Movable Locating-Domination in Graphs

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Abstract

In this paper, the 1-movable locating-dominating sets in the join, corona and composition of graphs are characterized. Also, the 1-movable \( L \)-domination numbers of these graphs are determined.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a simple connected graph and \( u \in V(G) \). The neighborhood of \( u \) is the set \( N_G(u) = N(u) = \{ v \in V(G) : uv \in E(G) \} \). The degree of a vertex \( u \in V(G) \) is equal to the cardinality of \( N_G(u) \) and the maximum degree of \( G \) is \( \Delta(G) = \max \{ \text{deg}_G(u) : u \in V(G) \} \). If \( X \subseteq V(G) \), then the open neighborhood of \( X \) is the set \( N_G(X) = N(X) = \bigcup_{v \in X} N_G(v) \). The closed neighborhood of \( X \) is \( N_G[X] = N[X] = X \cup N(X) \).

A connected graph \( G \) of order \( n \geq 3 \) is point distinguishing if for any two distinct vertices \( u \) and \( v \) of \( G \), \( N_G[u] \neq N_G[v] \). It is totally point determining if

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for any two distinct vertices \( u \) and \( v \) of \( G \), \( N_G(u) \neq N_G(v) \) and \( N_G[u] \neq N_G[v] \).

A subset \( S \) of \( V(G) \) is a dominating set of \( G \) if for every \( v \in (V(G) \setminus S) \), there exists \( w \in S \) such that \( vw \in E(G) \). The domination number \( \gamma(G) \) of \( G \) is the smallest cardinality of a dominating set of \( G \).

Let \( G \) be a connected graph. A set \( S \subseteq V(G) \) is a locating set of \( G \) if for every two vertices \( u \) and \( v \) of \( V(G) \setminus S \), \( N_G(u) \cap S \neq N_G(v) \cap S \). The locating number of \( G \) denoted by \( ln(G) \) is the smallest cardinality of locating set of \( G \). A set \( S \subseteq V(G) \) is a strictly locating set of \( G \) if it is a locating set and \( N_G(u) \cap S \neq S \) for all \( u \in V(G) \setminus S \). The strictly locating number of \( G \), denoted by \( sln(G) \), is the smallest cardinality of a strictly locating set of \( G \). A locating (resp. strictly locating) set \( S \subseteq V(G) \) which is dominating is called a locating-dominating (resp. strictly locating-dominating) set or simply \( LD \)-dominating (resp. \( SL \)-dominating) set in a graph \( G \). The minimum cardinality of a locating-dominating (resp. strictly locating-dominating) set of \( G \), denoted by \( \gamma_{LD}(G) \) (resp. \( \gamma_{SL}(G) \)), is called the \( LD \)-domination (resp. \( SL \)-domination) number of \( G \).

Let \( G \) be a connected graph. A non-empty \( S \subseteq V(G) \) is a 1-movable dominating set of \( G \) if \( S \) is a dominating set of \( G \) and for every \( v \in S \), either \( S \setminus \{v\} \) is a dominating set of \( G \) or there exists a vertex \( u \in (V(G) \setminus S) \cap N_G(v) \) such that \( (S \setminus \{v\}) \cup \{u\} \) is a dominating set of \( G \). The 1-movable domination number of a graph \( G \), denoted by \( \gamma^1_m(G) \) is the smallest cardinality of a 1-movable dominating set of \( G \).

Let \( G \) be a connected graph. A locating (resp. strictly locating) subset \( S \) of \( V(G) \) is a 1-movable locating (resp. 1-movable strictly locating) set of \( G \) if for every \( u \in S \), either \( S \setminus \{u\} \) is a locating (resp. strictly locating) set of \( G \), or there exists a vertex \( v \in N_G(u) \setminus S \) such that \( (S \setminus \{u\}) \cup \{v\} \) is a locating (resp. strictly locating) set of \( G \). The minimum cardinality of a 1-movable locating (resp. 1-movable strictly locating) set of \( G \), denoted by \( mln(G) \) (resp. \( msln(G) \)) is the 1-movable locating number (resp. 1-movable strictly locating number) of \( G \).

