psi, L)$-contraction. Then the following assertions hold:

(i) $T \circ f$ is a $(\psi, L)$-contraction on $C$ and has a unique fixed point in $C$;

(ii) for each $\alpha \in (0, 1)$ the mapping $x \mapsto \alpha f(x) \oplus (1-\alpha)T(x)$ is a Meir-Keeler type mapping on $C$ and it has a unique fixed point in $C$.

Proof. 1. $d((T \circ f)(x), (T \circ f)(y)) \leq d(f(x), f(y)) < \psi(d(x, y))$. for each $x, y \in C$ with $x \neq y$. The existence and uniqueness of the fixed point follow from Theorem 2.3.

2. Fix $\alpha \in (0, 1)$. For each $\varepsilon > 0$, by Lemma 2.6 there exist $r \in (0,1)$ such that for each $x, y \in C$ with $d(x, y) \geq \varepsilon$ we have $d(f(x), f(y)) \leq rd(x, y)$. Put

$$\delta := \frac{\alpha \varepsilon (1-r)}{1-\alpha + \alpha r} > 0.$$
Viscosity approximation method with Mier-Keeler Contraction

Fix \( x, y \in C \) with \( d(x, y) < \varepsilon + \delta \). In the case of \( d(x, y) \geq \varepsilon \), we have

\[
d((1 - \alpha)Tx \oplus \alpha f(x), (1 - \alpha)Ty \oplus \alpha f(y)) \\
\leq (1 - \alpha)d(Tx, Ty) + \alpha d(f(x), f(y)) \\
\leq (1 - \alpha + \alpha r)d(x, y) \\
< (1 - \alpha + \alpha r)(\varepsilon + \delta) = \varepsilon.
\]

In the other case of \( d(x, y) < \varepsilon \), we have

\[
d((1 - \alpha)Tx \oplus \alpha f(x), (1 - \alpha)Ty \oplus \alpha f(y)) \leq d(x, y) < \varepsilon.
\]

From Theorem 2.4, the proof is completes. \( \Box \)

From now on by a generalized contraction mapping we mean a Meir-Keeler type mapping or a \((\psi, L)\)-contraction. We suppose that the \( L \)-function from the characterization theorem (Theorem 2.5), as well as, the function \( \psi \) from the definition of the \((\psi, L)\)-contraction is continuous, strictly increasing and \( \lim_{t \to +\infty} \psi(t) = +\infty \), where \( \eta(t) := t - \psi(t), t \in \mathbb{R}_+ \). As a consequence, we have that \( \eta \) is a bijection on \( \mathbb{R}_+ \), \( \eta^{-1} \) is also.

The following lemmas play an important role in our paper.

**Lemma 2.8.** [2, Proposition 2.2] Let \( X \) be a \( \text{CAT}(0) \) space, \( p, q, r, s \in X \) and \( \lambda \in [0, 1] \). Then

\[
d(\lambda p \oplus (1 - \lambda)q, \lambda r \oplus (1 - \lambda)s) \leq \lambda d(p, r) + (1 - \lambda)d(q, s).
\]

**Lemma 2.9.** [9, Lemma 2.4] Let \( X \) be a \( \text{CAT}(0) \) space, \( x, y, z \in X \) and \( \lambda \in [0, 1] \). Then

\[
d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z).
\]

**Lemma 2.10.** [9, Lemma 2.5] Let \( X \) be a \( \text{CAT}(0) \) space, \( x, y, z \in X \) and \( \lambda \in [0, 1] \). Then

\[
d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y).
\]

The concept of \( \Delta \)-convergence introduced by Lim [16] in 1976 was shown by Kirk and Panyanak [15] in \( \text{CAT}(0) \) spaces to be very similar to the weak convergence in Banach space setting. Next, we give the concept of \( \Delta \)-convergence and collect some basic properties.

Let \( \{x_n\} \) be a bounded sequence in a \( \text{CAT}(0) \) space \( X \). For \( x \in X \), we set

\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in X \},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{ x \in X : r(x, \{x_n\}) = r(\{x_n\}) \}.$$

It is known from [8, Proposition 7] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

**Definition 2.11.** A sequence $\{x_n\} \subset X$ is said to $\Delta$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

**Lemma 2.12.** [15] Every bounded sequence in a complete CAT(0) space always has a $\Delta$-convergent subsequence.

**Lemma 2.13.** [7] If $C$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.

**Lemma 2.14.** [7] If $C$ is a closed convex subset of $X$ and $T : C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ $\Delta$-convergence to $x$ and $d(x_n, Tx_n) \to 0$, and imply $x \in C$ and $Tx = x$.

