Strong and $\Delta-$ Convergence Theorems under a Recent Iterative Scheme in CAT(0) Spaces

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Abstract

In this paper, we prove strong as well as $\Delta-$ convergence theorems in CAT(0) spaces for totally asymptotically nonexpansive nonself mappings under a recent iterative scheme essentially due to Agarwal et al. [3] which is relatively faster as well as independent to Ishikawa iterative scheme. Our results are improvements over several corresponding results contained in [1, 2, 14, 22, 30, 31, 33] and several others.

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1. Introduction

In recent years, CAT(0) spaces have attracted the attention of several researchers as newly proved results in such spaces may play a very important role due to the involvement of varied geometric aspects (cf.[15]). Fixed point theory in CAT(0) spaces was initiated by Kirk ([23], [24]) wherein it is shown that every nonexpansive mapping defined on a bounded, closed and convex subset of a complete CAT (0) space admits a fixed point.

In 1976, Lim [25] introduced a concept of convergence in metric spaces and termed the same as “$\Delta-$ convergence”. In 2008, Kirk and Panyanak [24] specialized Lim’s concept in CAT(0) spaces and also indicated that many Banach
space results involving weak convergence admit precise analogues in this setting. Thereafter, numerous existence results were proved employing \( \Delta \)– convergence of the underlying iterative schemes for several classes of mappings namely: nonexpansive mapping, asymptotically nonexpansive mapping nearly asymptotically nonexpansive and asymptotically nonexpansive mapping in intermediate sense. In the recent past, several convergence results for asymptotically nonexpansive nonself mapping using iteration schemes due to Picard, Mann [26], Ishikawa [21], Agarwal et al. [3] were proved in the framework of CAT(0) spaces and by now there exists considerable literature on this theme (e.g., [1, 2, 10, 14, 22, 33]).

Recently, Yang et al. [33], proved existence theorems besides results on convexity and closedness of fixed point sets in CAT(0) spaces for total asymptotically nonexpansive nonself mappings which is essentially more general than asymptotically nonexpansive nonself mappings. Furthermore, he also studied sufficient conditions for \( \Delta \) and strong convergence of sequence generated by finite (or infinite) family for totally asymptotically nonexpansive nonself mappings in CAT(0) spaces.

Motivated by the above work, the purpose of this paper is to discuss strong and \( \Delta \)– convergence theorems for totally asymptotically nonexpansive nonself mappings for modified Agarwal et al. iterative scheme [4]. Our results are improvements over corresponding results contained in [1, 2, 14, 30, 31, 22, 33].

2. Preliminaries

A metric space \( X \) is called a CAT(0) space (cf.[17]) provided it geodesically connected and every geodesic triangle in \( X \) is at least as “thin” as its comparison triangle in the Euclidean plane. For a rigorous discussion about such spaces, one can be referred to [6] and [9]. A complex Hilbert ball with Hyperbolic metric is a CAT(0) space (e.g., [19, 28]).

Let \((X, d)\) be a metric space. A geodesic path joining \( x \in X \) to \( y \in X \) (or, more briefly, a geodesic from \( x \) to \( y \) in \( X \)) is a map \( c \) from a closed interval \([0, l] \subset \mathbb{R} \) to \( X \) such that \( c(0) = x, c(l) = y, \) and \( d(c(t), c(t')) = |t - t'| \) for all \( t, t' \in [0, l] \).

In particular, \( c \) is an isometry and \( d(x, y) = l \). The image of \( c \) is called a geodesic (or metric) segment joining \( x \) and \( y \). In case this geodesic segment is unique, it is denoted by \([x, y] \). The space \((X, d)\) is said to be geodesic space if every two points of \( X \) are joined by a geodesic, and \( X \) is said to be uniquely geodesic if there is exactly one geodesic joining \( x \) and \( y \) for each \( x, y \in X \). A subset \( Y \subseteq X \) is said to be convex if \( Y \) includes every geodesic segment joining any two of its points. A geodesic triangle \( \Delta(x_1, x_2, x_3) \) in a geodesic metric space \((X, d)\) consists of three points \( x_1, x_2, x_3 \) in \( X \) (referred as the vertices of \( \Delta \)) and a geodesic segment between each pair of vertices (referred as the edges of \( \Delta \)).
A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X,d)$ is a triangle $\Delta(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $E^2$ such that $d_{E^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $(X,d)$ and let $\overline{\Delta}$ be a comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, such that $d(\overline{x}, \overline{y}) \leq d_{E^2}(\overline{x}, \overline{y})$.

