A Note on Annular Bound for the Zeros of a Polynomial

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Abstract
In this paper we obtain an annular bound for the zeros of a polynomial, based on the identity related to the generalized Fibonacci sequence. The result of this paper includes several recently reported ones as special cases.

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1 Introduction

During the last few decades, the region containing all the zeros of a polynomial has been an extensive area of research for both engineers and mathematicians. Among numerous results reported in the literature, the following theorem due to Cauchy [9] is classical and well known in the theory of zeros of a polynomial.

Theorem 1.1 A polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) has all its zeros in the annulus \( C = \{ z \in \mathbb{C} : \lambda_1 < |z| < \lambda_2 \} \), where

\[
\lambda_1 = \frac{1}{1 + \max_{1 \leq k \leq n} |d_k|/|d_0|},
\]

\[
\lambda_2 = 1 + \max_{0 \leq k \leq n-1} |d_k|/|d_n|.
\]
Theorem 1.1 is attractive in that it provides an explicit annular bound for the zeros of a polynomial in a very simple form via polynomial coefficients. However, the annular bound given in Theorem 1.1 is generally too crude, and some attempts have been made to get improved bounds [5], [7], [13].

Recently, several interesting results concerning the zeros of a polynomial have been reported in the literature [2], [4], [11]. In those works, an explicit annular bound for the zeros of a polynomial was computed, based on the identity related to the Fibonacci sequence \( \{F_n\}_{n=0}^\infty \) defined by [8]

\[
F_0 = 0, \; F_1 = 1, \; F_n = F_{n-1} + F_{n-2} \; (n \geq 2),
\]
or generalized Fibonacci sequence \( \{F_n^{(a,b,c)}\}_{n=0}^\infty \) of the form

\[
F_0^{(a,b,c)} = 0, \; F_1^{(a,b,c)} = 1,
\]

\[
F_n^{(a,b,c)} = \begin{cases} 
  aF_{n-1}^{(a,b,c)} + cF_{n-2}^{(a,b,c)}, & \text{if } n \text{ is even} \\
  bF_{n-1}^{(a,b,c)} + cF_{n-2}^{(a,b,c)}, & \text{if } n \text{ is odd} \quad (n \geq 2),
\end{cases}
\]

where \( a, b, c > 0 \).

Using the well-known identity [8]

\[
\sum_{k=1}^{n} 2^{n-k} 3^k F_k C_n^k = F_{4n}, \quad (1)
\]

where \( C_n^k = \frac{n!}{(n-k)!k!} \), Díaz-Barrero [2] showed that a polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) has all its zeros in the annulus \( C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\} \), where

\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{2^{n-k} 3^k F_k C_n^k |d_0|}{F_{4n}} \left| \frac{d_k}{|d_k|} \right| \right\}^{\frac{1}{k}},
\]

\[
r_2 = \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^{n-k} 3^k F_k C_n^k |d_n|} \left| \frac{d_{n-k}}{|d_n|} \right| \right\}^{\frac{1}{k}}.
\]

Bidkham and Shashahani [4] generalized the above result by proving the identity

\[
\sum_{k=1}^{n} (a^2 + 1)^{n-k} (a^3 + 2a^2)^k F_k^{(a,a,1)} C_n^k = F_{4n}^{(a,a,1)}. \quad (2)
\]


\[
\sum_{k=1}^{n} (abc + c^2)^{n-k} (ab + 2c^2)^k a^{\xi(k)} (ab)^{\frac{k}{2}} F_k^{(a,b,c)} C_n^k = F_{4n}^{(a,b,c)}, \quad (3)
\]
where \( \xi(k) = k - 2\left\lfloor \frac{k}{2} \right\rfloor \) is the parity function, and further generalized the result of [4] as follows.

**Theorem 1.2** All the zeros of a polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) lie in the annulus \( C = \{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \} \), where

\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{(abc+c^2)^{-k}(ab+2c)^k a^k(b)\xi(k)}{F_{4n}^{(a,b,c)}} \right\}^{\frac{1}{k}},
\]

\[
r_2 = \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}^{(a,b,c)}}{(abc+c^2)^{-k}(ab+2c)^k a^k(b)\xi(k)} \right\}^{\frac{1}{k}}.
\]

In fact, all the above-mentioned results are special cases of the following theorem due to Aziz and Qayoom [1] (see [5] also).

**Theorem 1.3** Let \( \{\lambda_k\}_{k=1}^{n} \) be arbitrary non-zero real or complex numbers satisfying \( \sum_{k=1}^{n} |\lambda_k| \leq 1 \). Then a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) (\( d_k \neq 0, 0 \leq k \leq n \)) has all its zeros in the annulus \( C = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[
r_1 = \min_{1 \leq k \leq n} \left| \frac{\lambda_k d_0}{d_k} \right|^{\frac{1}{k}},
\]

\[
r_2 = \max_{1 \leq k \leq n} \left| \frac{1}{\lambda_k} \right|^{\frac{1}{k}}.
\]

In this paper we obtain an annular bound for the zeros of a polynomial, based on the identity related to the further generalized Fibonacci sequence \( \{F_{n}^{(a,b,c,d)}\}_{n=0}^{\infty} \) defined in [12]. In fact, Theorem 2.1 in Section 2 is also a special case of Theorem 1.3. However it is stated as a theorem since it not only is meaningful in its own right but also provides a unified result in the direction of approaches taken in [5–7].

