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Jacobi Elliptic Function Solutions of a Fractional Nonlinear Evolution Equation

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Abstract

In this paper, we employ a mapping method to solve fractional Ostrovski equation. We derive Jacobi elliptic function solutions and deduce the trigonometric function solutions, solitary wave solutions and the singular wave solutions when the modulus of the elliptic functions approach 0 or 1. The solitary wave solutions and singular wave solutions have been plotted for different values of the parameters for both the Fractional Ostrovski equation as well as the Ostrovsky equation.

Keywords: Fractional Ostrovsky equation, Jacobi elliptic functions, solitary wave solutions, singular wave solutions

1 Introduction

One of the hot topics of research in applied mathematics is the study of nonlinear phenomena in different physical situations. There has been a significant progress in research on exact solutions of nonlinear evolution equations (NLEEs) [4,5,6,8,11,20] in the past few decades. NLEEs are the governing equations in various areas of physical, chemical, biological and geological sciences. For example in physics, NLEEs appear in the study of nonlinear optics, plasma physics, fluid dynamics etc. and in geological sciences in the dynamics of magma.

The important question that arises is about the integrability of these NLEEs. Several methods have been developed for finding exact solutions. Some of these commonly used techniques are tanh method [12], extended tanh method [1,19], exponential function method [18], G'/G expansion method [2,16,17], Mapping methods [9,14,15].

In this paper, we derive periodic wave solutions (PWSs) of a fractional Ostrovsky equation in terms of Jacobi elliptic functions (JEFs) [10] and deduce their infinite period counterparts in terms of hyperbolic functions such as solitary wave solutions (SWSs) and singular wave solutions using a mapping method. We also derive trigonometric functions solutions (TFSs) as a special case of the PWSs. The mapping method employed in this paper give a variety of solutions which other methods cannot.

The paper is organised as follows: In section 2, we give a definition of Riemann-Liouville fractional derivative [7], a mathematical analysis of the mapping method and also an introduction to JEFs. In section 3, we solve the fractional Ostrovski equation using the mapping method and obtain a variety of PWSs and their special cases such as TFSs, SWSs and singular wave solutions.

2 Mathematical analysis

In this section, we give an analysis of the mapping method which will be employed in this paper.

We consider the nonlinear FPDE

$$F(u, D_t^\alpha u, D_x^\alpha u, D_x^{2\alpha} u, D_x^{3\alpha} u, ...) = 0, \;\; 0 < \alpha \leq 1. \tag{1}$$

where the unknown function u depends on the space variable x and time variable t.

Here, the Riemann-Liouville fractional derivatives are given by

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \;\; D_x^\alpha x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha}. \tag{2}$$

We consider the TWS in the form

$$u(x,t) = u(\xi), \;\; \xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\, t^\alpha}{\Gamma(1+\alpha)}, \tag{3}$$

where c is the wave speed.

Substituting eq. (3) into eq. (1), the PDE reduces to an ODE and then we search for the solution of the ODE in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i(\xi), \tag{4}$$

where n is a positive integer which can be determined by balancing the linear term of the highest order with the nonlinear term. A_i are constants to be determined.

Here, f satisfies the equation

$$f'^2 = pf + qf^2 + rf^3, \tag{5}$$

where p, q, r are parameters to be determined.

After substituting eq. (4) into the reduced ODE and using eq. (5), the constants A_i, p, q, r can be determined.

The mapping relation is thus established through eq. (4) between the solution to eqn. (5) and that of eq. (1).

The motivation for the choice of f is from the fact that the squares of the first derivatives of JEFs satisfy eq. (5) and so we can express the solutions of eq. (1) in terms of those functions.

Some of the properties of JEFs with modulus $m (0 < m < 1)$ are as follows:

$$\begin{aligned} &\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1,\ \ \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1 \\ &\mathrm{ns}\,\xi = \frac{1}{\mathrm{sn}\,\xi}, \mathrm{nc}\,\xi = \frac{1}{\mathrm{cn}\,\xi}, \mathrm{nd}\,\xi = \frac{1}{\mathrm{dn}\,\xi} \\ &\mathrm{sc}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}, \mathrm{sd}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}, \mathrm{cd}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{dn}\,\xi} \\ &\mathrm{cs}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}, \mathrm{ds}\,\xi = \frac{\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}, \mathrm{dc}\,\xi = \frac{\mathrm{dn}\,\xi}{\mathrm{cn}\,\xi} \end{aligned} \tag{6}$$

