A General Control Method for Inverse Hybrid Function Projective Synchronization of a Class of Chaotic Systems

Gasri Ahlem$^{1,2}$ and Adel Ouannas$^2$

$^1$Department of Mathematics and Computer Science
Constantine University, Algeria

$^2$Department of Mathematics and Computer Science
Tebessa University, Algeria

Copyright © 2014 Gasri Ahlem and Adel Ouannas. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, a new type of chaos synchronization called inverse hybrid function projective synchronization (IHFPS) is investigated for a class of continuous chaotic systems. Based on Lyapunov stability theory and nonlinear control method, a new controller can be designed to achieve inverse hybrid function projective synchronization for $n-D$ chaotic systems in continuous-time. Simulation example validate the derived synchronization result.

Keywords: Hybrid function projective synchronization, inverse hybrid function projective synchronization, Lyapunov stability, nonlinear controllers, continuous-time chaotic systems

1 Introduction

In recent years, chaos synchronization has become an active research area, due to its potential applications in secure communication [1]. Various pow-
erful methods for chaos synchronization have been investigated such as active control method [2], backstepping design approach [3] and sliding mode technique [4], etc. and so on have been successfully applied to chaotic and hyperchaotic systems. Many type of synchronization for chaotic systems have been presented such as anti-synchronization [5], generalized synchronization [6] and hybrid function projective synchronization [7], etc. However, hybrid function projective synchronization (HFPS) scheme for chaotic systems is extensively considered [8]. In HFPS, the master system and the slave system could be synchronized with a scaling function matrix and the unpredictability of the scaling function matrix in HFPS scheme can enhance the security in tele-communication. On the other hand, studying the inverse problem of HFPS which produce a new synchronization type called inverse hybrid function projective synchronization (IHFPS) is an attractive and important idea.

In this paper, we introduce the inverse hybrid function projective synchronization type, (IHFPS), for \( n \)-dimensional general class of chaotic systems in continuous-time. To achieve IHFPS, we propose a general method based on new nonlinear controllers. The theoretical result derived in this paper, is proved using Lyapunov stability theory. In order to verify the effectiveness of the proposed method, we apply it to 5-D continuous-time chaotic systems.

This paper is organized as follows: in section 2, a description of the chaotic systems addressed in this paper is provided. In section 3, the definition of inverse hybrid function projective synchronization is introduced. In section 4, our synchronization method is described. In section 5, numerical example is used to show the effectiveness of the proposed method. Finally, we make conclusion in section 6.

## 2 Master-slave systems description

We consider the following master chaotic system described by

\[
\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{p_n=1}^{d} \ldots \sum_{p_1=1}^{d} \alpha_{p_1 \ldots p_n}^{(i)} x_{p_1}^{p_1} \ldots x_{p_n}^{p_n} + \gamma_i, \quad i = 1, 2, \ldots, n, \quad (1)
\]

where \((a_{ij}) \in \mathbb{R}^{n \times n}, \ (\alpha_{p_1 \ldots p_n}^{(i)}) \in \mathbb{R}^{d \times d}, \ (i = 1, 2, \ldots, n)\), such that: \(\alpha_{10\ldots0}^{(i)} = 0, \ \alpha_{01\ldots0}^{(i)} = 0, \ \alpha_{00\ldots1}^{(i)} = 0, \ (i = 1, 2, \ldots, n)\), \((\gamma_i)_{1 \leq i \leq n}\) are real numbers.