Berg and Nikolaev [1] introduced the concept of quasilinearization as follows:

Let us formally denote a pair $(a, b) \in X \times X$ by $\vec{ab}$ and call it a vector. Then **quasilinearization** is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \vec{ab}, \vec{cd} \rangle = \frac{1}{2} \left( d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right), \quad (a, b, c, d \in X). \quad (5)$$

It is easily seen that $\langle \vec{ab}, \vec{cd} \rangle = \langle \vec{cd}, \vec{ab} \rangle$, $\langle \vec{ab}, \vec{cd} \rangle = -\langle \vec{ba}, \vec{cd} \rangle$ and $\langle \vec{ax}, \vec{cd} \rangle + \langle \vec{xb}, \vec{cd} \rangle = \langle \vec{ab}, \vec{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that $X$ satisfies the Cauchy-Schwarz inequality if

$$\langle \vec{ab}, \vec{cd} \rangle \leq d(a, b)d(c, d) \quad (6)$$

for all $a, b, c, d \in X$. It known [1, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Having the notion of quasilinearization, Kakavandi and Amini [12] introduced the following notion of convergence.
Definition 2.15. A sequence $\{x_n\}$ in the complete CAT(0) space $(X,d)$ $w$-converges to $x \in X$ if $\lim_{n \to \infty} \langle xx_n, xy \rangle = 0$, i.e. $\lim_{n \to \infty} (d^2(x_n,x) - d^2(x_n,y) + d^2(x,y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies $w$-convergence, and it is easy to check that $w$-convergence implies $\Delta$-convergence [12, Proposition 2.5], but it is showed in ([13, Example 4.7]) that the converse is not valid. However the following lemma shows another characterization of $\Delta$-convergence as well as, more explicitly, a relation between $w$-convergence and $\Delta$-convergence.

Lemma 2.16. [13, Theorem 2.6] Let $X$ be a complete CAT(0) space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then $\{x_n\}$ $\Delta$-converges to $x$ if and only if $\limsup_{n \to \infty} \langle xx_n, xy \rangle \leq 0$ for all $y \in X$.

By using the concept of quasilinearization, Dehghan and Rooin [6] presented a characterization of metric projection in CAT(0) spaces as follows:

Theorem 2.17. [6, Theorem 2.4] Let $C$ be a nonempty convex subset of a complete CAT(0) space $X$, $x \in X$ and $u \in C$. Then

$$u = P_Cx \quad \text{if and only if} \quad \langle yu, xu \rangle \geq 0, \quad \text{for all} \quad y \in C.$$ 

The following two lemmas can be found in [28, 29].

Lemma 2.18. Let $X$ be a CAT(0) space. Then for all $x, y, z \in X$, the following inequality holds

$$d^2(x,u) \leq d^2(y,u) + 2 \langle xu, uy \rangle.$$

Lemma 2.19. Let $X$ be a CAT(0) space. For any $t \in [0,1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,

(i) $\langle u_t \bar{x}, u_t \bar{y} \rangle \leq t \langle \bar{x}, \bar{y} \rangle + (1-t) \langle \bar{v}, \bar{w} \rangle$;

(ii) $\langle u_t \bar{x}, u_t \bar{y} \rangle \leq t \langle \bar{x}, \bar{y} \rangle + (1-t) \langle v, w \rangle$ and $\langle u_t \bar{x}, u_t \bar{y} \rangle \leq t \langle \bar{x}, \bar{y} \rangle + (1-t) \langle \bar{v}, \bar{w} \rangle$.

Lemma 2.20. [33, Lemma 2.1] Let $\{a_n\}$ be a sequence of non-negative real number satisfying the property

$$a_{n+1} \leq (1 - \alpha_n)a_n + a_n \beta_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \beta_n \leq 0$ or $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\{a_n\}$ converges to zero, as $n \to \infty$. 
3 Main Results

Now, we are a position to state and prove our main results.