Let \( G \) be a connected graph. A locating-dominating (resp. strictly locating-dominating) set \( S \) of \( G \) is a 1-movable locating-dominating (resp. 1-movable strictly locating-dominating) set of \( G \) if for every \( v \in S \), either \( S \setminus \{v\} \) is a locating-dominating (resp. strictly locating-dominating) set of \( G \) or there exists a vertex \( u \in (V(G) \setminus S) \cap N_G(v) \) such that \( (S \setminus \{v\}) \cup \{u\} \) is a locating-dominating (resp. strictly locating-dominating) set of \( G \). The minimum cardinality of a 1-movable locating-dominating (resp. 1-movable strictly locating-dominating) set of \( G \), denoted by \( \gamma^1_{mLD}(G) \) (resp. \( \gamma^1_{mSL}(G) \)) is the 1-movable \( LD \)-domination (resp. 1-movable \( SL \)-domination number) of \( G \).
2 Results

Remark 2.1 Let $G$ be a connected graph. Then $\gamma(G) \leq \gamma_{mL}^1(G)$ and $\gamma_L(G) \leq \gamma_{mL}^1(G)$.

Remark 2.2 Let $G$ be a connected non-trivial graph. Then $1 \leq \gamma_{mL}^1(G) \leq n$.

Remark 2.3 For $n \geq 2$, $\gamma_{mL}^1(K_n) = n - 1$.

Lemma 2.4 Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma_{mL}^1(G) = 1$ if and only if $G \cong P_2$.

Proof: Suppose that $\gamma_{mL}^1(G) = 1$, say $S = \{v\}$ is a minimum 1-movable locating-dominating set of $G$. Clearly, $|V(G)| \geq 2$. Suppose that $|V(G)| > 2$. Since $G$ is connected and $S$ is a dominating set of $G$, it follows that $wv \in E(G)$ for all $w \in V(G) \setminus \{v\}$. Since $|V(G)| > 2$, there exist $x$ and $y$ with $x \neq y$ such that $N_G(x) \cap S = S = N_G(y) \cap S$. Hence, $S$ is not a locating-dominating set of $G$, contrary to the assumption. Therefore, $|V(G)| = 2$. Since $G$ is connected, $G \cong P_2$.

The converse is easy. □

Theorem 2.5 Let $G$ be a connected graph. If $\gamma_{mL}^1(G) = 2$, then $3 \leq |V(G)| \leq 5$.

Proof: By Lemma 2.4, $|V(G)| \geq 3$. Suppose that $|V(G)| > 5$. Let $S = \{a, b\}$ be a minimum 1-movable locating-dominating set of $G$. Let $u_1, u_2, u_3, u_4 \in V(G) \setminus S$. Since $S$ is a dominating set of $G$, $N_G(u_i) \cap S \neq \emptyset$ for all $i \in \{1, 2, 3, 4\}$. Thus, $N_G(u_i) \cap S$ is either $\{a\}$, $\{b\}$ or $S$. This implies that there exists distinct vertices $u_i$ and $u_j$, where $i, j \in \{1, 2, 3, 4\}$ such that $N_G(u_i) \cap S = N_G(u_j) \cap S$. Hence, $S$ is not a locating-dominating set of $G$, contrary to the assumption of $S$. Thus, $|V(G)| \leq 5$. □

Theorem 2.6 Let $G$ be a connected graph of order $n = 4$. Then $\gamma_{mL}^1(G) = 2$ if and only if $G \cong P_4$ or $G$ has a single end-vertex.

Proof: Suppose that $\gamma_{mL}^1(G) = 2$. Let $S = \{x, y\}$ be a minimum locating-dominating set of $G$. Since $S$ is a locating-dominating set of $G$, $N_G(x) \cap (V(G) \setminus S) \neq \emptyset$ and $N_G(y) \cap (V(G) \setminus S) \neq \emptyset$. Assume that $ax, by \in E(G)$, where $V(G) = \{x, y, a, b\}$. Consider the following cases:

Case 1. Suppose that $ay \in E(G)$.

Since $S$ is a locating set of $G$, $bx \notin E(G)$. Suppose that $xy \in E(G)$. Since $S$ is a 1-movable locating set of $G$, it follows that $ab \notin E(G)$. Thus, $b$ is an end-vertex of $G$. Suppose that $xy \notin E(G)$. If $ab \in E(G)$, then $x$ is an end-vertex of $G$. If $ab \notin E(G)$, then $G \cong P_4$. 