As the slave system, we consider the following chaotic system described by

\[
\dot{y}_i = \sum_{j=1}^{n} b_{ij} y_j + \sum_{p_n=1}^{d} \ldots \sum_{p_1=1}^{d} \beta_{p_1 \ldots p_n}^{(i)} y_{p_1}^{p_1} \ldots y_{p_n}^{p_n} + \delta_i + u_i, \quad i = 1, 2, \ldots, n, \quad (2)
\]
where \((d_{ij}) \in \mathbb{R}^{n \times n}, (\beta^{i}_{p_1...p_n}) \in \mathbb{R}^{d \times d}, (i = 1, 2, ..., n)\), such that: \(\beta^{(i)}_{00...0} = 0, \beta^{(i)}_{01...0} = 0, ..., \beta^{(i)}_{00...1} = 0, i = 1, 2, ..., n, (\delta_i)_{1 \leq i \leq n}\) are real numbers and \((u_i)_{1 \leq i \leq n}\) are controllers to be determined.

3 Definition of IHFPS

First of all, we call the definition of hybrid function projective synchronous (HFPS) then we present the definition of inverse hybrid function projective synchronous (IHFPS) for master-slave chaotic systems given in Eqs. (1) and (2).

Definition 1 The master system (1) and the slave system (2) are said to be hybrid function projective synchronized (HFPS), if there exists controllers \((u_i)_{1 \leq i \leq n}\) and continuously differentiable bounded functions \(h_i(t) \neq 0\), for all \(t, 1 \leq i \leq n\), such that the synchronization errors \(e_i = y_i - h_i x_i, 1 \leq i \leq n\), satisfies that \(\lim_{t \to +\infty} e_i = 0, i = 1, 2, ..., n\).

Remark 2 HFPS of chaotic dynamical systems, based on Lyapunov stability theory, have been studied and carried out, for example, in Ref. [7].

Definition 3 The master system (1) and the slave system (2) are said to be inverse hybrid function projective synchronized (IHFPS), if there exists controllers \((u_i)_{1 \leq i \leq n}\) and continuously differentiable bounded functions \(h_i(t) \neq 0\), for all \(t, 1 \leq i \leq n\), such that the synchronization errors \(e_i(t) = x_i - h_i(t) y_i, 1 \leq i \leq n\), satisfies that \(\lim_{t \to +\infty} e_i = 0, i = 1, 2, ..., n\).

4 Synchronization Method

The error system, according to definition 3, between the master system (1) and the slave system (2), can be derived as

\[
\dot{e}_i(t) = \sum_{j=1}^{n} a_{ij} e_j(t) + \phi_i + \varphi_i + \psi_i - h_i(t) u_i, \quad i = 1, 2, ..., n, \quad (3)
\]

where

\[
\phi_i = \sum_{p_n=1}^{d} \sum_{p_1=1}^{d} \alpha^{(i)}_{p_1...p_n} x_1^{p_1} ... x_n^{p_n}, \quad i = 1, 2, ..., n, \quad (4)
\]
\[
\varphi_i = -h_i(t) \sum_{p_n=1}^{d} \ldots \sum_{p_1=1}^{d} \beta_{p_1 \ldots p_n}^{(i)} y_1^{p_1} \ldots y_n^{p_n}, \quad i = 1, 2, \ldots, n, \quad (5)
\]

and

\[
\psi_i = h_i(t) \sum_{j=1}^{n} (a_{ij} - b_{ij}) y_j - h_i(t) y_i + \gamma_i - h_i(t) \delta_i, \quad i = 1, 2, \ldots, n. \quad (6)
\]

To achieve inverse HFPS between the master system (1) and the slave system (2), we choose the control law as follow

\[
u_i = \frac{1}{h_i(t)} \left( \sum_{j=1}^{i-1} (2a_{ij} + a_{ji}) e_j(t) + k_i e_i(t) - \sum_{j=i+1}^{n} a_{ji} e_j(t) + \phi_i + \varphi_i + \psi_i \right), \quad i = 1, 2, \ldots, n, \quad (7)
\]

where \((k_i)_{1 \leq i \leq n}\) are control constants to be designed.