**Theorem 3.1.** Let $X$ be a complete CAT(0) space and let $C$ be a nonempty closed convex subset of $X$. Let $T: C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$, and $f: C \rightarrow C$ a generalized contraction on $C$. For each $t \in (0, 1)$, let $\{x_t\}$ be given by

\[ x_t = tf(x_t) \oplus (1 - t)Tx_t. \]  

Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to $\tilde{x}$ such that $\tilde{x} = P_{\text{Fix}(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

\[ \langle \tilde{x}f(\tilde{x}), x \rangle \geq 0, \quad x \in \text{Fix}(T). \]  

**Proof.** We first show that $\{x_t\}$ is bounded. For any $p \in \text{Fix}(T)$, we have that

\[ d^2(x_t, p) = \langle \tilde{x}_t, \tilde{x}_t \rangle \quad \leq \quad t\langle f(x_t)p, \tilde{x}_t \rangle + (1 - t)\langle \tilde{x}_t, \tilde{x}_t \rangle \]

\[ \leq \quad t\langle f(x_t)p, \tilde{x}_t \rangle + t\langle f(p)p, \tilde{x}_t \rangle + (1 - t)\langle \tilde{x}_t, \tilde{x}_t \rangle \]

\[ \leq \quad td(f(x_t), f(p))d(x_t, p) + td(f(p), p)d(x_t, p) + (1 - t)d(Tx_t, p)d(x_t, p) \]

\[ \leq \quad t\psi(d(x_t, p))d(x_t, p) + td(f(p), p)d(x_t, p) + (1 - t)d^2(x_t, p), \]

and hence

\[ d^2(x_t, p) \leq \psi(d(x_t, p))d(x_t, p) + d(f(p), p)d(x_t, p). \]

Then

\[ \eta(d(x_t, p)) := d(x_t, p) - \psi(d(x_t, p)) \leq d(f(p), p), \]

equivalent to

\[ d(x_t, p) \leq \eta^{-1}(d(f(p), p)). \]

Hence $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$. We get that

\[ \lim_{t \to 0} d(x_t, Tx_t) = \lim_{t \to 0} d(tf(x_t) \oplus (1 - t)Tx_t, Tx_t) \]

\[ \leq \lim_{t \to 0} [td(f(x_t), Tx_t) + (1 - t)d(Tx_t, Tx_t)] \]

\[ \leq \lim_{t \to 0} td(f(x_t), Tx_t) = 0. \]

Assume that $\{t_n\} \subset (0, 1)$ is such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{\text{Fix}(T)}f(\tilde{x})$ which is equivalent to the following variational inequality

\[ \langle \tilde{x}f(\tilde{x}), x \rangle \geq 0, \quad x \in \text{Fix}(T). \]
Since \( \{x_n\} \) is bounded, by Lemma 2.12, 2.14, we may assume that \( \{x_n\} \) \( \Delta \)-converges to a point \( \bar{x} \), and \( \bar{x} \in Fix(T) \). By Lemma 2.16, we have

\[
\limsup_{n \to \infty} \langle \overrightarrow{f(\bar{x})}, \overrightarrow{x_n} \rangle \leq 0. \tag{10}
\]

We want to show that \( d(x_n, \bar{x}) \to 0 \) as \( n \to \infty \). Suppose not, there exists \( \varepsilon > 0 \) and subsequence \( d(x_{n_j}, \bar{x}) \) of \( d(x_n, \bar{x}) \) such that \( d(x_{n_j}, \bar{x}) \geq \varepsilon \) for all \( j \in \mathbb{N} \). By Lemma 2.6, there exists \( r \in (0, 1) \) such that

\[
d(f(x_{n_j}), f(\bar{x})) \leq rd(x_{n_j}, \bar{x}) \quad \text{for all } j \in \mathbb{N}. \tag{11}
\]

Put \( x_j = x_{n_j} \) and \( \alpha_j = \alpha_{n_j} \) for all \( j \). It follows from Lemma 2.19 (i) that

\[
d^2(x_j, \bar{x}) = \langle \overrightarrow{x_j}, \overrightarrow{x_j} \rangle = \alpha_j \langle \overrightarrow{f(x_j)}, \overrightarrow{x_j} \rangle + (1 - \alpha_j) \langle \overrightarrow{T f(x_j)}, \overrightarrow{x_j} \rangle \\
\leq \alpha_j \langle \overrightarrow{f(x_j)}, \overrightarrow{x_j} \rangle + (1 - \alpha_j) d(T f(x_j), \bar{x}) d(x_j, \bar{x}) \\
\leq \alpha_j \langle \overrightarrow{f(x_j)}, \overrightarrow{x_j} \rangle + (1 - \alpha_j) d^2(x_j, \bar{x}).
\]

