On $\sigma$-$\delta$-Continuity on $\sigma$-Structures

Young Key Kim

Department of Mathematics, MyongJi University
Youngin 449-728, Korea

Won Keun Min\textsuperscript{1}

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea

Copyright \textcopyright{} 2015 Young Key Kim and Won Keun Min. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The purpose this paper is to introduce the notion of $\sigma$-$\delta$-continuity and to investigate basic properties. We also introduce the notion of $\sigma$-$G$-regular structure, and study the relations among $\sigma$-$\delta$-continuity and the other $\sigma$-continuities on the $\sigma$-$G$-regular structures.

Mathematics Subject Classification: 54A05

Keywords: $\sigma$-structure, $\sigma$-$\delta$-open, $\sigma$-$G$-regular, $\sigma$-$\delta$-continuous, almost $\sigma$-continuous

1 Introduction

The notions of generalized topology and generalized open sets are introduced by Császár [1] as the following: Let $X$ be a nonempty set and $\mu$ be a collection of subsets of $X$. Then $\mu$ is called a generalized topology (briefly GT) on $X$ iff

\textsuperscript{1}Corresponding author
∅ ∈ μ and G_i ∈ μ for i ∈ I ≠ ∅ implies G = \bigcup_{i \in I} G_i ∈ μ. The elements of μ are called g-open sets and the complements are called g-closed sets. Kim and Min [2] introduced the notion of σ-structures which is an extended notion of generalized topology: s ⊆ 2^X is called a σ-structure on X if for i ∈ I ≠ ∅, U_i ∈ s implies \bigcup_{i \in I} U_i ∈ s. The elements of s are called σ-open sets and the complements are called σ-closed sets. And the two operators i_s and c_s [2] were defined as the following:

i_s(A) = \bigcup\{S ⊆ X : S ⊆ A, S \text{ is } σ-open\};
c_s(A) = \bigcap\{F ⊆ X : A ⊆ F, F \text{ is } σ-closed\}.

Then we showed that:

1. The collection μ = \{A ⊆ X : i_s A = A\} is a generalized topology on X.
2. x ∈ i_s A iff there exists a σ-open set S containing x such that S ⊆ A.
3. x ∈ c_s A iff S ∩ A ≠ ∅ for every σ-open set S containing x.

We also introduced the notions of σ-semiopen sets [3], σ-preopen sets [4] and σ-β-open sets [5]. In particular, we showed that the family σPO(X) (resp., σβO(X)) of all σ-preopen sets (resp., σ-β-open sets) in (X, σ) is a generalized topology in sense of Császár. In [4], we showed that the family σSO(X) of all σ-semiopen subsets is strong but σPO(X) may not be strong in (X, σ). In [5], we studied σ-β-open sets which are the generalized σ-open sets of σ-semiopen sets and σ-preopen sets, and showed that the family σβO(X) of all σ-β-open sets in (X, σ) is a strong σ-structure. The notions of σ-regular open sets and σ-δ-open sets were introduced in [6,7]. In [2], we introduced the notion of σ-continuity and obtained characterizations of σ-continuity by using the two operators i_s and c_s. In [7], we studied the notion of almost σ-continuity, and investigated characterizations by using σ-regular open sets. The purpose this paper is to introduce the notion of σ-δ-continuity and to investigate basic properties. We also introduce the notion of σ-G-regular structure, and study the relations among σ-δ-continuity and the other σ-continuities on the σ-G-regular structures (see Theorem 3.12 and Theorem 3.13).

2 Preliminaries

Definition 2.1 ([6, 7]). Let s be a σ-structure on a nonempty set X and A ⊆ X. Then a subset A in a σ-structure s is said to be

1. σ-regular open [7] if A = i_s(c_s(A));
2. δ-σ-open [6] if for each x ∈ A, there exists a σ-regular open set G such that x ∈ G ⊆ A. The collection of all σ-regular open sets (respectively, σ-δ-open sets) is denoted by σRO(X) (respectively, δ_σ).

Then we showed that every σ-regular open set is σ-δ-open, and if A is σ-
closed, then \(i_\delta(A)\) is \(\sigma\)-regular open. Furthermore, every non-empty \(\sigma\)-\(\delta\)-open set is clearly \(\sigma\)-open but the converse is not true in general.