The derivatives of JEFs are given by

$$(\mathrm{sn}\,\xi)' = \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi,\ \ (\mathrm{cn}\,\xi)' = -\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi,\ \ (\mathrm{dn}\,\xi)' = -\mathrm{m}^2\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi. \tag{7}$$

When $m \to 0$, the JEFs degenerate to the triangular functions, that is,

$$\mathrm{sn}\,\xi \to \sin\xi,\ \ \mathrm{cn}\,\xi \to \cos\xi,\ \ \mathrm{dn}\,\xi \to 1 \tag{8}$$

and when $m \to 1$, the JEFs degenerate to the hyperbolic functions, that is,

$$\mathrm{sn}\,\xi \to \tanh\xi,\ \ \mathrm{cn}\,\xi \to \mathrm{sech}\,\xi,\ \ \mathrm{dn}\,\xi \to \mathrm{sech}\,\xi. \tag{9}$$

3 Fractional Ostrovski equation

The Ostrovski equation [3,13]

$$\left(u_t - \lambda u_{xxx} + (u^2)_x\right)_x = \mu u \tag{10}$$

is a model equation for the unidirectional propagation of weakly nonlinear long surface and internal waves of small amplitude in a rotating fluid. In this equation, $u(x,t)$ represents the surface of the incompressible and inviscid liquid, λ and μ measure the effect of dispersion and rotation respectively.

The fractional Ostrovski equation can be written as

$$D_x^{\alpha}\left(D_t^{\alpha}u - \lambda D_x^{3\alpha}u + D_x^{\alpha}(u^2)\right) = \mu u. \tag{11}$$

Substituting the TWS given by eq. (3) into eq. (11), we obtain the ODE

$$-c\,\frac{d^2u}{d\xi^2} - \lambda\,\frac{d^4u}{d\xi^4} + 2u\,\frac{d^2u}{d\xi^2} + 2\left(\frac{du}{d\xi}\right)^2 = \mu\, u. \tag{12}$$

Substitution of eq. (4) into the above ODE and balancing the highest order linear term with the nonlinear terms, we get $n = 1$.

So, we can assume the solution of eq. (12) in the form

$$u(\xi) = A_0 + A_1 f \tag{13}$$

where f and its higher order derivatives are given by

$$f'^2 = pf + qf^2 + rf^3, \tag{14}$$

$$f'' = \frac{p}{2} + qf + \frac{3}{2}rf^2 \tag{15}$$

$$f^{(4)} = \frac{pq}{2} + \left(q^2 + \frac{9}{2}pr\right)f + \frac{15}{2}qrf^2 + \frac{15}{2}r^2f^3. \tag{16}$$

Substituting eq. (13) into eq. (12) and using eqs. (14 - 16), we arrive at an algebraic equation in powers of f. Equating the coefficients of powers of f on both sides leads us to a set of algebraic equations given by

$$-\frac{15}{2}\lambda A_1 r^2 + 5A_1^2 r = 0, \tag{17}$$

$$-\frac{3}{2}cA_1 r - \frac{15}{2}\lambda A_1 qr + 2A_1^2 q + 3A_0A_1 r + 2A_1^2 q = 0, \tag{18}$$

$$-cA_1q - \lambda A_1\left(q^2 + \frac{9}{2}pr\right) + 2A_0A_1q + 3A_1^2p = \mu A_1, \tag{19}$$

$$-\frac{c}{2}A_1p - \frac{\lambda}{2}A_1pq + A_0A_1p = \mu A_0. \tag{20}$$

Eqs. (17) and (18) give

$$A_0 = \frac{c+\lambda q}{2}, \quad A_1 = \frac{3}{2}\lambda r. \tag{21}$$

Substituting for A_0 and A_1 in eq. (19), we can easily see that $\mu = 0$. So, the solutions we derive in this paper are all for the case of no rotation. Eq. (20) also gives the same A_0 as given above.