It follows that

\[
d^2(x_j, \bar{x}) \leq \langle \overrightarrow{f(x_j)}, \overrightarrow{x_j} \rangle = \langle \overrightarrow{f(x_j)}, \overrightarrow{x_j} \rangle + \langle \overrightarrow{f(\bar{x})}, \overrightarrow{x_j} \rangle \\
\leq d(f(x_j), f(\bar{x})) d(x_j, \bar{x}) + \langle \overrightarrow{f(\bar{x})}, \overrightarrow{x_j} \rangle \\
\leq rd^2(x_j, \bar{x}) + \langle \overrightarrow{f(\bar{x})}, \overrightarrow{x_j} \rangle,
\]

and thus

\[
d^2(x_j, \bar{x}) \leq \frac{1}{1 - r} \langle \overrightarrow{f(\bar{x})}, \overrightarrow{x_j} \rangle. \tag{12}
\]

It follows from (10) and (12) that \( \{x_j\} \) converge strongly to \( \bar{x} \), which is a contradiction. This implies that \( d(x_n, \bar{x}) \to 0 \) as \( n \to \infty \).

Next, we show that \( \bar{x} \) solves the variational inequality (8). Applying Lemma 2.10, for any \( q \in Fix(T) \),

\[
d^2(x_t, q) = d^2(tf(x_t) \oplus (1 - t)Tx_t, q) \\
\leq td^2(f(x_t), q) + (1 - t)d^2(Tx_t, q) - t(1 - t)d^2(f(x_t), Tx_t) \\
\leq td^2(f(x_t), q) + (1 - t)d^2(x_t, q) - t(1 - t)d^2(f(x_t), Tx_t).
\]

It implies that

\[
d^2(x_t, q) \leq d^2(f(x_t), q) - (1 - t)d^2(f(x_t), Tx_t)
\]
Taking the limit through \( t = t_n \to 0 \), we can get that
\[
d^2(\hat{x}, q) \leq d^2(f(\hat{x}), q) - d^2(f(\hat{x}), \hat{x}).
\]
Hence
\[
0 \leq \frac{1}{2} \left[ d^2(\hat{x}, \tilde{x}) + d^2(f(\hat{x}), q) - d^2(\hat{x}, q) - d^2(\tilde{x}, \hat{x}) \right] = \langle \tilde{x} f(\hat{x}), q \tilde{x} \rangle, \quad \forall q \in \text{Fix}(T).
\]
That is, \( \hat{x} \) solves the inequality (8).

Finally, we show that the entire net \( \{x_t\} \) converges to \( \hat{x} \), assume \( x_{s_n} \to \hat{x} \), where \( s_n \to 0 \) as \( n \to \infty \). By the same argument, we get that \( \hat{x} \in \text{Fix}(T) \) and solves the variational inequality (8), consequently we have
\[
\langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle \leq 0,
\]
and
\[
\langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle \leq 0.
\]
We want to show that \( \tilde{x} = \hat{x} \), if not, there exists \( \varepsilon > 0 \) such that \( d(\tilde{x}, \hat{x}) \geq \varepsilon \). By Lemma 2.6 there exists \( r \in (0, 1) \) such that \( d(f(\tilde{x}), f(\hat{x})) \leq rd(\tilde{x}, \hat{x}) \). Adding up (13) and (14), we get that
\[
0 \geq \langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle - \langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle
\]
\[
= \langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle + \langle f(\tilde{x})f(\hat{x}), \tilde{x} \hat{x} \rangle - \langle \tilde{x} \hat{x}, \tilde{x} \hat{x} \rangle - \langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle
\]
\[
= \langle \tilde{x} \hat{x}, \tilde{x} \hat{x} \rangle - \langle \tilde{x} \hat{x}, \tilde{x} \hat{x} \rangle - \langle \tilde{x} f(\hat{x}), \tilde{x} \hat{x} \rangle
\]
\[
\geq \langle \tilde{x} \hat{x}, \tilde{x} \hat{x} \rangle - d(f(\tilde{x}), f(\hat{x}))d(\tilde{x}, \hat{x})
\]
\[
\geq d^2(\tilde{x}, \hat{x}) - rd(\tilde{x}, \hat{x})d(\tilde{x}, \hat{x})
\]
\[
\geq d^2(\tilde{x}, \hat{x}) - rd^2(\tilde{x}, \hat{x})
\]
\[
\geq (1 - r)d^2(\tilde{x}, \hat{x}) \geq (1 - r)\varepsilon^2 > 0.
\]
This is a contradiction, so \( \tilde{x} = \hat{x} \). Hence the net \( x_t \) converge strongly to \( \hat{x} \) which is the unique solution to the variational inequality (8). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( C \) be a closed convex subset of a complete \( \text{CAT}(0) \) space \( X \), and let \( T : C \to C \) be a nonexpansive mapping with \( \text{Fix}(T) \neq \emptyset \). Let \( f \) be a generalized contraction on \( C \). For the arbitrary initial point \( x_0 \in C \), let \( \{x_n\} \) be generated by
\[
x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)Tx_n, \quad \forall n \geq 0,
\]
where \( \{\alpha_n\} \subset (0, 1) \) satisfies the following conditions:
1. \( \lim_{n \to \infty} \alpha_n = 0; \)
2. \( \sum_{n=0}^{\infty} \alpha_n = \infty \) and
3. either \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \) or \( \lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1. \)