Case 2. Suppose that $ay \notin E(G)$.
Since $G$ is connected, then $xy \in E(G)$ or $bx \in E(G)$ or $ab \in E(G)$. Suppose that $xy \in E(G)$. Suppose further that $bx \in E(G)$. Then $(S \setminus \{y\}) \cup \{b\} = \{x, b\}$ cannot be a locating-dominating set of $G$ if $ab \in E(G)$. Thus, since $S$ is a 1-movable locating-dominating set of $G$, $ab \notin E(G)$. This implies that $a$ is an end-vertex of $G$. Suppose that $bx \notin E(G)$. Again, since $S$ is a 1-movable locating-dominating set of $G$, $ab \notin E(G)$. This implies that $G \cong P_4$. Next, suppose that $xy \notin E(G)$. Then $ab \in E(G)$ or $bx \in E(G)$. Thus, either $y$ is an end-vertex of $G$ or $G \cong P_4$.

Accordingly, $G \cong P_4$ or $G$ has a single end-vertex.

The converse is straightforward. □

**Lemma 2.7** Let $G$ be a connected non-trivial graph. If $S$ is a strictly locating set of $G$, then for any $z \in V(G) \setminus S$, $S \cup \{z\}$ is a strictly locating set of $G$.

**Proof:** Let $u, v \in V(G) \setminus (S \cup \{z\})$. Since $V(G) \setminus (S \cup \{z\}) \subseteq V(G) \setminus S$ and $S$ is a locating set, it follows that $u, v \in V(G) \setminus S$ and $N_G(u) \cap S \neq N_G(v) \cap S$. Hence, $N_G(u) \cap (S \cup \{z\}) = (N_G(u) \cap S) \cup (N_G(u) \cap \{z\}) \neq (N_G(v) \cap S) \cup (N_G(v) \cap \{z\}) = N_G(v) \cap (S \cup \{z\})$, that is, $S \cup \{z\}$ is a locating set of $G$.

Finally, let $y \in V(G) \setminus (S \cup \{z\})$. Then $y \in V(G) \setminus S$. Since $S$ is a strictly locating set, $N_G(y) \cap S \neq S$. Thus, $N_G(y) \cap (S \cup \{z\}) = (N_G(y) \cap S) \cup (N_G(y) \cap \{z\}) \neq S \cup \{z\}$. Therefore, $S \cup \{z\}$ is a strictly locating set of $G$. □

### 3 Movable Locating-Dominating Sets in the Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The disjoint union of $A$ and $B$, denoted by $A \uplus B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex-set $V(G + H) = V(G) \uplus V(H)$ and edge-set $E(G + H) = E(G) \uplus E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

**Theorem 3.1** [2] Let $G$ and $H$ be connected non-trivial graphs. Then $S \subseteq V(G + H)$ is a locating dominating set of $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets of $G$ and $H$, respectively, where $S_1$ or $S_2$ is a strictly locating set.

**Theorem 3.2** Let $G$ and $H$ be connected non-trivial graphs. Then $S \subseteq V(G + H)$ is a 1-movable locating-dominating set of $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are 1-movable locating sets of $G$ and $H$, respectively, and one of the following statements holds:
(i) $S_1$ and $S_2$ are strictly locating sets of $G$ and $H$, respectively;

(ii) $S_1$ is a strictly locating set of $G$ and for each $u \in S_1$, $S_1 \setminus \{u\}$ or $(S_1 \setminus \{u\}) \cup \{w\}$ is a strictly locating set of $G$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $S_1 \setminus \{u\}$ is a locating set of $G$ and $S_2 \cup \{z\}$ is a strictly locating set of $H$ for some $z \in V(H) \setminus S_2$;

(iii) $S_2$ is a strictly locating set of $G$ and for each $v \in S_2$, $S_2 \setminus \{v\}$ or $(S_2 \setminus \{v\}) \cup \{y\}$ is a strictly locating set of $H$ for some $y \in N_H(v) \cap (V(H) \setminus S_2)$ or $S_2 \setminus \{v\}$ is a locating set of $H$ and $S_1 \cup \{x\}$ is a strictly locating set of $G$ for some $x \in V(G) \setminus S_1$.