So, our equation under consideration reduces to

$$D_x^{\alpha}\left(D_t^{\alpha}u - \lambda D_x^{3\alpha}u + D_x^{\alpha}(u^2)\right) = 0. \tag{22}$$

Case 1 : $p = 4,\ q = -4(1+m^2),\ r = 4m^2.$

Eq. (5) has two solutions $f(\xi) = \mathrm{sn}^2(\xi)$ and $f(\xi) = \mathrm{cd}^2(\xi)$. So, we obtain the PWSs of eq. (22) as

$$u(x,t) = \frac{c-4\lambda(1+m^2)}{2} + 6m^2\lambda\,\mathrm{sn}^2\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}\right) \tag{23}$$

and

$$u(x,t) = \frac{c-4\lambda(1+m^2)}{2} + 6m^2\lambda\,\mathrm{cd}^2\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}\right). \tag{24}$$

As $m \to 1$, eqs. (23) will give rise to the SWS

$$u(x,t) = \frac{c+4\lambda}{2} - 6\lambda\,\mathrm{sech}^2\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}\right). \tag{25}$$

Case 2 : $p = 4(1-m^2),\ q = 4(2m^2-1),\ r = -4m^2.$

Here, eq. (5) has the solution $f(\xi) = \mathrm{cn}^2(\xi)$. So, we have the PWS of eq. (22) as

$$u(x,t) = \frac{c+4\lambda(2m^2-1)}{2} - 6m^2\lambda\,\mathrm{cn}^2\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}\right). \tag{26}$$

As $m \to 1$, eq. (26) leads us to the same SWS given by eq. (25).

Case 3 : $p = 4(m^2 - 1),\ \ q = 4(2 - m^2),\ \ r = -4.$

In this case, eq. (5) has the solution $f(\xi) = \mathrm{dn}^2(\xi)$. So, we have the PWS of eq. (22) as

$$u(x,t) = \frac{c + 4\lambda(2 - m^2)}{2} - \lambda\,\mathrm{dn}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{27}$$

In the infinite period limit, eq. (27) leads us to the same SWS as eq. (25).

Case 4 : $p = 4m^2,\ \ q = -4(1 + m^2),\ \ r = 4.$

Thus eq. (5) has two solutions $f(\xi) = \mathrm{ns}^2(\xi)$ and $f(\xi) = \mathrm{dc}^2(\xi)$. So, the PWSs of eq. (22) are

$$u(x,t) = \frac{c - 4\lambda(1 + m^2)}{2} + 6\lambda\,\mathrm{ns}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{28}$$

and

$$u(x,t) = \frac{c - 4\lambda(1 + m^2)}{2} + 6\lambda\,\mathrm{dc}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{29}$$

As $m \to 0$, eqs. (28) and (29) lead us to the TFSs

$$u(x,t) = \frac{c - 4\lambda}{2} + 6\lambda\csc^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{30}$$

and

$$u(x,t) = \frac{c - 4\lambda}{2} + 6\lambda\sec^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{31}$$

As $m \to 1$, eqs. (28) gives rise to the singular wave solution along the curve $x^\alpha = c\,t^\alpha$ given by

$$u(x,t) = \frac{c + 4\lambda}{2} + 6\lambda\,\mathrm{csch}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{32}$$

Case 5 : $p = -4m^2,\ \ q = 4(2m^2 - 1),\ \ r = 4(1 - m^2).$

Here, eq. (5) has the solution $f(\xi) = \mathrm{nc}^2(\xi)$. So, the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2m^2 - 1)}{2} + 6\lambda(1 - m^2)\,\mathrm{nc}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{33}$$

which as $m \to 0$ gives rise to the TFS (31).

Case 6 : $p = -4,\ \ q = 4(2 - m^2),\ \ r = 4(m^2 - 1).$

Thus eq. (5) has the solution $f(\xi) = \mathrm{nd}^2(\xi)$. In this case, the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2 - m^2)}{2} + 6\lambda(m^2 - 1)\,\mathrm{nd}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{34}$$

which gives rise to only trivial solutions in the limiting cases.

Case 7 : $p = 4,\ \ q = 4(2 - m^2),\ \ r = 4(1 - m^2).$

Here, eq. (5) has the solution $f(\xi) = \mathrm{sc}^2(\xi)$. So, the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2 - m^2)}{2} + 6\lambda(1 - m^2)\,\mathrm{sc}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{35}$$

which gives the solution (31) when $m \to 0$.

Case 8 : $p = 4(1 - m^2),\ \ q = 4(2 - m^2),\ \ r = 4.$

Thus eq. (5) has the solution $f(\xi) = \mathrm{cs}^2(\xi)$. In this case, the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2 - m^2)}{2} + 6\lambda\,\mathrm{cs}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{36}$$

As $m \to 0$, eq. (36) will give rise to the TFS

$$u(x,t) = \frac{c + 4\lambda}{2} + 6\lambda\cot^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{37}$$

and as $m \to 1$, eq. (36) will degenerate to the singular wave solution (32).