Then \( \{x_n\} \) converges strongly as \( n \to \infty \) to \( \tilde{x} \) such that \( \tilde{x} = P_{\text{Fix}(T)} f(\tilde{x}) \) which is equivalent to the variational inequality (8).

**Proof.** First we show that \( \{x_n\} \) is bounded. Indeed, if we take a fixed point \( p \in \text{Fix}(T) \). We will prove by induction that

\[
d(x_n, p) \leq M \text{ for all } n \geq 0,
\]

where \( M := \{d(x_0, p), \eta^{-1}(d(f(p), p))\} \). The case \( n = 0 \) is obvious.

Suppose that \( d(x_n, p) \leq M \), we have

\[
d(x_{n+1}, p) \leq \alpha_n d(f(x_n), p) + (1 - \alpha_n) d(Tx_n, p)
\]

\[
\leq \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) + (1 - \alpha_n) d(Tx_n, p)
\]

\[
\leq \alpha_n \psi(d(x_n, p)) + \alpha_n d(f(p), p) + (1 - \alpha_n) d(x_n, p)
\]

\[
\leq \alpha_n \psi(d(x_n, p)) + \alpha_n \eta(\eta^{-1}(d(f(p), p))) + (1 - \alpha_n) d(x_n, p)
\]

\[
\leq \alpha_n \psi(M) + \alpha_n \eta(M) + (1 - \alpha_n) M
\]

\[
= \alpha_n \psi(M) + \alpha_n (M - \psi(M)) + (1 - \alpha_n) M = M
\]

By induction,

\[
d(x_n, p) \leq \max \{d(x_0, p), \eta^{-1}(d(f(p), p))\}, \quad \forall n \geq 0.
\]

Thus \( \{x_n\} \) is bounded, and so are \( \{Tx_n\} \) and \( \{f(x_n)\} \).

Next, we claim that

\[
\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \quad (16)
\]

If not, there exists \( \varepsilon > 0 \) and subsequence \( d(x_{n_j+1}, x_{n_j}) \) of \( d(x_{n+1}, x_n) \) such that \( d(x_{n_j+1}, x_{n_j}) \geq \varepsilon \) for all \( j \in \mathbb{N} \). By Lemma 2.6, there exists \( r \in (0, 1) \) such that

\[
d(f(x_{n_j+1}), f(x_{n_j})) \leq rd(x_{n_j+1}, x_{n_j}) \quad \text{for all } j \in \mathbb{N}. \quad (17)
\]

Put \( x_j = x_{n_j} \) and \( \alpha_j = \alpha_{n_j} \) for all \( j \). Observing that

\[
d(x_{j+1}, x_j) = \alpha_j d(f(x_j), p) \oplus (1 - \alpha_j) Tx_j, \alpha_j f(x_j) \oplus (1 - \alpha_j) Tx_{j-1}
\]

\[
\leq d(\alpha_j f(x_j) \oplus (1 - \alpha_j) Tx_j, \alpha_j f(x_j) \oplus (1 - \alpha_j) Tx_{j-1})
\]

\[
+ d(\alpha_j f(x_j) \oplus (1 - \alpha_j) Tx_{j-1}, \alpha_j f(x_{j-1}) \oplus (1 - \alpha_j) Tx_{j-1})
\]