By substituting Eq. (7) into Eq. (3), the error system can be written as

\[
\begin{pmatrix}
\dot{e}_1(t) \\
\dot{e}_2(t) \\
\vdots \\
\dot{e}_n(t)
\end{pmatrix} =\begin{pmatrix}
a_{11} - k_1 & a_{12} + a_{12} & \cdots & a_{n1} + a_{1n} \\
-(a_{21} + a_{12}) & a_{22} - k_2 & \cdots & a_{n2} + a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-(a_{n1} + a_{1n}) & -(a_{n2} + a_{2n}) & \cdots & a_{nn} - k_n
\end{pmatrix}
\begin{pmatrix}
e_1(t) \\
e_2(t) \\
\vdots \\
e_n(t)
\end{pmatrix}, \quad (8)
\]

Now, rewriting the error system described in Eq. (8) in the compact form

\[
\dot{e}(t) = (L + K - L^T) e(t), \quad (9)
\]

where \(K = \text{diag} \{(a_{11} - k_1), \ldots, (a_{nn} - k_n)\}\) and \(L = (l_{ij})_{n \times n}\) such that

\[
l_{ij} = \begin{cases} 
-(a_{ij} + a_{ji}) & \text{if } j < i \\
0 & \text{if } j \geq i
\end{cases} \quad (10)
\]

**Theorem 4** If the control constants \((k_i)_{1 \leq i \leq n}\) are chosen such that: \(k_i > a_{ii}, \ i = 1, 2, \ldots, n\). Then, the two systems (1) and (2) are globally inverse hybrid function projective synchronized under the controllers (7).
Proof. Consider the Lyapunov function in the form: \( V(e(t)) = \frac{1}{2} e^T(t) e(t) \), then we get
\[
\dot{V}(e(t)) = \frac{1}{2} (\dot{e}^T(t) e(t) + e^T(t) \dot{e}(t)) = \frac{1}{2} (e^T(t) (L^T + K^T - L) e(t) + e^T(t) (L + K - L^T) e(t)) = e^T(t) K e(t) = \sum_{i=1}^{n} (a_i - k_i) e_i^2(t) < 0.
\]

Thus, by Lyapunov stability theory it is immediate that \( \lim_{t \to \infty} e_i(t) = 0 \), \( i = 1, 2, ..., n \). We conclude that the systems (1) and (2) are globally inverse hybrid function projective synchronized.

5 Simulation Example

In this section, an example of chaotic system is considered to validate the proposed chaos synchronization approach. The master system and the slave system are described as, respectively,
\[
\begin{align*}
\dot{x}_1 &= a_1 (x_2 - x_1) + x_2 x_3 x_4 x_5 \\
\dot{x}_2 &= a_2 (x_1 + x_2) - x_1 x_3 x_4 x_5 \\
\dot{x}_3 &= -x_3 + 0.1 x_1^2 \\
\dot{x}_4 &= -a_3 x_4 + x_1 x_2 x_3 x_5 \\
\dot{x}_5 &= -a_4 (x_5 - x_4) - a_5 x_1 + x_1 x_2 x_3 x_4
\end{align*}
\tag{11}
\]
and
\[
\begin{align*}
\dot{y}_1 &= a_1 (y_2 - y_1) + y_2 y_3 y_4 y_5 + u_1 \\
\dot{y}_2 &= a_2 (y_1 + y_2) - y_1 y_3 y_4 y_5 + u_2 \\
\dot{y}_3 &= -y_3 + 0.1 y_1^2 + u_3 \\
\dot{y}_4 &= -a_3 y_4 + y_1 y_2 y_3 y_5 + u_4 \\
\dot{y}_5 &= -a_4 (y_5 - y_4) - a_5 y_1 + y_1 y_2 y_3 y_4 + u_5
\end{align*}
\tag{12}
\]
where \( a_1, a_2, a_3, a_4, a_5 \) are bifuraction parameters and \( (u_i)_{1 \leq i \leq 5} \) are the synchronization controllers. The system (11), has a chaotic attractor, when \( (a_1, a_2, a_3, a_4, a_5) = (37, 14.5, 10.5, 15, 9.5) \), [8].