## 2 Main Result

Before presenting our main result, we state the following lemma [8].

**Lemma 2.1** For \( p, q > 0 \), define the sequence \( \{Q_n\}_{n=0}^{\infty} \) by

\[
Q_0 = 0, \ Q_1 = 1, \ Q_n = pQ_{n-1} + qQ_{n-2} \ (n \geq 2).
\]
Then
\[ Q_n = \frac{u^n - v^n}{u - v}, \quad n \geq 0, \]
where \( u \) and \( v \) are roots of the equation \( x^2 - px - q = 0 \).

Now, consider the generalized Fibonacci sequence \( \{F_n^{(a,b,c,d)}\}_{n=0}^{\infty} \) defined by [12]
\[
F_0^{(a,b,c,d)} = 0, \quad F_1^{(a,b,c,d)} = 1,
\]
\[
F_n^{(a,b,c,d)} = \begin{cases} 
  aF_{n-1}^{(a,b,c,d)} + cF_{n-2}^{(a,b,c,d)}, & \text{if } n \text{ is even} \\
  bF_{n-1}^{(a,b,c,d)} + dF_{n-2}^{(a,b,c,d)}, & \text{if } n \text{ is odd (} n \geq 2) 
\end{cases}
\]
where \( a, b, c, d > 0 \). Our main result is stated in Theorem 2.1 below.

**Theorem 2.1** Define the sequence \( \{B_n\}_{n=0}^{\infty} \) by
\[
B_n = \frac{r^n - s^n}{r - s},
\]
where \( r \) and \( s \) are roots of the equation
\[
x^2 - \sqrt{ab + c + d - 2\sqrt{cd}}x - \sqrt{cd} = 0.
\]
Then, for \( j \geq 2 \) and \( l \geq 0 \), a polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) has all its zeros in the annulus \( C = \{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \} \), where
\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C_n \sqrt{cd} B_{j-1}^{n-k}(B_j)^k \gamma_{j+n+l} F_k^{(a,b,c,d)} |d_k|}{\gamma_{j+n+l} F_k^{(a,b,c,d)} |d_k|} \right\}^{1/k},
\]
\[
r_2 = \max_{1 \leq k \leq n} \left\{ \frac{C_n \sqrt{cd} B_{j-1}^{n-k}(B_j)^k \gamma_{j+n+l} F_k^{(a,b,c,d)} |d_k|}{\gamma_{j+n+l} F_k^{(a,b,c,d)} |d_k|} \right\}^{1/k},
\]
and
\[
\gamma_m = \sqrt{ab + c + d - 2\sqrt{cd}} \left( \frac{B_{m+1} - cB_{m-1}}{B_m} \right)^{\xi(m)} \left( \frac{B_{m+1} - cB_{m-1}}{B_m} \right)^{\xi(m)},
\]
\[
m = l + 1, l + 2, \ldots, l + n.
\]

**Proof.** It is easily seen that
\[
F_{2n}^{(a,b,c,d)} = (ab + c + d)F_{2n-2}^{(a,b,c,d)} - cdF_{2n-1}^{(a,b,c,d)} \quad (n \geq 2),
\]
with \( F_0^{(a,b,c,d)} = 0 \) and \( F_2^{(a,b,c,d)} = a \). Applying Lemma 2.1 to \( F_{2n}^{(a,b,c,d)}/a \), we have
\[
F_{2n}^{(a,b,c,d)} = a \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right),
\]
where \( \alpha \) and \( \beta \) are roots of the equation

\[
x^2 - (ab + c + d)x + cd = 0.
\]

Let \( r = \sqrt{\alpha} \) and \( s = -\sqrt{\beta} \). Then \( r \) and \( s \) are roots of the equation

\[
x^2 - \sqrt{ab + c + d}x - \sqrt{cd} = 0,
\]

and

\[
F_{2n}^{(a,b,c,d)} = \frac{a}{r + s} B_{2n} = \frac{a}{\sqrt{ab + c + d - 2\sqrt{cd}}} B_{2n}.
\]

On the other hand, for \( n \geq 0 \)

\[
F_{2n+1}^{(a,b,c,d)} = \frac{1}{a} \left( F_{2n+2}^{(a,b,c,d)} - cF_{2n}^{(a,b,c,d)} \right) = \frac{1}{\sqrt{ab + c + d - 2\sqrt{cd}}} \left( B_{2n+2}^{a,b,c,d} - cB_{2n}^{a,b,c,d} \right) = \frac{1}{\sqrt{ab + c + d - 2\sqrt{cd}}} \left( \frac{B_{2n+2}^{a,b,c,d} - cB_{2n}^{a,b,c,d}}{B_{2n+1}^{a,b,c,d}} \right) B_{2n+1}^{a,b,c,d}.
\]

Hence, \( F_n^{(a,b,c,d)} \) can be written as

\[
F_n^{(a,b,c,d)} = \gamma_n B_n,
\]

where \( \gamma_n \) is as defined in (5).