\[
+ d(\alpha_j f(x_{j-1}) \oplus (1 - \alpha_j) Tx_{j-1}, \alpha_j f(x_{j-1}) \oplus (1 - \alpha_j) Tx_{j-1})
\]

\[
\leq (1 - \alpha_j) d(Tx_j, Tx_{j-1}) + \alpha_j d(f(x_j), f(x_{j-1})) + |\alpha_j - \alpha_{j-1}| d(f(x_{j-1}), Tx_{j-1})
\]

\[
\leq (1 - \alpha_j) d(x_j, x_{j-1}) + \alpha_j d(f(x_j), f(x_{j-1})) + |\alpha_j - \alpha_{j-1}| d(f(x_{j-1}), Tx_{j-1})
\]

\[
\leq (1 - \alpha_j) d(x_j, x_{j-1}) + \alpha_j d(x_j, x_{j-1}) + |\alpha_j - \alpha_{j-1}| d(f(x_{j-1}), Tx_{j-1})
\]

\[
= (1 - \alpha_j (1 - r)) d(x_n, x_{n-1}) + |\alpha_j - \alpha_{j-1}| d(f(x_{n-1}), Tx_{n-1}).
\]
By the conditions (ii) and (iii) and Lemma 2.20, we have \( \lim_{n \to \infty} d(x_{j+1}, x_j) = 0 \), which is a contradiction, so (16) is proved. It then follows that

\[
d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(\alpha_nf(x_n) + (1 - \alpha_n)T x_n, Tx_n) \leq d(x_n, x_{n+1}) + \alpha_n d(f(x_n), Tx_n) \to 0 \text{ as } n \to \infty.
\]  

(18)

Let \( \{x_t\} \) be a net in \( C \) such that

\[
x_t = tf(x_t) + (1 - t)Tx_t.
\]

By Theorem 3.1, we have that \( \{x_t\} \) converges strongly as \( t \to 0 \) to a fixed point \( \bar{x} \in \text{Fix}(T) \) which solves the variational inequality (8). Now, we claim that

\[
\lim \sup_{n \to \infty} \langle f(\bar{x})\bar{x}, x_n\bar{x} \rangle \leq 0.
\]

(19)

Notice that \( f \) is a nonexpansive mapping. It follows from Lemma 2.19 (i) that

\[
d^2(x_t, x_n) = \langle \bar{x}_t\bar{x}_n, \bar{x}_t\bar{x}_n \rangle \\
\leq t \langle f(x_t)x_n, \bar{x}_t\bar{x}_n \rangle + (1 - t) \langle Tx_t, x_t \bar{x}_n \rangle \\
= t \langle f(x_t)f(\bar{x}), x_t\bar{x}_n \rangle + t \langle f(\bar{x})\bar{x}, x_t\bar{x}_n \rangle + t \langle \bar{x}_t\bar{x}_n, x_t\bar{x}_n \rangle + t \langle \bar{x}_t\bar{x}_n, \bar{x}_t\bar{x}_n \rangle \\
+ (1 - t) \langle Tx_t, x_t \bar{x}_n \rangle + (1 - t) \langle Tx_t, x_t \bar{x}_n \rangle \\
\leq td(x_t, \bar{x})d(x_t, x_n) + t \langle f(\bar{x})\bar{x}, x_t\bar{x}_n \rangle + td(\bar{x}, x_t)d(x_t, x_n) + td^2(x_t, x_n) \\
+ (1 - t) d^2(x_t, x_n) + (1 - t) d(T x_t, x_t)M \\
\leq d^2(x_t, x_n) + td(x_t, \bar{x})M + td(\bar{x}, x_t)M + d(T x_t, x_n)M + t \langle f(\bar{x})\bar{x}, x_t\bar{x}_n \rangle,
\]

where \( M \geq \sup_{m,n \geq 1} \{d(x_t, x_n)\} \). This implies that

\[
\langle f(\bar{x})\bar{x}, x_n\bar{x}_t \rangle \leq 2d(x_t, \bar{x})M + \frac{d(T x_t, x_n)}{t} M.
\]

(20)

Taking the limit as \( n \to \infty \) first, and then \( t \to 0 \) inequality (20) yields that

\[
\lim \sup_{n \to \infty} \lim \sup_{t \to 0} \langle f(\bar{x})\bar{x}, x_n\bar{x}_t \rangle \leq 0.
\]