**Theorem 2.2** ([6]). Let \(s\) be a \(\sigma\)-structure on a nonempty set \(X\). Then

1. the empty set is \(\sigma\)-\(\delta\)-open.
2. the any union of \(\sigma\)-\(\delta\)-open sets is \(\sigma\)-\(\delta\)-open.

The two operators \(i_\delta\) and \(c_\delta\) were defined as the following:

\[
i_\delta A = \bigcup\{S \subseteq X : S \subseteq A, \ S \text{ is } \sigma\text{-\(\delta\)-open }\};
\]

\[
c_\delta A = \bigcap\{F \subseteq X : A \subseteq F, \ F \text{ is } \sigma\text{-\(\delta\)-closed }\}.
\]

**Theorem 2.3** ([6]). Let \(s\) be a \(\sigma\)-structure on a nonempty set \(X\) and \(A \subseteq X\). Then

1. \(i_\delta \emptyset = \emptyset\) and \(c_\delta X = X\).
2. \(i_\delta A \subseteq A\) and \(A \subseteq c_\delta A\).
3. If \(A \subseteq B\), then \(i_\delta A \subseteq i_\delta B\) and \(c_\delta A \subseteq c_\delta B\).
4. \(i_\delta i_\delta A = i_\delta A\) and \(c_\delta c_\delta A = c_\delta A\).
5. \(c_\delta(A) = X - i_\delta(X - A)\) and \(i_\delta(A) = X - c_\delta(X - A)\).
6. \(A\) is \(\sigma\)-\(\delta\)-open iff \(A = i_\delta(A)\).
7. \(A\) is \(\sigma\)-\(\delta\)-closed iff \(A = c_\delta(A)\).

**Theorem 2.4** ([6]). Let \(s\) be a \(\sigma\)-structure on a nonempty set \(X\). Then

1. the non-empty elements of \(\delta_s\) coincide with the unions of \(\sigma\)-regular open sets;
2. \(x \in i_\delta A\) iff there exists a \(\sigma\)-regular open set \(S\) containing \(x\) such that \(S \subseteq A\);
3. \(x \in c_\delta A\) iff \(S \cap A \neq \emptyset\) for every \(\sigma\)-regular open set \(S\) containing \(x\).

## 3 Main Results

**Definition 3.1.** Let \(s, s'\) be \(\sigma\)-structures on \(X\) and \(Y\), respectively. Then a function \(f : X \to Y\) is said to be \(\sigma\)-\(\delta\)-continuous at \(x \in X\) if for each \(\sigma\)-open set \(V\) containing \(f(x)\), there is a \(\sigma\)-open set \(U\) containing \(x\) such that \(f(i_\delta(c_\delta(U))) \subseteq i_\delta(c_\delta(V))\). A function \(f : X \to Y\) is said to be \(\sigma\)-\(\delta\)-continuous if it has the property at each point of \(X\).

**Remark 3.2.** Let \(s, s'\) be \(\sigma\)-structures on \(X\) and \(Y\), respectively. Then \(f : X \to Y\) is said to be almost \(\sigma\)-continuous [7] at \(x \in X\) if for each \(\sigma\)-open subset \(V\) containing \(f(x)\), there is a \(\sigma\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq i_\delta(c_\delta(V))\). A function \(f : (X, \tau) \to (Y, \mu)\) is said to be almost \(\sigma\)-continuous if it has the property at each point of \(X\). From the notion of \(\sigma\)-\(\delta\)-continuity, we have the following relationships.
\( \sigma \)-continuous

\( \forall \ \not\exists \) almost \( \sigma \)-continuous

\( \sigma \)-\( \delta \)-continuous

In general, the converses are not true as shown in the next examples.

**Example 3.3.** (1) Let \( X = \{a, b, c, d\} \) and let \( s = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \) and \( s' = \{\{a\}, \{c\}, \{a, c\}, \{a, b, c\}\} \) be \( \sigma \)-structures on \( X \). Then the identity function \( f : (X, s) \rightarrow (X, s') \) is \( \sigma \)-continuous. Let \( V = \{a\} \) be a \( \sigma \)-open set containing \( f(a) \). Note that:

\[
\begin{align*}
    f(i_s(c_s(\{a\}))) &= f(i_s(\{a, b, d\})) = f(\{a, b\}) = \{a, b\}; \\
    i_s(c_s(V)) &= i_s(c_s(\{a\})) = i_s(\{a, b, d\}) = \{a\} \text{ for } \sigma \text{-structure } s'.
\end{align*}
\]

So there is no any \( \sigma \)-open set \( U \) containing \( a \) such that \( f(i_s(c_s(U))) \subseteq i_s(c_s(V)) \). So \( f \) is not \( \sigma \)-\( \delta \)-continuous.