If $x, y_1, y_2$ are points in a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$ (CN)

Indeed, this is the (CN) inequality due to Bruhat and Tits [8]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf [9], p. 163).

Following are some elementary facts about CAT(0) spaces available in [14].

**Lemma 2.1.** Let $(X,d)$ be a CAT(0) space. Then

(i) $(X,d)$ is uniquely geodesic.

(ii) Let $p, x, y$ be points of $X$ and $\alpha \in [0, 1]$. Let $m_1$ and $m_2$ be respectively the points in $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then

$$d(m_1, m_2) \leq \alpha d(x, y).$$

(iii) Let $x, y \in X$, $x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$.

(iv) Let $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1-t)d(x, y).$$ (2.1)

For convenience, from now on, we use the notation $(1-t)x \oplus ty$ for the unique point $z$ satisfying (2.1).

Now, we give some definitions and known results needed in the sequel.

**Definition 2.2.** Let $C$ be a nonempty subset of a metric space $X$ and $T : C \to C$ a mapping. A sequence $\{x_n\}$ in $C$ is said to be approximating fixed point sequence of $T$ if

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$
Definition 2.3. Let $C$ be a nonempty subset of a metric space $X$. The mapping $T : C \to C$ is said to be:

(a) nonexpansive if
$$d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in C;$$

(b) uniformly $L$-Lipschitzian if for each $n \in \mathbb{N}$, there exists a positive number $L > 0$ such that
$$d(T^n x, T^n y) \leq L d(x, y) \text{ for all } x, y \in C;$$

(c) asymptotically nonexpansive (cf. [18]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
$$d(T^n x, T^n y) \leq k_n d(x, y) \text{ for all } x, y \in C \text{ and } n \in \mathbb{N};$$

(d) asymptotically nonexpansive in the intermediate sense (cf. [7]) if $T$ is uniformly continuous and following inequality holds:
$$\limsup_{n \to \infty} \sup_{x, y \in C} \{d(T^n x, T^n y) - d(x, y)\} \leq 0 \text{ for all } x, y \in C \text{ and } n \in \mathbb{N};$$

(e) $(\{\mu_n\}, \{v_n\}, \zeta)$-total asymptotically nonexpansive mappings (cf. [5]), if there exist nonnegative sequences $\{\mu_n\}$ and $\{v_n\}$ with $\mu_n \to 0$ and $v_n \to 0$ as $n \to \infty$ and
strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that
$$d(T^n x, T^n y) \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n, \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Remark 2.4. In view of Definition 2.3, every nonexpansive mapping is an asymptotically nonexpansive (set sequence $\{k_n\} = 1$) and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{v_n\}, \zeta)$-total asymptotically nonexpansive mapping (choose $\{\mu_n\} = 0, \{v_n\} = k_n - 1$, for all $n \geq 1$ and $\zeta(t) = t, t \geq 0$).

Let $C$ be nonempty closed subset of a metric space $X$. Recall that $C$ is said to be a retract of $X$ if there exists a continuous maps $P : X \to C$ such that $Px = x, \forall x \in C$. A map $P : X \to C$ is said to be a retraction if $P^2 = P$. It follows that if a map $P$ is retraction, then $Py = y$ for all $y$ in the range of $P$.

Definition 2.5. [10] Let $C$ be nonempty closed subset of a metric space $X$. Let $P : X \to C$ be the nonexpansive retraction of $X$ onto $C$. A map $T : C \to X$ is said to be:

(a) asymptotically nonexpansive nonself if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
$$d(T(PT)^{n-1} x, T(PT)^{n-1} y) \leq k_n d(x, y),$$
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for all $x, y \in C$ and $n \in \mathbb{N}$,

(b) uniformly $L-$ Lipschitzian if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq L \, d(x, y)$$

for all $x, y \in C$ and $n \in \mathbb{N}$ and

(c) $(\{v_n\}, \{\mu_n\}, \zeta) -$ total asymptotically nonexpansive ([33]), if there exist nonnegative sequences $\{\mu_n\}$ and $\{v_n\}$ with $\mu_n \to 0$ and $v_n \to 0$ as $n \to \infty$ and

strictly increasing function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n \quad (2.2)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let $C$ be a nonempty closed convex subset of a CAT(0) space $X$. The following iteration scheme is studied:

$$x_{n+1} = P[(1 - \alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n]$$

$$y_n = P[(1 - \beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n] \quad n \geq 1, \quad (2.3)$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$. This scheme is called modified Agrawal et al. iterative scheme, where $T$ and $P$ are as in Definition 2.5 (c).