Since \( x_t \to \bar{x} \) as \( t \to 0 \) and continuity of metric distance \( d \), we have for any fixed \( n \geq 0 \),

\[
\lim_{t \to 0} \langle f(\bar{x})\bar{x}, x_n\bar{x}_t \rangle = \lim_{t \to 0} \frac{1}{2} \left[ d^2(f(\bar{x}), x_t) + d^2(\bar{x}, x_n) - d^2(f(\bar{x}), x_n) - d^2(\bar{x}, x_t) \right] \\
= \frac{1}{2} \left[ d^2(f(\bar{x}), \bar{x}) + d^2(\bar{x}, x_n) - d^2(f(\bar{x}), x_n) - d^2(\bar{x}, \bar{x}) \right] \\
= \langle f(\bar{x})\bar{x}, x_n\bar{x} \rangle.
\]
It implies that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that
\[
\langle f(\bar{x}), \bar{x}_n \bar{x} \rangle < \langle f(\bar{x}), \bar{x}_n \bar{x}_t \rangle + \varepsilon, \quad \forall t \in (0, \delta).
\tag{21}
\]
Thus, by the limit as $n \to \infty$ first, and then $t \to 0$ inequality in (21), we get that
\[
\limsup_{n \to \infty} \langle f(\bar{x}), \bar{x}_n \bar{x} \rangle \leq \varepsilon.
\]
Since $\varepsilon$ is arbitrary, it follows that
\[
\limsup_{n \to \infty} \langle f(\bar{x}), \bar{x}_n \bar{x} \rangle \leq 0.
\]
Hence we have desired. Finally, we prove that $x_n \to \bar{x}$, i.e., $d(x_n, \bar{x}) \to 0$, as $n \to \infty$. Suppose not, there exists $\varepsilon > 0$ and subsequence $d(x_{n_j}, \bar{x})$ such that $d(x_{n_j}, \bar{x}) \geq \varepsilon$ for all $j \in \mathbb{N}$. By Lemma 2.6, there exists $r \in (0, 1)$ such that
\[
d(f(x_{n_j}), f(\bar{x})) \leq rd(x_{n_j}, \bar{x}) \quad \text{for all } j \in \mathbb{N}. \tag{22}
\]
Put $x_j = x_{n_j}$ and $\alpha_j = \alpha_{n_j}$ for all $j$. It follows from Lemma 2.19 (i) and Cauchy-Schwarz inequality that
\[
d^2(x_{j+1}, \bar{x}) \leq (1 - \alpha_j) \langle f(x_j), x_{j+1} \bar{x} \rangle + \alpha_j \langle f(x_j), \bar{x}_{j+1} \bar{x} \rangle = (1 - \alpha_j) \langle f(x_j), x_{j+1} \bar{x} \rangle + \alpha_j \langle f(x_j), \bar{x}_{j+1} \bar{x} \rangle \leq (1 - \alpha_j) \langle f(x_j), x_{j+1} \bar{x} \rangle + \alpha_j \langle f(x_j), \bar{x}_{j+1} \bar{x} \rangle \leq (1 - \alpha_j) \langle f(x_j), x_{j+1} \bar{x} \rangle + \alpha_j \langle f(x_j), \bar{x}_{j+1} \bar{x} \rangle \leq \frac{(1 - \alpha_j)}{2} \left( d^2(x_j, \bar{x}) + d^2(x_{j+1}, \bar{x}) \right) + \frac{\alpha_j}{2} \left( r^2 d^2(x_j, \bar{x}) + d^2(x_{j+1}, \bar{x}) \right).
\]
which implies that
\[
d^2(x_{j+1}, \bar{x}) \leq (1 - \alpha_j (1 - r^2) d^2(x_j, \bar{x}) + 2\alpha_j \langle f(\bar{x}), \bar{x}_{j+1} \bar{x} \rangle \]
Applying Lemma 2.20 and (19), we can conclude that $x_j \to \bar{x}$. This is a contradiction. Hence $x_n \to \bar{x}$ as $n \to \infty$. \qed
4 Conclusion

In this paper, we established the strong convergence theorem of Moudafi’s viscosity approximation method for a nonexpansive mapping $T$ with a generalized contraction $f$ in a complete CAT(0) space, which solves the variational inequality (4). Theorem 3.1 and 3.2 are more general than the results in Theorem 3.1 and 3.4 of [28] in the case where a mapping $f$ is a generalized contraction.

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