(2) Let \( X = \{a, b, c, d\} \) and a \( \sigma \)-structure \( s = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \). Consider a function \( f : (X, s) \rightarrow (X, s) \) defined by \( f(a) = b, f(b) = a, f(c) = c, f(d) = d \). Then \( f \) is \( \sigma \)-\( \delta \)-continuous but not \( \sigma \)-continuous.

(3) Let \( X = \{a, b, c, d\} \), and let \( s = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \) and \( s' = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) be \( \sigma \)-structures on \( X \). Consider a function \( f : (X, s) \rightarrow (X, s') \) defined as \( f(a) = b, f(b) = f(d) = d, f(c) = c \). Then \( f \) is almost \( \sigma \)-continuous but not \( \sigma \)-\( \delta \)-continuous.

**Theorem 3.4.** Let \( s, s' \) be \( \sigma \)-structures on \( X \) and \( Y \), respectively. Let \( f : X \rightarrow Y \) be a function. If \( Y \not\in s' \) and \( \emptyset \not\in s' \), then the following are equivalent:

1. \( f \) is \( \sigma \)-\( \delta \)-continuous at \( x \in X \).
2. \( x \in i_\delta(f^{-1}(i_s c_s(V))) \) for every \( \sigma \)-open subset \( V \) containing \( f(x) \).
3. \( x \in i_\delta(f^{-1}(V)) \) for every \( \sigma \)-regular open subset \( V \) containing \( f(x) \).
4. For \( \sigma \)-regular open set \( V \) containing \( f(x) \), there exists a \( \sigma \)-regular open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

**Proof.**

(1) \( \Rightarrow \) (2) Let \( V \) be any \( \sigma \)-open set containing \( f(x) \). Then by (1), there exists a \( \sigma \)-open set \( U \) of \( X \) containing \( x \) such that \( f(i_s c_s(U)) \subseteq i_s c_s(V) \). It implies that \( x \in U \subseteq i_s c_s(U) \subseteq f^{-1}(i_s c_s(V)) \). Since \( i_s c_s(U) \) is \( \sigma \)-regular open, by Theorem 2.13 (1), \( x \in i_\delta(f^{-1}(i_s c_s(V))) \).

(2) \( \Rightarrow \) (3) Let \( V \) be any \( \sigma \)-regular open set containing \( f(x) \). Then \( V = i_s c_s(V) \), and by (2), it is obtained \( x \in i_\delta(f^{-1}(V)) \).

(3) \( \Rightarrow \) (4) Let \( V \) be any \( \sigma \)-regular open set containing \( f(x) \). From the condition (3), there exists a \( \sigma \)-regular open set \( U \) containing \( x \) such that \( U \subseteq f^{-1}(V) \).
(4) ⇒ (1) Let $V$ be any $\sigma$-open set containing $f(x)$. Then $i_\delta c_\delta (V)$ is a $\sigma$-regular open set and $f(x) \in V \subseteq i_\delta c_\delta (V)$. By (4), there exists a $\sigma$-regular open set $U$ containing $x$ such that $f(U) \subseteq i_\delta c_\delta (V)$. Since every $\sigma$-regular open set is $\sigma$-open and $U = i_\delta c_\delta (U)$, $f$ is $\sigma$-$\delta$-continuous at $x \in X$. \qed

**Theorem 3.5.** Let $s, s'$ be $\sigma$-structures on $X$ and $Y$, respectively. Let $f : X \rightarrow Y$ be a function. If $Y \notin s'$ and $\emptyset \notin s'$, then the following are equivalent:

1. $f$ is $\sigma$-$\delta$-continuous.
2. For each $x \in X$ and each $\sigma$-regular open set $V$ containing $f(x)$, there exists a $\sigma$-regular open set $U$ containing $x$ such that $f(U) \subseteq V$.
3. $f(c_\delta A) \subseteq c_\delta f(A)$ for every $A \subseteq X$.
4. $c_\delta f^{-1}(B) \subseteq f^{-1}(c_\delta B)$ for every $B \subseteq Y$.
5. For every $V \in \delta$ in $Y$, $f^{-1}(V) \in \delta$.
6. For every $\delta$-closed set $F$ in $Y$, $f^{-1}(F)$ is $\delta$-closed.
7. For every $\sigma$-regular open subset $V$ in $Y$, $f^{-1}(V) \in \delta$.
8. For every $\sigma$-regular closed subset $F$ in $Y$, $f^{-1}(F)$ is $\delta$-closed.