**Remark 2.6.** If $T$ is a selfmap, then $P$ becomes identity map so that in Definition 2.5 (a), (b) and (c) coincide with (b), (c) and (e) of Definition 2.3 respectively. Moreover, (2.3) reduces to Agrawal et al. iterative scheme in [3].

In the sequel, we need the following lemmas.

**Lemma 2.7.** (cf.[14], Lemma 2.4) Let $X$ be a CAT(0) space. Then

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

**Lemma 2.8.** (cf.[10], Lemma 3.2) Let $X$ be a CAT(0) space and $x \in X$ a given point. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1 - c) \leq \frac{1}{2}$, if $\{x_n\}$ and $\{y_n\}$ are any sequences in $X$ such that

$$\limsup_{n \to \infty} d(x_n, x) \leq r, \quad \limsup_{n \to \infty} d(y_n, x) \leq r,$$

and

$$\lim_{n \to \infty} d((1 - t_n)x_n \oplus t_ny_n), x) = r,$$

for some $r \geq 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$.

**Lemma 2.9.** (cf.[10] Lemma 3.3) Let $\{a_n\}, \{\lambda_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that

$$a_{n+1} \leq (1 + \lambda_n)a_n + c_n,$$
for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and \( \sum_{n=1}^{\infty} c_n < \infty \), then \( \lim_{n \to \infty} a_n \) exists. If there exists a subsequence of \( \{a_n\} \) which converges to 0, then \( \lim_{n \to \infty} a_n = 0 \).

Now, we recall the concept of \( \Delta- \) convergence besides collecting some of its basic properties.

Let \( C \) be a nonempty subset of a metric space \((X, d)\) and \( \{x_n\} \) a bounded sequence in \( X \). Let \( \text{diam}(C) \) denote the diameter of \( C \). Consider the function \( r_a(., \{x_n\}) \) defined by
\[
 r_a(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x), \quad x \in X.
\]

The infimum of \( r_a(., \{x_n\}) \) over \( C \) is called the asymptotic radius of \( \{x_n\} \) with respect to \( C \) and is denoted by \( r_a(C, \{x_n\}) \). We denote the asymptotic radius of \( \{x_n\} \) with respect to \( X \) by \( r_a(\{x_n\}) \). A point \( z \in C \) is said to be an asymptotic center of the sequence \( \{x_n\} \) with respect to \( C \) if
\[
 r_a(z, \{x_n\}) = \inf_{x \in C} r_a(x, \{x_n\}).
\]

The set of all asymptotic centers of \( \{x_n\} \) with respect to \( C \) is denoted by \( Z_a(C, \{x_n\}) \). The set of all asymptotic centers of \( \{x_n\} \) with respect to \( X \) is the set
\[
 Z_a(\{x_n\}) = \{ z \in X : r_a(z, \{x_n\}) = r_a(\{x_n\}) \}.
\]

It is known (cf.[12], Proposition 7, [13]) that if \( C \) is a nonempty closed, convex subset of a complete CAT(0) space, \( Z_a(C, \{x_n\}) \) (and hence \( Z_a(\{x_n\}) \)) consists of exactly one point.

Next, we define \( \Delta- \) convergence of a sequence in a CAT(0) space.

**Definition 2.10.** (cf.[24], [25]) Let \( \{x_n\} \) be a bounded sequence in a complete CAT(0) space \( X \). Then \( \{x_n\} \) is said to be \( \Delta- \) convergent to \( x \) in \( X \) if \( x \) is the unique asymptotic center of \( \{x_m\} \) for every subsequence \( \{x_m\} \) of \( \{x_n\} \). In this case we write \( \Delta- \lim_{n \to \infty} x_n = x \) and call \( x \) to be the \( \Delta- \) limit of \( \{x_n\} \).