Proof: Suppose that $S$ is a 1-movable locating-dominating set of $G + H$. Then by Theorem 3.1, $S_1$ and $S_2$ are locating sets of $G$ and $H$, respectively, where $S_1$ or $S_2$ is a strictly locating set. Moreover, since $G$ and $H$ are non-trivial graphs, $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Let $u \in S_1$. By assumption, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup S_2$ or $(S \setminus \{u\}) \cup \{w\} = [(S_1 \setminus \{u\}) \cup \{w\}] \cup S_2$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup (S_2 \cup \{z\})$ for some $z \in V(H) \setminus S_2$ is a locating-dominating set of $G + H$. Thus, by Theorem 3.1, $(S \setminus \{u\}) \cup \{w\}$ is a locating set of $G$. This implies that $S_1$ is a 1-movable locating set of $G$. Similarly, $S_2$ is a 1-movable locating set of $H$.

Now, if $S_1$ and $S_2$ are strictly locating sets, then (i) holds. Suppose $S_2$ is not a strictly locating set of $G$. Then $S_1$ is strictly locating set of $G$. Let $u \in S_1$. Since $S$ is a 1-movable locating-dominating set, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup S_2$ or $(S \setminus \{u\}) \cup \{w\} = [(S_1 \setminus \{u\}) \cup \{w\}] \cup S_2$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup (S_2 \cup \{z\})$ for some $z \in V(H) \setminus S_2$ is a locating-dominating set of $G + H$. Since $S_2$ is not a strictly locating set, it follows from Theorem 3.1 that $S_1 \setminus \{u\}$ or $(S \setminus \{u\}) \cup \{w\}$ is a strictly locating set of $G$ or $S_1 \setminus \{u\}$ is a locating set in $G$ and $S_2 \cup \{z\}$ is a strictly locating of $H$. Thus, (ii) holds. Similarly, if $S_1$ is not a strictly locating set and $S_2$ is a strictly locating set, then (iii) holds.

For the converse, suppose that $S_1$ and $S_2$ are 1-movable locating sets of $G$ and $H$, respectively. Suppose that (i) holds. Then $S = S_1 \cup S_2$ is a locating-dominating set of $G + H$ by Theorem 3.1. Let $u \in S$. If $u \in S_1$, then by assumption, Lemma 2.7 and Theorem 3.1, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup S_2$ or $(S \setminus \{u\}) \cup \{w\} = [(S_1 \setminus \{u\}) \cup \{w\}] \cup S_2$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup (S_2 \cup \{z\})$ for some $z \in V(H) \setminus S_2$ is a locating-dominating set of $G + H$. Similarly, if $u \in S_2$, then $S \setminus \{u\} = (S_2 \setminus \{u\}) \cup S_1$ or $(S \setminus \{u\}) \cup \{y\} = [(S_2 \setminus \{u\}) \cup \{y\}] \cup S_1$ for some $y \in N_H(u) \cap (V(H) \setminus S_2)$ or $(S \setminus \{u\}) \cup \{x\} = (S_2 \setminus \{u\}) \cup (S_1 \cup \{x\})$ for some $x \in V(G) \setminus S_1$ is a locating-dominating set of $G + H$. Hence, $S$ is a 1-movable locating-dominating set of $G + H$.