Applying a slightly extended version of [Theorem 1, 10] (see [10] also) to \( B_n = F_n^{(a,b,c,d)}/\gamma_n \), we obtain the following identity

\[
\sum_{k=1}^{\lfloor n/2 \rfloor} C_n^k (\sqrt{cd} B_j)^{n-k} (B_j)^k \frac{\gamma_{j+l}}{\gamma_{k+l}} F_k^{(a,b,c,d)} = F_{(j+l)}^{(a,b,c,d)}.
\]

Then the proof is completed by Theorem 1.3.

**Remark 2.1** \( r_1 \) and \( r_2 \) in Theorem 2.1 can be expressed in terms of \( B_n \) as

\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C_n^k (\sqrt{cd} B_j)^{n-k} (B_j)^k B_{j+k+l}}{B_{j+n+l}} \right\}^{\frac{1}{2}},
\]

\[
r_2 = \max_{1 \leq k \leq n} \left\{ \frac{B_{j+n+l}}{C_n^k (\sqrt{cd} B_j)^{n-k} (B_j)^k B_{j+k+l}} \right\}^{\frac{1}{2}}.
\]
which in turn implies that the conclusion of Theorem 2.1 is independent of the initial condition of $F_{n}^{(a,b,c,d)}$, i.e., $F_{0}^{(a,b,c,d)}$ and $F_{1}^{(a,b,c,d)}$.

**Remark 2.2** As already pointed out in [11], $r_{1}$ and $r_{2}$ in Theorem 2.1 can be computed independently of each other using different values of free parameters.

**Remark 2.3** It is easily seen that the sequence $\{B_{n}\}_{n=0}^{\infty}$ in (4) satisfies the recurrence relation

$$B_{n} = \sqrt{ab + c + d - 2\sqrt{cd}B_{n-1}} + \sqrt{cd}B_{n-2} \quad (n \geq 2).$$

If $c = d$, then $B_{n+1} - cB_{n-1} = \sqrt{ab}B_{n}$. Hence, $\gamma_{m}$ in (5) reduces to

$$\gamma_{m} = a^{1-x(m)}\left(\sqrt{ab}\right)^{\xi(m)-1},$$

and the identity (6) becomes

$$\sum_{k=1}^{n} C_{n}^{k}(cB_{j-1})^{n-k}(B_{j})^{k - \frac{a^{\xi(k+l)-\xi(jn+l)}}{\sqrt{ab}^{\xi(k+l)-\xi(jn+l)}}F_{j+1}^{(a,b,c)}} = F_{j}^{(a,b,c)}.$$  (8)

Since $B_{3} = ab + c$ and $B_{4} = \sqrt{ab}(ab + 2c)$, then, setting $j = 4$, $l = 0$, (8) reduces to (3), and so Theorem 1.2 is a special case of Theorem 2.1.

### 3 Example

Consider the polynomial $[2] \ P(z) = z^{3} + 0.1z^{2} + 0.3z + 0.7$. The annular bounds for the zeros of $P(z)$ obtained by existing methods and Theorem 2.1 of this paper are compared in Table 1.

It can be seen that the classical Cauchy’s bound in Theorem 1.1 is improved by the methods of [2], [5], [7], [13]. On the other hand, Theorem 2.1 indicates that all the zeros of $P(z)$ are contained in the annulus $C_{1} = \{z \in \mathbb{C} : r_{1} \leq |z| \leq r_{2}\}$, where $r_{1} \simeq 0.7113$ for $j = 5$, $l = 0$, $(a, b, c, d) = (1.2, 0.2, 3.4, 4.2)$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Annular bound for the zeros of $P(z)$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1.1</td>
<td>$0.41 \leq</td>
<td>z</td>
</tr>
<tr>
<td>[2]</td>
<td>$0.5833 \leq</td>
<td>z</td>
</tr>
<tr>
<td>[5](Theorem 2)</td>
<td>$0.7803 \leq</td>
<td>z</td>
</tr>
<tr>
<td>[7](Theorem 1)</td>
<td>$0.5146 \leq</td>
<td>z</td>
</tr>
<tr>
<td>[13](Theorem 1)</td>
<td>$0.4350 \leq</td>
<td>z</td>
</tr>
<tr>
<td>Theorem 2.1</td>
<td>$0.7113 \leq</td>
<td>z</td>
</tr>
</tbody>
</table>
and $r_2 \simeq 1.0540$ for $j = 3$, $l = 2$, $(a, b, c, d) = (2, 2, 0.2, 4.2, 4.2)$. Consequently Theorem 2.1 yields a further improved result for this example.

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References


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