The following Lemmas can be found in [14].

**Lemma 2.11.** (cf.[14], Lemma 2.7)

(i) Every bounded sequence in a complete CAT(0) space \( X \) has a \( \Delta- \) convergent subsequence.

(ii) If \( C \) is a closed convex subset of a complete CAT(0) space \( X \) and \( \{x_n\} \) is a bounded sequence in \( C \), then the asymptotic center of \( \{x_n\} \) lies in \( C \).

Recall that a bounded sequence \( \{x_n\} \) in a complete CAT(0) space \( X \) is said to be regular if \( r_a(X, \{x_n\}) = r_a(X, \{u_n\}) \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). It is also known that every bounded sequence in a Banach space has a regular subsequence. Since every regular sequence \( \Delta- \) converges, we
see immediately that every bounded sequence in a complete CAT(0) space admits a ∆-convergent subsequence. Notice that (e.g., [12], Proposition 7) given a bounded sequence \( \{x_n\} \) in a complete CAT(0) space \( X \) such that \( \{x_n\} \) \( \Delta \)-converges to \( x \) and given \( y \in X \) with \( y \neq x \),

\[
\limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, y).
\]

Clearly, \( X \) satisfies a condition which is well known in Banach space theory as Opail property. We denote \( w_w(x_n) = \bigcup Z_a(\{u_n\}) \), where the union is taken over all subsequences \( \{u_n\} \) of \( \{x_n\} \).

We recall the definition of weak convergence in CAT(0) space.

**Definition 2.12.** (cf, [20]) Let \( C \) be a closed convex subset of a CAT(0) space \( X \). A bounded sequence \( \{x_n\} \) in \( C \) is said to be weakly convergent to \( w \in C \) if and only if \( \phi(w) = \inf_{x \in C} \phi(x) \), where \( \phi(x) = \limsup_{n \to \infty} d(x_n, x) \).

Notice that \( \{x_n\} \rightharpoonup w \) if and only if \( Z_a(C, \{x_n\}) = \{w\} \).

The following Lemma establishes a relationship in between ∆- convergence and weak convergence in a CAT(0) space.

**Lemma 2.13** ([27], Proposition 3.12). Let \( \{x_n\} \) be a bounded sequence in a CAT(0) space \( X \) and \( C \) be a closed subset of \( X \) which contains \( \{x_n\} \). Then

(i) \( \Delta \lim_{n} x_n = x \) implies \( \{x_n\} \rightharpoonup x \),

(ii) The converse of (i) is true \( \{x_n\} \) is regular.

**Lemma 2.14.** [10] Let \( C \) be a closed and convex subset of a complete CAT(0) space \( X \) and \( T : C \to X \) a uniformly \( L \)-Lipschitzian and \( (\{\mu_n\}, \{v_n\}, \zeta) \)-totally asymptotically nonexpansive nonself mapping. If \( \{x_n\} \) is a bounded sequence in \( C \) such that \( \{x_n\} \rightharpoonup q \) and \( d(x_n, Tx_n) = 0 \), then \( Tq = q \).

**Lemma 2.15.** [10] Let \( C \) be a closed and convex subset of a complete CAT(0) space \( X \), and \( T : C \to X \) a asymptotically nonexpansive nonself mapping. if \( \{x_n\} \) is a bounded sequence in \( C \) such that \( \{x_n\} \rightharpoonup q \) and \( d(x_n, Tx_n) = 0 \), then \( Tq = q \).

Recently, Yang et al. [33] established the following existence theorem besides results on convexity and closedness of a fixed point set in CAT(0) spaces in respect of totally asymptotically nonexpansive nonself mappings.

**Theorem 2.16.** ([33], Theorem 3.1, 3.2) Let \( X \) be a complete CAT(0) space and \( C \) a nonempty bounded closed and convex subset of \( X \). If \( T : C \to X \) is a uniformly Lipschitzian and totally asymptotically nonexpansive nonself mapping, then \( T \) has fixed point in \( C \) and set of fixed point is closed as well as convex.

**Theorem 2.17.** Let \( C \) be a nonempty, closed and convex subset of a complete CAT(0) space \( X \) and \( T : C \to X \) a uniformly \( L \)-Lipschitzian \( (\{\mu_n\}, \{v_n\}, \zeta) \)-
totally asymptotically nonexpansive mapping. If \( \{x_n\} \) is a bounded sequence in \( C \) which \( \Delta^- \) converges to \( x \) and \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \), then \( x \in C \) and \( x = Tx \).