According to our control method presented in section 2, the synchronization errors between the master system (11) and the slave system (12) can be derived as
\[
\begin{align*}
\dot{e}_1(t) &= a_1 (e_2(t) - e_1(t)) + R_1 - h_1(t) u_1 \\
\dot{e}_2(t) &= a_2 (e_1(t) + e_2(t)) + R_1 - h_2(t) u_2 \\
\dot{e}_3(t) &= -e_3(t) + R_3 - h_3(t) u_3 \\
\dot{e}_4(t) &= -a_3 e_4(t) + R_4 - h_4(t) u_4 \\
\dot{e}_5(t) &= -a_4 (e_5(t) - e_4(t)) - a_5 e_1(t) + R_5 - h_5(t) u_5
\end{align*}
\tag{13}
\]
where
\[
\begin{aligned}
R_1 &= x_2 x_3 x_4 x_5 - h_1(t) y_2 y_3 y_4 y_5 - \dot{h}_1(t) y_1 \\
R_2 &= -x_1 x_2 x_4 x_5 + h_2(t) y_1 y_3 y_4 y_5 - \dot{h}_2(t) y_2 \\
R_3 &= 0.1x_1^2 - h_3(t) (0.1y_1^2) - \dot{h}_3(t) y_3 \\
R_4 &= x_1 x_2 x_3 x_5 - h_4(t) y_1 y_2 y_3 y_5 - \dot{h}_4(t) y_4 \\
R_5 &= x_1 x_2 x_3 x_4 - h_5(t) y_1 y_2 y_3 y_4 - \dot{h}_5(t) y_5
\end{aligned}
\]

and \((h_i(t))_{1 \leq i \leq 5}\) are the scaling functions.

To achieve IHFPS between systems (11) and (12), the synchronization controllers are proposed as follow

\[
\begin{aligned}
u_1 &= \frac{1}{h_1(t)} (a_2 e_2(t) + a_5 e_5(t) + R_1) \\
u_2 &= \frac{1}{h_2(t)} (a_1 e_1(t) + k e_2(t) + R_2) \\
u_3 &= \frac{1}{h_3(t)} R_3 \\
u_4 &= \frac{1}{h_4(t)} (a_4 e_5(t) R_4) \\
u_5 &= \frac{1}{h_5(t)} R_5
\end{aligned}
\]

where \(k\) is a constant control to be designed.

Then, the synchronization errors (13) can be described as

\[
\begin{bmatrix}
\dot{e}_1(t) \\
\dot{e}_2(t) \\
\dot{e}_3(t) \\
\dot{e}_4(t) \\
\dot{e}_5(t)
\end{bmatrix} =
\begin{bmatrix}
-a_1 & a_1 & -a_2 & 0 & 0 & a_5 \\
a_2 & a_2 & -k & 0 & 0 & 0 \\
0 & a_1 & -1 & 0 & 0 & 0 \\
0 & 0 & a_4 & -a_3 & -a_4 & 0 \\
a_5 & 0 & 0 & a_4 & -a_4 & 0
\end{bmatrix}
\begin{bmatrix}
e_1(t) \\
e_2(t) \\
e_3(t) \\
e_4(t) \\
e_5(t)
\end{bmatrix},
\]

and by using the same procedure of proof of Theorem 4, we can get the following result.

**Corollary 5** If the control constant \(k\) is chosen such that: \(k > a_2\), then the two systems (11) and (12) are globally inverse hybrid function projective synchronized under the control law (15).

Finally, we get the numeric result that is shown in Fig. 1.

## 6 Conclusion

In this paper, a new synchronization method was presented for some class of chaotic dynamical systems in continuous-time to investigate a new type of synchronization called inverse hybrid function projective synchronization (IHFPS). Numerical example was utilized to illustrate the effectiveness of the proposed method.
Figure 1: Time evolution of synchronization errors between systems (11) and (12).

References


Received: July 15, 2014; Published: February 9, 2015