**Proof.** (1) ⇔ (2) From Theorem 3.4, it is obtained.

(2) ⇒ (3) For $A \subseteq X$, we will show that for every $x \in c_\delta A$, $f(x) \in c_\delta f(A)$. For each $x \in c_\delta A$ and each $\sigma$-regular open set $V$ containing $f(x)$, by (2), there exists a $\sigma$-regular open set $U$ containing $x$ such that $f(U) \subseteq V$. From $x \in c_\delta A$, obviously $U \cap A \neq \emptyset$. Thus from this fact and $f(U) \subseteq V$, it follows $V \cap f(A) \neq \emptyset$, and so $f(x) \in c_\delta f(A)$.

(3) ⇒ (4) By (3), $f(c_\delta f^{-1}(B)) \subseteq c_\delta f(f^{-1}(B)) \subseteq c_\delta B$ for each $B \subseteq Y$. It implies $c_\delta f^{-1}(B) \subseteq f^{-1}(c_\delta B)$.

(4) ⇒ (5) Obvious.

(5) ⇔ (6) It is obvious.

(5) ⇒ (7) Let $V$ be any $\sigma$-regular open set in $Y$. By Remark 2.14, $V$ is $\delta$-open and so, $f^{-1}(V) \in \delta$.

(7) ⇔ (8) Obvious.

(7) ⇒ (2) It is obvious from Theorem 2.3. \qed

**Definition 3.6.** Let $s$ be $\sigma$-structures on $X$ and $S_s = \bigcup_{U \in s} U$. Then $X$ is said to be relative $\sigma$-$G$-regular (simply, $\sigma$-$G$-regular) on $S_s$ if for $x \in S_s$ and $\sigma$-closed set $F$ with $x \notin F$, there exist $\sigma$-open sets $U$ and $V$ such that $x \in U$, $F \cap S_s \subseteq V$ and $U \cap V = \emptyset$. 
Theorem 3.7. Let \( s \) be \( \sigma \)-structures on \( X \). Then \( X \) is \( \sigma \)-G-regular if and only if for each \( x \in S_s \) and each \( \sigma \)-open set \( V \) containing \( x \), there is a \( \sigma \)-open set \( U \) containing \( x \) such that \( x \in U \subseteq c_s(U) \cap S_s \subseteq V \).

Proof. Assume that \( X \) is \( \sigma \)-G-regular. Then for \( x \in S_s \) and any \( \sigma \)-open set \( V \) containing \( x \), \( x \) and the \( \sigma \)-closed set \( V^c \) have disjoint \( \sigma \)-open sets \( U, W \) such that \( x \in U \) and \( V^c \cap S_s \subseteq W \). Since \( U \subseteq W^c \), we have \( c_s(U) \cap (V^c \cap S_s) \subseteq c_s(U) \cap W = \emptyset \). So \( c_s(U) \cap S_s \subseteq V \).

For the converse, let \( F \) be any \( \sigma \)-closed set and \( x \notin F \) for \( x \in S_s \). Then since \( F^c \) is a \( \sigma \)-open set containing \( x \), by hypothesis, there is a \( \sigma \)-open set \( U \) containing \( x \) such that \( x \in U \subseteq c_s(U) \cap S_s \subseteq F^c \), so \( c_s(U) \cap S_s \cap F = \emptyset \). Put \( V = (c_s(U))^c \). Then \( V \) is \( \sigma \)-open, \( U \cap V = \emptyset \) and \( S_s \cap F \subseteq V \). Hence \( X \) is \( \sigma \)-G-regular. \( \square \)

Theorem 3.8. Let \( s \) be \( \sigma \)-structures on \( X \). If \( X \) is \( \sigma \)-G-regular, every \( \sigma \)-open set is \( \delta \)-open.