**Proof.** Let \( \{x_n\} \) be a bounded sequence in \( C \) which is an approximating fixed point sequence of \( T \) and \( \Delta^- \) converges to \( x \). One can notice (by Lemma 2.11) that \( x \in C \). Also, \( Z_a(\{x_n\}) = \{x\} \), so that \( r_a(x, \{x_n\}) = r_a(\{x\}) \). By Theorem 2.16, we conclude that \( x = Tx \). \( \square \)

### 3. Strong and \( \Delta^- \) convergence Theorems

In this section, we establish \( \Delta^- \) convergence and strong convergence theorems for modified Agrawal et al. sequence for \( (\{v_n\}, \{\mu_n\}, \zeta) \)-totally asymptotically nonexpansive nonself mapping. Let \( F(T) = \{x \in C, Tx = x\} \) denotes the set of fixed points of \( T \).

Now, we are equipped to prove our main result as follows.

**Theorem 3.1.** Let \( C \) be a nonempty closed, convex and bounded subset of a complete CAT(0) space \( X \). If \( T : C \to X \) is a uniformly \( L^- \) Lipschitzian and \( (\{v_n\}, \{\mu_n\}, \zeta) \)-totally asymptotically nonexpansive nonself mapping satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} v_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty; \)

(ii) there exists a constant \( M^* > 0 \) such that \( \zeta(r) \leq M^* r, \forall r \geq 0; \)

(iii) there exist constants \( a, b \in (0, 1) \) with \( 0 < b(1 - c) \leq \frac{1}{2} \) such that \( 0 < a \leq \alpha_n, \beta_n \leq b < 1; \)

If \( F(T) \neq \phi \). Then the following holds:

(a) \( \lim_{n \to \infty} d(x_n, x^*) = \lim_{n \to \infty} d(y_n, x^*) \) exists for \( x^* \in F(T); \)

(b) \( \lim_{n \to \infty} d(x_n, T(PT)^{n-1}x_n) = 0; \)

(c) if \( T \) is uniformly \( L^- \) Lipschitzian, it follows that \( \{x_n\} \) has the property \( \lim_{n \to \infty} d(x_n, Tx_n) = 0; \)

(d) the sequence \( \{x_n\} \) defined by (2.3) \( \Delta^- \) converges to a fixed point of \( T \).

**Proof.** Since \( T : C \to X \) be a \( (\{\mu_n\}, \{v_n\}, \zeta) \)-totally asymptotically nonexpansive nonself mapping, by the condition (ii) ( for any \( x, y \in C \)), we have

\[
\begin{align*}
d(T(PT)^{n-1}x, T(PT)^{n-1}y) & \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n \\
& \leq (1 + v_n M^*) d(x, y) + \mu_n, \quad \forall n \geq 1. \quad (3.1)
\end{align*}
\]
Firstly, we show that \( \lim_{n \to \infty} d(x_n, x^*) \) exists for each \( x^* \in F(T) \). In fact, for each \( x^* \in F(T) \), using (2.3) and (3.1), we have

\[
\begin{align*}
d(y_n, x^*) &= d(P((1 - \beta_n)x_n + \beta_nT(T)^{n-1}x_n), x^*) \\
&\leq d((1 - \beta_n)x_n + \beta_nT(T)^{n-1}x_n), x^*) \\
&\leq (1 - \beta_n)d(x_n, x^*) + \beta_n d(T(T)^{n-1}x_n, x^*) \\
&\leq (1 - \beta_n)d(x_n, x^*) + \beta_n[(1 + v_nM^*d(x_n, x^*) + \mu_n] \\
&\leq d(x_n, x^*) + \beta_n v_nM^*d(x_n, x^*) + \beta_n\mu_n \\
&\leq (1 + \beta_n v_nM^*)d(x_n, x^*) + \beta_n\mu_n \tag{3.2}
\end{align*}
\]