Now, suppose that (ii) holds. Then $S = S_1 \cup S_2$ is a locating-dominating set of $G + H$. If $u \in S_1$, then by assumption, Lemma 2.7 and Theorem 3.1, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup S_2$ or $(S \setminus \{u\}) \cup \{w\} = [(S_1 \setminus \{u\}) \cup \{w\}] \cup S_2$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup (S_2 \cup \{z\})$ for some $z \in V(H) \setminus S_2$ is a locating-dominating set of $G + H$. Similarly, if $u \in S_2$, then $S \setminus \{u\} = (S_2 \setminus \{u\}) \cup S_1$ or $(S \setminus \{u\}) \cup \{y\} = [(S_2 \setminus \{u\}) \cup \{y\}] \cup S_1$ for some $y \in N_H(u) \cap (V(H) \setminus S_2)$ or $(S \setminus \{u\}) \cup \{x\} = (S_2 \setminus \{u\}) \cup (S_1 \cup \{x\})$ for some $x \in V(G) \setminus S_1$ is a locating-dominating set of $G + H$. Hence, $S$ is a 1-movable locating-dominating set of $G + H$.
set of $G + H$ by Theorem 3.1. Let $u \in S$. If $u \in S_1$, then by assumption and Theorem 3.1, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup S_2$ or $(S_1 \setminus \{u\}) \cup \{w\} = [(S_1 \setminus \{u\}) \cup \{w\}] \cup S_2$ for some $w \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S_1 \setminus \{u\}) \cup \{y\} = [(S_1 \setminus \{y\}) \cup \{y\}] \cup S_1$ for some $y \in N_H(u) \cap (V(H) \setminus S_2)$ is a locating-dominating set of $G + H$. Now, suppose that $u \in S_2$. Since $S_2$ is a 1-movable locating set and $S_1$ is a strictly locating set of $G$, it follows from Theorem 3.1 that $S \setminus \{u\} = (S_2 \setminus \{u\}) \cup S_1$ or $(S_2 \setminus \{u\}) \cup \{y\} = [(S_2 \setminus \{y\}) \cup \{y\}] \cup S_1$ for some $y \in N_H(u) \cap (V(H) \setminus S_2)$ is a locating-dominating set of $G + H$. Therefore, $S$ is a 1-movable locating-dominating set of $G + H$. Similarly, $S$ is a 1-movable locating-dominating set of $G + H$ if (iii) holds. □

Corollary 3.3 Let $G$ and $H$ be connected non-trivial graphs.

(i) If $G$ has a 1-movable strictly locating set, then $\gamma^1_{mL}(G + H) \leq \text{msln}(G) + \text{mln}(H)$.

(ii) If $H$ has a 1-movable strictly locating set, then $\gamma^1_{mL}(G + H) \leq \text{msln}(H) + \text{mln}(G)$.

(iii) If $G$ and $H$ have 1-movable strictly locating set, then $\gamma^1_{mL}(G + H) \leq \min \{\text{msln}(G) + \text{mln}(H), \text{msln}(H) + \text{mln}(G)\}$.

Proof: (i) : Suppose $G$ has a 1-movable strictly locating set. Let $S_1$ be a minimum 1-movable strictly locating set of $G$ and $S_2$ be a minimum 1-movable locating set of $H$. Then $S = S_1 \cup S_2$ is a 1-movable locating-dominating set of $G + H$ by Theorem 3.2. Thus, $\gamma^1_{mL}(G + H) \leq |S| = |S_1| + |S_2| = \text{msln}(G) + \text{mln}(H)$.

Statement (ii) is proved similarly and (iii) follows from (i) and (ii). □

Remark 3.4 The bounds given in Corollary 3.3 are tight.

To see this, consider the graphs in Figure 1 and Figure 2. Note that $\text{msln}(C_4) = 4$, $\text{mln}(C_4) = 4$, $\text{mln}(P_2) = 1$, $\text{msln}(G) = 4$ and $\text{mln}(G) = 2$.

Also, $\gamma^1_{mL}(C_4 + P_2) = 5 = \text{msln}(C_4) + \text{mln}(P_2)$ and $\gamma^1_{mL}(G + C_4) = 6 = \min \{\text{msln}(G) + \text{mln}(C_4), \text{msln}(C_4) + \text{mln}(G)\} = \min \{8, 6\}$.

\[
C_4 + P_2:
\]

Figure 1: The graph $C_4 + P_2$ with $\gamma^1_{mL}(C_4 + P_2) = 5$
Theorem 3.5 [2] Let $G$ be a connected non-trivial graph $K_1 = \langle v \rangle$. Then $S \subseteq V(G + K_1)$ is a locating-dominating set of $G + K_1$ if and only if either $S = S_1 \cup \{v\}$ where $S_1$ is a locating set of $G$, or $v \notin S$ and $S$ a strictly locating-dominating set of $G$.