Case 9 : $p = 4,\ \ q = 4(2m^2 - 1),\ \ r = -4m^2(1 - m^2).$

Here, eq. (5) has the solution $f(\xi) = \mathrm{sd}^2(\xi)$. Thus the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2m^2 - 1)}{2} - 6\lambda m^2(1 - m^2)\,\mathrm{sd}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{38}$$

which gives trivial constant solutions in both limiting cases.

Case 10 : $p = -4m^2(1-m^2), \;\; q = 4(2m^2-1), \;\; r = 4.$

Thus eq. (5) has the solution $f(\xi) = \mathrm{ds}^2(\xi)$. In this case, the PWS of eq. (22) is,

$$u(x,t) = \frac{c + 4\lambda(2m^2-1)}{2} + 6\lambda\,\mathrm{ds}^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right). \tag{39}$$

As $m \to 0$, eq. (39) will lead to the TFS

$$u(x,t) = \frac{c - 4\lambda}{2} + 6\lambda\csc^2\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{c\,t^\alpha}{\Gamma(1+\alpha)}\right) \tag{40}$$

and as $m \to 0$, eq. (39) will give rise to the singular wave solution (32).

4 Conclusion

The fractional Ostrovski equation has been solved using a mapping method which involves JEFs. When the modulus m of the elliptic functions approaches 0, it gives rise to TFSs and when m approches 1, it leads to SWSs as well as singular wave solutions. However, since the solutions are in terms of squared JEFs, the equation under consideration cannot lead to shock wave solutions. Also, the solutions obtained are for the case of no rotation. The effect of rotation is worth investigating in the future.

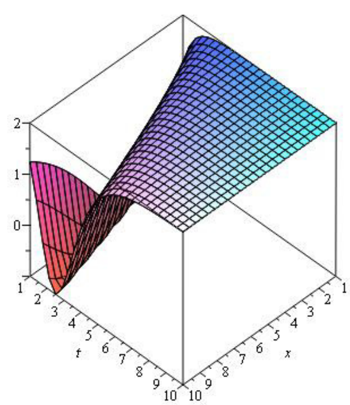

Figure 1: Solitary wave solution (25) with $\alpha = 0.5, \lambda = 0.5, c = 2$

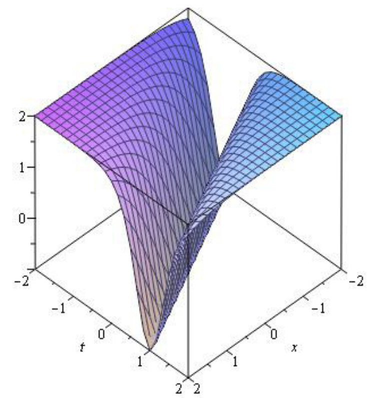

Figure 2: Solitary wave solution (25) with $\alpha = 1, \lambda = 0.5, c = 2$

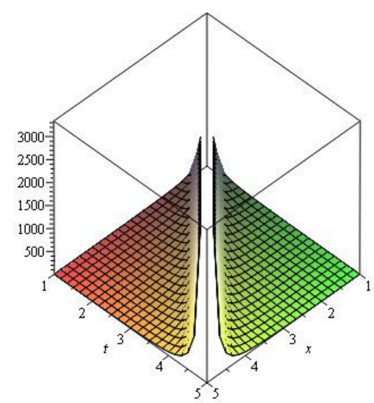

Figure 3: Singular wave solution (32) with $\alpha = 0.5, \lambda = 1, c = 1$

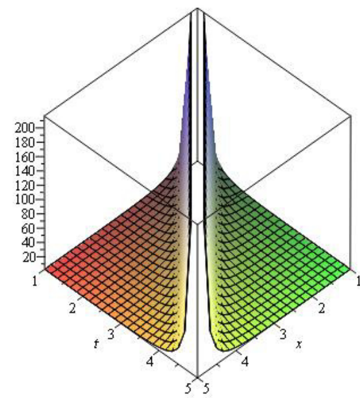

Figure 4: Singular wave solution (32) with $\alpha = 1, \lambda = 1, c = 1$

References

[1] M. Alquran and K. Al Khaled, Mathematical methods for a reliable treatment of the (2 + 1)-dimensional Zoomeron equation, *Mathematical Sciences*, **6** (2012), no. 1, 11. http://dx.doi.org/10.1186/2251-7456-6-11

[2] M. Alquran and A. Qawasmeh, Soliton solutions of shallow water wave equations by means of G'/G expansion method, *Journal of Applied Analysis and Computation*, **4** (2014), 221-229.