Again, using (2.3) and (3.1), we have

\[
\begin{align*}
d(x_{n+1}, x^*) &= d(P((1 - \alpha_n)T(T)^{n-1}x_n + \alpha_nT(T)^{n-1}y_n), x^*) \\
&\leq d((1 - \alpha_n)T(T)^{n-1}x_n + T(T)^{n-1}y_n), x^*) \\
&\leq (1 - \alpha_n)d(T(T)^{n-1}x_n, x^*) + \alpha_n d(T(T)^{n-1}y_n, x^*) \\
&\leq (1 - \alpha_n)[(1 + v_nM^*)d(x_n, x^*) + \mu_n] \\
&\quad + \alpha_n[(1 + v_nM^*)d(y_n, x^*) + \mu_n] \\
&\leq (1 + v_nM^*)[(1 - \alpha_n)d(x_n, x^*) + \alpha_n d(y_n, x^*) + \mu_n]. \tag{3.3}
\end{align*}
\]

Owing (3.2) and (3.3), we have

\[
\begin{align*}
d(x_{n+1}, x^*) &\leq (1 + v_nM^*)[(1 - \alpha_n)d(x_n, x^*) + \alpha_n(1 + v_n\beta_nM^*)d(x_n, x^*)] \\
&\quad + \mu_n(1 + (1 + v_nM^*)\alpha_n\beta_n) \\
&\leq (1 + v_nM^*)(1 + \alpha_n\beta_n(1 + v_nM^*))d(x_n, x^*) + \mu_n(1 + (1 + v_nM^*)\alpha_n\beta_n) \\
&\leq (1 + v_nM)d(x_n, x^*) + M_1\mu_n. \tag{3.4}
\end{align*}
\]

For some \( M \) and \( M_1 \geq 0 \), (owing to \( \sum_{n=1}^{\infty} v_n < \infty \) and \( \sum_{n=1}^{\infty} \mu_n < \infty \) ), \( \{v_n\} \) and \( \{\mu_n\} \) are bounded. Now, in view of Lemma 2.9, \( \lim_{n \to \infty} d(x_n, x^*) \) exists.

(b) It follows from part (a) that \( \lim_{n \to \infty} d(x_n, x^*) \) exists. Set \( \lim_{n \to \infty} d(x_n, x^*) = r \geq 0 \). Since

\[
d(T(T)^{n-1}x_n, x^*) \leq (1 + v_nM^*)d(x_n, x^*) + \mu_n,
\]

we have that

\[
\limsup_{n \to \infty} d(T(T)^{n-1}x_n, x^*) \leq r
\]

so that

\[
\limsup_{n \to \infty} d(y_n, x^*) \leq r. \tag{3.5}
\]

Hence

\[
\limsup_{n \to \infty} d(T(T)^{n-1}y_n, x^*) \leq \limsup_{n \to \infty}[(1 + v_nM^*)d(y_n, x^*) + \mu_n] \leq r.
\]
Since
\[ \lim_{n \to \infty} d(x_{n+1}, x^*) = \lim_{n \to \infty} d(P((1 - \alpha_n)T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n), x^*) \]
\[ \leq (1 + Mv_n)d(x_n, x^*) + M_1\mu_n = r, \]
it follows (owing to Lemma 2.8) that
\[ \lim_{n \to \infty} d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) = 0. \quad (3.6) \]

From (2.3) and (3.6), we have
\[ d(x_{n+1}, T(PT)^{n-1}x_n) = d(P((1 - \alpha_n)T(PT)^{n-1}x_n \oplus (PT)^{n-1}y_n), T(PT)^{n-1}x_n) \]
\[ \leq \alpha_n d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) \]
\[ \leq b d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) \to 0 \text{ as } n \to \infty. \quad (3.7) \]

Now,
\[ d(x_{n+1}, T(PT)^{n-1}y_n) \leq d(x_{n+1}, T(PT)^{n-1}x_n) + d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) \]
so that from (3.6) and (3.7), we have
\[ d(x_{n+1}, T(PT)^{n-1}y_n) \to 0 \text{ as } n \to \infty. \]

Also,
\[ d(x_{n+1}, x^*) \leq d(x_{n+1}, T(PT)^{n-1}y_n) + d(T(PT)^{n-1}y_n, x^*) \]
\[ \leq d(x_{n+1}, T(PT)^{n-1}y_n) + (1 + v_nM^*)d(y_n, x^*) + \mu_n \quad (3.8) \]
which gives rise (using (3.8)0 that
\[ r \leq \liminf_{n \to \infty} d(y_n, x^*). \quad (3.9) \]

Using (3.5) and (3.9), we obtain
\[ r = \lim_{n \to \infty} d(y_n, x^*) \]
\[ = \lim_{n \to \infty} d(P((1 - \beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n), x^*). \quad (3.10) \]

On making use of Lemma 2.9 in (3.10), we obtain
\[ \lim_{n \to \infty} d(x_n, T(PT)^{n-1}x_n) = 0 \]
so that part (b) is proved.