Theorem 3.6 Let $G$ be a connected non-trivial graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(G + K_1)$ is a 1-movable locating-dominating set of $G + K_1$ if and only if either

(i) $S = S_1 \cup \{v\}$, where $S_1$ is a 1-movable locating set of $G$ and either $S_1$ is a strictly locating-dominating set of $G$ for some $z \in V(G) \setminus S_1$; or

(ii) $S = S_1$, where $S_1$ is a strictly locating-dominating set of $G$ such that for each $u \in S_1$, $S_1 \setminus \{u\}$ or $(S \setminus \{u\}) \cup \{z\}$ is a strictly locating-dominating set of $G$ for some $z \in N_G(u) \cap (V(G) \setminus S_1)$ or $S_1 \setminus \{u\}$ is a locating set of $G$.

Proof: Let $S \subseteq V(G + K_1)$ be a 1-movable locating-dominating set of $G + K_1$. Let $S_1 = V(G) \cap S$. Suppose first that $S = S_1 \cup \{v\}$. Then by Theorem 3.5, $S_1$ is a locating set of $G$. Moreover, since $G$ is non-trivial, $S_1 \neq \emptyset$. Let $u \in S_1$. Since $S$ is a 1-movable locating-dominating set of $G + K_1$, $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup \{v\}$ or $(S \setminus \{u\}) \cup \{z\} = [(S_1 \setminus \{u\}) \cup \{z\}] \cup \{v\}$ is a locating-dominating set of $G + K_1$ for some $z \in N_G(u) \cap (V(G) \setminus S_1)$. Thus, by Theorem 3.5, $S_1 \setminus \{u\}$ or $(S_1 \setminus \{u\}) \cup \{z\}$ is a locating set of $G$. Hence, $S_1$ is a 1-movable locating set of $G$. Now, since $v \in S$, either $S_1 \setminus \{v\} = S_1$ is a strictly locating-dominating set of $G$ or there exists $z \in V(G) \setminus S_1$ such that $(S \setminus \{v\}) \cup \{z\} = S_1 \cup \{z\}$ is a strictly locating-dominating set of $G$ by Theorem 3.5. Therefore, (i) holds.

Next, suppose that $v \notin S$. Then by Theorem 3.5, $S_1$ is a strictly locating-dominating set of $G$. Let $u \in S_1$. Since $S$ is a 1-movable locating-dominating set of $G + K_1$, $S \setminus \{u\} = S_1 \setminus \{u\}$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup \{z\}$ for some $z \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{v\} = (S_1 \setminus \{u\}) \cup \{v\}$ is a locating-dominating set of $G + K_1$. Thus, by Theorem 3.5, $S_1 \setminus \{u\}$ or $(S_1 \setminus \{u\}) \cup \{z\}$ is a strictly locating-dominating set of $G$ or $S_1 \setminus \{u\}$ is a locating set of $G$. Hence,
(ii) holds.

For the converse, suppose first that (i) holds. Then by Theorem 3.5, $S = S_1 \cup \{v\}$ is a locating-dominating set of $G + K_1$. Let $u \in S$. Suppose that $u \in S_1$. By Theorem 3.5 and since $S_1$ is a 1-movable locating set, it follows that $S \setminus \{u\} = (S_1 \setminus \{u\}) \cup \{v\}$ or $(S \setminus \{u\}) \cup \{z\} = ([S_1 \setminus \{u\}] \cup \{z\}) \cup \{v\}$ is a locating-dominating set of $G + K_1$ for some $z \in N_G(u) \cap (V(G) \setminus S_1)$. Suppose that $u = v$. If $S_1$ is a strictly locating-dominating set, then $S \setminus \{v\} = S_1$ is a locating-dominating set of $G + K_1$. If $S_1$ is not a strictly locating-dominating set, then by assumption, there exists $z \in V(G) \setminus S_1$ such that $S_1 \cup \{z\}$ is a strictly locating-dominating set of $G + K_1$. Hence, by Theorem 3.5, $(S \setminus \{v\}) \cup \{z\}$ is a locating-dominating set of $G + K_1$. Therefore, $S$ is a 1-movable locating-dominating set of $G + K_1$.