Proof. Let \( A \) be any \( \sigma \)-open set in \( X \) and \( x \in A \). Then from the above theorem, there exists a \( \sigma \)-open set \( V \) such that \( x \in V \subseteq c_s(V) \cap S_s \subseteq A \). Then it implies \( x \in V \subseteq i_s c_s(V) \cap V \subseteq i_s c_s V \cap (c_s(V) \cap S_s) \subseteq A \). Since \( i_s c_s(V) \cap S_s = i_s c_s(V) \) and \( i_s c_s(V) \) is \( \sigma \)-regular open, from Theorem 2.3, \( A \) is \( \delta \)-open. \( \square \)

Theorem 3.9. Let \( s, s' \) be \( \sigma \)-structures on \( X \) and \( Y \), respectively. Let \( f : X \to Y \) be a function. If \( X \) is \( \sigma \)-G-regular and \( \emptyset \notin s \), then

\( f \) is \( \sigma \)-\( \delta \)-continuous if and only if for every \( \sigma \)-regular open set \( V \neq \emptyset \) in \( Y \), \( f^{-1}(V) \) is \( \sigma \)-open.

Proof. It follows from Theorem 3.5 and Theorem 3.8. \( \square \)

Theorem 3.10. ([7]) Let \( s, s' \) be \( \sigma \)-structures on \( X \) and \( Y \), respectively. If \( f : X \to Y \) is a function, then \( f \) is almost \( \sigma \)-continuous iff for every \( \sigma \)-regular open subset \( V \) in \( Y \), \( f^{-1}(V) \) is \( \sigma \)-open.

Remark 3.11. Let \( s \) be a \( \sigma \)-structure on a nonempty set \( X \) and \( A \subseteq X \). Then \( A \) is said to be \( \sigma \)-semiopen [3] (resp., \( \sigma \)-preopen [4], \( \sigma \)-\( \beta \)-open [5], \( \sigma \)-regular open [7]) if \( A \subseteq c_s(i_s(A)) \) (resp., \( A \subseteq i_s(c_s(A)) \), \( A \subseteq c_s(i_s(c_s(A))) \), \( A = i_s(c_s(A)) \)). The family of all \( \sigma \)-preopen sets (resp., \( \sigma \)-semiopen sets, \( \sigma \)-\( \beta \)-open sets, \( \sigma \)-regular open sets) in \( X \) is denoted by \( \sigma PO(X) \) (resp., \( \sigma SO(X) \), \( \sigma \beta O(X) \), \( \sigma RO(X) \)). Then, in [6], we obtained the following relations:

1. In case \( X \notin s \) and \( \emptyset \notin s \):
σRO(X) ⊆ σδ; \quad σRO(X)−{∅} ⊆ σδ−{∅} ⊆ s;

s ⊆ σSO(X) ⊆ σβO(X); \quad s ⊆ σPO(X) ⊆ σβO(X).

(2) In case \( X ∈ s \) and \( ∅ /∈ s \):

σRO(X)−{X} ⊆ σδ; \quad σRO(X)−{∅} ⊆ σδ−{∅} ⊆ s;

σRO(X)−{X, ∅} ⊆ σδ−{∅} ⊆ s;

s ⊆ σSO(X) ⊆ sβO(X); \quad s ⊆ σPO(X) ⊆ sβO(X).

(3) In case \( ∅ ∈ s \):

σRO(X) ⊆ σδ ⊆ s ⊆ σSO(X) ⊆ σβO(X);

σ ⊆ σPO(X) ⊆ σβO(X).

**Theorem 3.12.** Let \( s, s' \) be \( σ \)-structures on \( X \) and \( Y \), respectively. Let \( f : X → Y \) be a function. If \( X \) and \( Y \) are \( σ \)-\( G \)-regular and \( ∅ /∈ s \), then

(1) \( f \) is \( σ-δ \)-continuous.

(2) For every \( σ \)-regular open set \( V /≠ ∅ \) in \( Y \), \( f^{-1}(V) \) is \( σ \)-open.

(3) For every \( σ \)-open set \( V /≠ ∅ \) in \( Y \), \( f^{-1}(V) \) is \( σ \)-open.

**Proof.** It follows from Theorem 3.9 and Remark 3.11.

**Theorem 3.13.** Let \( s, s' \) be \( σ \)-structures on \( X \) and \( Y \), respectively. Let \( f : X → Y \) be a function. If \( X \) and \( Y \) are \( σ \)-\( G \)-regular and \( ∅ ∈ s \), then the following are equivalent:

(1) \( f \) is \( σ-δ \)-continuous.

(2) \( f \) is \( σ \)-continuous.

(3) \( f \) is almost \( σ \)-continuous.

**Proof.** It follows from Remark 3.11 and Theorem 3.12.

**References**


Received: November 14, 2015; Published: January 14, 2016