(c) Since $T$ is uniformly $L-$Lipschitzian, making use of (3.7) and part (b), we have
\[ d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T(PT)^{n}x_{n+1}) \]
\[ + d(T(PT)^{n}x_{n+1}, T(PT)^{n}x_{n}) + d(T(PT)^{n}x_n, x_n) \]
\[ \leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, T(PT)^{n}x_{n+1}) \]
\[ + Ld(x_n, T(PT)^{n-1}x_n) \to 0 \text{ as } n \to \infty \]
so that
\[ \lim_{n \to \infty} d(x_n Tx_n) = 0. \quad (3.11) \]
Hence part (c) is proved.

(d) Finally, we prove that the sequence \( \{x_n\} \) \( \Delta \)-converges to a fixed point of \( T \). For this, first we show that \( w_w(\{x_n\}) \subseteq F(T) \). Let \( u \in w_w(\{x_n\}) \). Then there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( Z_a(C, \{u_n\}) = \{u\} \). By Lemma 2.11, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_n v_n = v \) for some \( v \in C \). In the view of (3.11), \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) and by Theorem 2.17, \( v \in F(T) \) and by part (a), \( \lim_{n \to \infty} d(x_n, v) \) exists. We now claim that \( u = v \). Suppose, to the contrary, that \( u \neq v \). Then by uniqueness of asymptotic centers, we have

\[
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, u) \leq \limsup_{n \to \infty} d(u_n, u) \\
\leq \limsup_{n \to \infty} d(u_n, v) = \limsup_{n \to \infty} d(x_n, v) \\
= \limsup_{n \to \infty} d(v_n, v),
\]
a contradiction. Thus, \( u = v \in F(T) \) and hence \( w_w(\{x_n\}) \subseteq F(T) \). To show that \( \{x_n\} \) \( \Delta \)-converges to a fixed point of \( T \), it suffices to show that \( w_w(\{x_n\}) \) consists of exactly one point. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \). By Lemma 2.11, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_n v_n = v \) for some \( v \in C \). Let \( Z_a(C, \{u_n\}) = \{u\} \) and \( Z_a(C, \{x_n\}) = \{x\} \). We have already seen that \( u = v \) and \( v \in F(T) \). Finally we claim that \( x = v \). Suppose not, then by the existence of \( \lim_{n \to \infty} d(x_n, v) \) and uniqueness of asymptotic centers, we have that

\[
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, x) \leq \limsup_{n \to \infty} d(x_n, x) \\
\leq \limsup_{n \to \infty} d(x_n, v) = \limsup_{n \to \infty} d(v_n, v),
\]
a contradiction and hence \( x = v \in F(T) \). Therefore, \( w_w(\{x_n\}) = \{x\} \).

Thus, the proof of Theorem 3.1 is completed.

Notice that Theorem 3.1 extends corresponding results of Chang et al. [11], Abbas et al. [22] and Yang et al. [33] to modified Agarwal et al. [3] which is faster and independent of Ishikawa iterative scheme.

Next, we prove strong convergence theorem under modified Agrawal et al. iteration process defined by (2.3) for \( \{\mu_n\}, \{v_n\}, \zeta \) - totally asymptotically nonexpansive nonself mapping.

**Theorem 3.2.** Let \( X \) be a complete CAT(0) space with \( C, T, \{\alpha_n\} \) and \( \{\beta_n\} \) be the same as in Theorem 3.1. Then the sequence \( \{x_n\} \) defined by (2.3) converges strongly to a fixed point of \( T \) if and only if \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \).