Suppose that (ii) holds. Then by Theorem 3.5, $S = S_1$ is a locating-dominating set of $G + K_1$. Let $u \in S$. Then by assumption and Theorem 3.5, $S \setminus \{u\} = S_1 \setminus \{u\}$ or $(S \setminus \{u\}) \cup \{z\} = (S_1 \setminus \{u\}) \cup \{z\}$ for some $z \in N_G(u) \cap (V(G) \setminus S_1)$ or $(S \setminus \{u\}) \cup \{v\} = (S_1 \setminus \{u\}) \cup \{v\}$ is a locating-dominating set of $G + K_1$. Therefore, $S$ is a 1-movable locating-dominating set of $G + K_1$. □

**Corollary 3.7** Let $G$ be a connected non-trivial graph having a 1-movable strictly locating-dominating set. Then $\gamma_{mL}(G + K_1) \leq \gamma_{mSL}(G)$.

**Proof:** Let $S_1$ be a minimum 1-movable strictly locating-dominating set of $G$. Then by Theorem 3.6, $S = S_1$ is a 1-movable locating-dominating set of $G + K_1$. Thus, $\gamma_{mL}^1(G + K_1) \leq |S| = \gamma_{mSL}^1(G)$. □

**Remark 3.8** The bound given in Corollary 3.7 is tight and the strict inequality can be attained.

To see this, consider the graphs in Figure 3 and Figure 4. Note that $\gamma_{mSL}^1(C_5) = 3$ and $\gamma_{mSL}^1(P_4) = 4$ with $\gamma_{mL}^1(C_5 + K_1) = 3 = \gamma_{mSL}^1(C_5)$ and $\gamma_{mL}^1(P_4 + K_1) = 3 < \gamma_{mSL}^1(P_4)$.

![Figure 3](image-url)
4 Movable Locating-Dominating Sets in the Corona of Graphs

Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding the join $(\{v\}) + H^v$, where $v \in V(G)$.

**Theorem 4.1** [3] Let $G$ and $H$ be non-trivial connected graphs. Then $S \subseteq V(G \circ H)$ is a locating-dominating set of $G \circ H$ if and only if $S = A \cup B \cup C \cup D$, where $A \subseteq V(G)$, $B = \bigcup \{B_v : v \in A \text{ and } B_v \text{ is a locating set of } H^v\}$, $C = \bigcup \{E_v : v \notin A, N_G(v) \cap A \neq \emptyset \text{ and } E_v \text{ is a locating-dominating set of } H^v\}$ and $D = \bigcup \{D_v : v \notin A, N_G(v) \cap A = \emptyset \text{ and } D_v \text{ is strictly locating-dominating set of } H^v\}$.

**Theorem 4.2** Let $G$ and $H$ be non-trivial connected graphs such that $H$ has a 1-movable strictly locating-dominating set. Then $\gamma_{mL}^1(G \circ H) \leq |V(G)| \gamma_{mSL}^1(H)$.

**Proof:** Let $S$ be a minimum 1-movable strictly locating-dominating set of $H$. For each $v \in V(G)$, let $S_v \subseteq V(H^v)$ such that $\langle S_v \rangle \cong \langle S \rangle$. Let $C = \bigcup_{v \in V(G)} S_v$. Then by Theorem 4.1, $C$ is a locating-dominating set of $G \circ H$. Let $u \in C$ and let $w \in V(G)$ such that $u \in V(H^w)$. Then $u \in S_w$. Since $S_w$ is a 1-movable strictly locating-dominating set of $H^w$, $S_w \setminus \{u\}$ or $(S_w \setminus \{u\}) \cup \{z\}$ is a strictly locating-dominating set of $H^w$ for some $z \in N_{H^w}(u) \cap (V(H^w) \setminus S_w)$. Thus, by Theorem 4.1, $C \setminus \{u\} = (\bigcup_{v \in V(G) \setminus \{w\}} S_v) \cup (S_w \setminus \{u\})$ or $(C \setminus \{u\}) \cup \{z\} = (\bigcup_{v \in V(G) \setminus \{w\}} S_v) \cup ((S_w \setminus \{u\}) \cup \{z\})$ is a locating-dominating set of $G \circ H$ for some $z \in N_{G \circ H}(u) \cap (V(G \circ H) \setminus C)$. Hence $C$ is a 1-movable locating-dominating set of $G \circ H$. Therefore,

$$\gamma_{mL}^1(G \circ H) \leq |C| = \sum_{v \in V(G)} |S_v| = |V(G)| \gamma_{mSL}^1(H). \quad \Box$$
5 Movable Locating-Dominating Sets in the Composition of Graphs