[3] A. Biswas and E.V. Krishnan, Exact solutions for Ostrovsky equation, *Indian Journal of Physics*, **85** (2011), 1513-1521. http://dx.doi.org/10.1007/s12648-011-0169-5

[4] A. Biswas and H. Triki, 1-soliton solution of the Klein-Gordon-Schrodinger's equation with power law nonlinearity, *Applied Mathematics and Computation*, **217** (2010), 3869-3874. http://dx.doi.org/10.1016/j.amc.2010.09.046

[5] M.W. Coffey, On series expansions giving closed-form solutions of Korteweg-de Vries-like equations, *SIAM Journal of Applied Mathematics*, **50** (1990), 1580-1592. http://dx.doi.org/10.1137/0150093

[6] C. Deng and Y. Shang, Construction of exact periodic wave and solitary wave solutions for the long-short wave resonance equations by VIM, *Communications in Nonlinear Science and Numerical Simulation*, **14** (2009), 1186-1195. http://dx.doi.org/10.1016/j.cnsns.2008.01.005

[7] M. Eslami, B.F. Vajargah, M. Mirzazadeh and A. Biswas, Application of first integral method to fractional partial differential equations, *Indian*

Journal of Physics, **88** (2014), 177-184.
http://dx.doi.org/10.1007/s12648-013-0401-6

[8] W. Hereman, P.P. Banerjee, A. Korpel, G. Assanto, A. Van Immerzeele and A. Meerpoel, Exact solitary wave solutions of non-linear evolution and wave equations using a direct algebraic method, *Journal of Physics A*, **19** (1986), 607-628. http://dx.doi.org/10.1088/0305-4470/19/5/016

[9] E.V. Krishnan and Y. Peng, A new solitary wave solution for the new Hamiltonian amplitude equation, *Journal of the Physical Society of Japan*, **74** (2005), 896-897. http://dx.doi.org/10.1143/jpsj.74.896

[10] D.F. Lawden, *Elliptic Functions and Applications*, Springer Verlag, New York, 1989. http://dx.doi.org/10.1007/978-1-4757-3980-0

[11] M. Labidi, H. Triki, E.V. Krishnan and A. Biswas, Soliton Solutions of the Long-Short Wave Equation with Power Law Nonlinearity, *Journal of Applied Nonlinear Dynamics*, **1** (2012), 125-140.
http://dx.doi.org/10.5890/jand.2012.05.002

[12] W. Malfliet, The tanh method: 1. Exact solutions of nonlinear evolution and wave equations, *Physica Scripta*, **54** (1996), 563-568.
http://dx.doi.org/10.1088/0031-8949/54/6/003

[13] L.A. Ostrovsky, Nonlinear internal waves in a rotating ocean, *Oceanology*, **18** (1978), 119-125.

[14] Y. Peng, Exact periodic wave solutions to a new Hamiltonian amplitude equation, *J. of the Phys. Soc. of Japan.*, **72** (2003), 1356-1359.
http://dx.doi.org/10.1143/jpsj.72.1356

[15] Y.Z. Peng and E.V. Krishnan, Exact travelling wave solutions to the (3+1)-dimensional Kadomtsev-Petviashvili equation, *Acta Physica Polonica*, **108** (2005), 421-428.

[16] A. Qawasmeh and M. Alquran, Soliton and periodic solutions for $(2+1)$-dimensional dispersive long water-wave system, *Applied Mathematical Sciences*, **8** (2014), 2455-2463. http://dx.doi.org/10.12988/ams.2014.43170

[17] A. Qawasmeh and M. Alquran, Reliable study of some fifth-order nonlinear equations by means of G'/G expansion method and rational sine-cosine method, *Applied Mathematical Sciences*, **8** (2014), 5985-5994.
http://dx.doi.org/10.12988/ams.2014.48669

[18] K.R. Raslan, The Application of Hes Exp-function Method for MKdV and Burgers Equations with Variable Coefficients, *International Journal of Nonlinear Science*, **7** (2009), 174-181.

[19] S. Shukry and K. Al Khaled, The extended tanh method for solving systems of nonlinear wave equations, *Applied Mathematics and Computation*, **217** (2010), 1997-2006. http://dx.doi.org/10.1016/j.amc.2010.06.058

[20] T. Ying-Hui, C. Han-Lin and L. Xi-Qiang, New exact solutions to long-short wave interaction equations, *Communications in Theoretical Physics*, **46** (2006), 397-402. http://dx.doi.org/10.1088/0253-6102/46/3/004

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