**Proof.** Necessity part is obvious. Conversely, suppose that \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \). From (3.4), we have

\[
d(x_{n+1}, F(T)) \leq (1 + Mv_n) d(x_n, F(T)) + M_1 \mu_n, \ n \in \mathbb{N},
\]
so that \( \lim_{n \to \infty} d(x_n, F(T)) \) and \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \). Following arguments similar to those given in [Lemma 5, [16], Theorem 4.3, [1] ], we obtain
\[
d(x_{n+m}, p) \leq L \left[ d(x_n, p) + \sum_{j=n}^{\infty} b_j \right],
\]
for every \( p \in F(T) \) and for all \( m, n \geq 1 \), where \( L = e^{M(\sum_{j=n}^{n+m-1} v_n)} > 0 \) and \( b_j = M_1 \mu_j \). As, \( \sum_{n=1}^{\infty} v_n < \infty \) so \( L^* = e^{\sum_{n=1}^{\infty} v_n} \geq L = e^{\sum_{j=n}^{n+m-1} v_j} > 0 \). Let \( \epsilon > 0 \) be arbitrarily chosen. Since \( \lim_{n \to \infty} d(x_n, F) = 0 \) and \( \sum_{n=1}^{\infty} \mu_n < \infty \), there exists a positive integer \( n_0 \) such that
\[
d(x_n, F) < \frac{\epsilon}{4L^*} \text{ and } \sum_{j=n_0}^{\infty} b_j < \frac{\epsilon}{6L^*}, \forall n \geq n_0.
\]
In particular, \( \inf \{d(x_{n_0}, p) : p \in F\} < \frac{\epsilon}{4L^*} \). Thus there must exist \( p^* \in F \) such that
\[
d(x_{n_0}, p^*) < \frac{\epsilon}{3L^*}.
\]
Hence for \( n \geq n_0 \), we have
\[
d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(p^*, x_n)
\leq 2L^* \left[ d(x_{n_0}, p^*) + \sum_{j=n_0}^{\infty} b_j \right]
\leq 2L^* \left( \frac{\epsilon}{3L^*} + \frac{\epsilon}{6L^*} \right) = \epsilon.
\]
Hence \( \{x_n\} \) is a Cauchy sequence in closed subset \( C \) of a complete \( \text{CAT}(0) \) space which converges strongly to a point \( q \) in \( C \). Now, \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) gives rise \( d(q, F(T)) = 0 \). Since \( F(T) \) is closed, we have \( q \in F(T) \). This concludes the proof. \( \square \)

Recall that a mapping \( T \) from a subset of a metric space \( (X, d) \) into itself with \( F(T) \neq \emptyset \) is said to be satisfy condition (A) (see [32]) if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0, f(t) > 0 \) for \( t \in (0, \infty) \) such that
\[
d(x, Tx) \geq f(d(x, F(T))) \text{ for all } x \in C.
\]
Finally, we prove the following:
Theorem 3.3. Let $X$ be a complete $\text{CAT}(0)$ space with $C$, $T$, $\{\alpha_n\}$ and $\{\beta_n\}$ be the same as in Theorem 3.1. Suppose that $T$ satisfies conditions (A). Then the sequence $\{x_n\}$ defined by (2.3) converges strongly to a fixed point of $T$.

Proof. By part (c) of Theorem 3.1, we have $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Further, by the condition (A),

$$\lim_{n \to \infty} d(x_n, Tx_n) \geq \lim_{n \to \infty} f(d(x_n, F(T))),$$

so that $\lim_{n \to \infty} d(x_n, F(T)) = 0$. Therefore, this result follows from Theorem 3.2. □

In view of Definitions 2.5, every nonexpansive mapping is an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} = 1$ while every asymptotically nonexpansive nonself mapping is a $(\{\mu_n\}, \{v_n\}, \zeta)$-totally asymptotically nonexpansive nonself mapping with $\mu_n = 0$, $v_n = k_n - 1$, for all $n \geq 1$ and $\zeta(t) = t, t \geq 0$. Thus, Theorems 3.1, 3.2 and 3.3 extends corresponding results of Chidume et al. [11] from Banach spaces to $\text{CAT}(0)$ spaces. It also extends corresponding results of Dhompongsa et al. [14] from the class of nonexpansive mapping to the class of nonexpansive nonself mapping. Our results are improvements over corresponding results contained in [1, 2, 10, 14, 22, 24, 33] due to the involvement of a relatively faster and independent iteration scheme as compared to Ishikawa iterative scheme.

References


Strong and $\Delta-$ convergence theorems


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