The composition (lexicographic product) $G[H]$ of two graphs $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

**Theorem 5.1** [3] Let $G$ and $H$ be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $C = \bigcup_{x \in S} \{ x \} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a locating-dominating set of $G[H]$ if and only if

(i) $S = V(G)$;

(ii) $T_x$ is a locating set in $H$ for every $x \in V(G)$;

(iii) $T_x$ or $T_y$ is strictly locating in $H$ whenever $x$ and $y$ are adjacent vertices of $G$ with $N_G(x) = N_G(y)$; and

(iv) $T_x$ or $T_y$ is (locating) dominating in $H$ whenever $x$ and $y$ are distinct non-adjacent vertices of $G$ with $N_G(x) = N_G(y)$.

Observe that the condition $\Delta(H) \leq |V(H)| - 2$ in Theorem 5.1 can be dropped.

**Theorem 5.2** Let $G$ be a connected totally point determining graph and let $H$ be a connected non-trivial graph. Then $C = \bigcup_{x \in S} \{ x \} \times T_x$ is a 1-movable locating-dominating set of $G[H]$ if and only if

(i) $S = V(G)$; and

(ii) $T_x$ is a 1-movable locating set of $H$ for every $x \in V(G)$.

**Proof:** Suppose that $C$ is a 1-movable locating-dominating set of $G[H]$. By Theorem 5.1, $S = V(G)$ and $T_x$ is a locating set for each $x \in S$. Let $a \in T_x$. Since $C$ is a 1-movable locating-dominating set of $G[H]$, $C \setminus \{(x, a)\} = \bigcup_{v \in S \setminus \{x\}} \{v\} \times T_v \cup \{x\} \times (T_x \setminus \{a\})$ or $C \setminus \{(x, a)\} = \bigcup_{z \in S \setminus \{x\}} \{z\} \times T_z \cup \{x\} \times ((T_x \setminus \{a\}) \cup \{b\})$ for some $b \in N_H(a) \cap (V(H) \setminus T_x)$ or $C \setminus \{(x, a)\} \cup \{(y, b)\} = \bigcup_{z \in S \setminus \{x\}} \{z\} \times T_z \cup \{x\} \times (T_x \setminus \{a\}) \cup \{(y) \times (T_y \cup \{b\})\}$ for some $y \in N_G(x)$ with $b \notin T_y$ is a locating-dominating set of $G[H]$. Thus, by Theorem 15, $T_x \setminus \{a\}$ or $(T_x \setminus \{a\}) \cup \{b\}$ is a locating set of $H$. Hence, $T_x$ is a 1-movable locating set of $H$.

For the converse, suppose that (i) and (ii) hold. Then by Theorem 5.1,
Let \( S \subseteq V(G) \) denote the set of vertices in \( G \). Then, for any vertex \( v \in V(G) \), the set \( T_v \) contains all vertices adjacent to \( v \) in \( G \), and \( T_v \setminus \{a\} \cup \{b\} \) is a locating set of \( H \) for some \( b \in (V(H) \setminus T_x) \cap N_H(a) \). Thus, \( \gamma_{1mL}(G[H]) = \sum_{x \in V(G)} |T_x| = |V(G)| \text{mln}(H) \). □

Corollary 5.3 Let \( G \) be a connected totally point determining graph and let \( H \) be a connected non-trivial graph. Then \( \gamma_{1mL}(G[H]) = |V(G)| \text{mln}(H) \).

Proof: Let \( C = \bigcup_{x \in S} \{x\} \times T_x \) be a minimum 1-movable locating-dominating set of \( G[H] \). Then by Theorem 5.2, \( S = V(G) \) and \( T_x \) is a minimum 1-movable locating set of \( H \). Therefore, \( \gamma_{1mL}(G[H]) = |C| = \sum_{x \in V(G)} |T_x| = |V(G)| \text{mln}(H) \). □

References


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