The Multiple-sets Split Equality
Fixed Point Problem for Countable Families of
Multi-valued Demi-contractive Mappings

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Abstract
algorithm for solving the multiple-sets split equality fixed point problem. Weak and strong convergence theorems are proved for two countable families of multi-valued demi-contractive mappings in real Hilbert spaces. Our theorems extend and complement some recent results of Chang et al., Chidume et al., Wu et al. and a host of other recent important results.

**Mathematics Subject Classification:** 47J25, 47H06, 49J53, 90C25

**Keywords:** Multiple-sets split equality fixed point problem, Multi-valued mappings, Demicontractive mappings, iterative scheme, Fixed point

1 Introduction

The split feasibility problem (SFP) arises in many areas of applications such as phase retrieval, medical image reconstruction, image restoration, computer tomography and radiation therapy treatment planning (see e.g., Byrne [1], Censor et al. [2], Censor et al. [3], Censor and Elfving [4], and Palta and Mackie [16]). The SFP was introduced in 1994 by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modelling inverse problems arising from phase retrieval and medical image reconstruction. It takes the following form:

Let $C$ and $Q$ be two nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, $A : H_1 \rightarrow H_2$ be a bounded linear map. The split feasibility problem (SFP) is formulated as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q.$$  \hfill (1.1)

Censor et al. [3] proposed the multiple-sets split feasibility problem (MSSFP) which arises in areas of applications such as intensity-modulated radiation therapy [16] and which is formulated as follows:

$$\text{Find } x^* \in C = \bigcap_{i=1}^{N} C_i \text{ such that } Ax^* \in Q = \bigcap_{j=1}^{M} Q_j,$$  \hfill (1.2)

where $N$ and $M$ are positive integers, $\{C_1, \ldots, C_N\}$ and $\{Q_1, \ldots, Q_M\}$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear map.

Recently, Moudafi et al. [13] introduced the following split equality fixed point problem as a generalization of the split feasibility problem (1.1):

$$\text{Find } x \in C := F(U) \text{ and } y \in Q := F(T) \text{ such that } Ax = By,$$  \hfill (1.3)

where $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are two bounded linear maps, $U : H_1 \rightarrow H_1$, $T : H_2 \rightarrow H_2$, $F(U)$ and $F(T)$ denote the fixed point sets of $U$.
and $T$, respectively, and the operators $U$ and $T$ are assumed to be firmly quasi-nonexpansive. Note that problem (1.3) reduces to problem (1.1) if $H_2 = H_3$ and $B = I$ (where $I$ is the identity map on $H_2$) in (1.3).

A mapping $T : H \to H$ is said to be firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$
\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|x - Tx\|^2 \quad \forall x^* \in F(T), \quad x \in H. \tag{1.4}
$$

Let $D$ be a nonempty subset of $H$. $T : D \to D$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

$$
\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \quad \forall x \in D, \quad x^* \in F(T).
$$

Clearly, the class of demi-contractive mappings properly contains that of firmly quasi-nonexpansive mappings.

Recently, motivated by the works of Moudafi [12], Moudafi et al. [13], Moudafi [14] and Yuan-Fang et al. [19], Chidume et al. [9] defined the following iterative algorithm to solve the split equality fixed point problem (1.3) in the case where $U$ and $T$ are demi-contractive:

\[
\begin{align*}
\forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\
x_{n+1} &= (1 - \alpha)\left(x_n - \gamma A^*(Ax_n - By_n)\right) + \alpha U\left(x_n - \gamma A^*(Ax_n - By_n)\right) \\
y_{n+1} &= (1 - \alpha)\left(y_n + \gamma B^*(Ax_n - By_n)\right) + \alpha T\left(y_n + \gamma B^*(Ax_n - By_n)\right), \quad \forall n \geq 1.
\end{align*}
\] \tag{1.5}

Let $H$ be a real Hilbert space. We denote by $CB(H)$ the collection of all nonempty closed and bounded subsets of $H$. The Hausdorff metric $D$ on $CB(H)$ is defined by

$$
D(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \text{ for all } A, B \in CB(H),
$$

where $d(x, K) := \inf_{y \in K} d(x, y)$.

**Definition 1.1** Let $T : H \to CB(H)$ be a multi-valued mapping. An element $x^* \in H$ is said to be a fixed point of $T$ if $x^* \in T(x^*)$. We denote by $F(T)$ the fixed points set of $T$ defined by

$$
F(T) := \{ x \in H; \ x \in Tx \}.
$$
Definition 1.2 Let $H$ be a real Hilbert space. A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$D(Tx, Tx^*) \leq \|x - x^*\| \forall x^* \in F(T), \ x \in D(T).$$  \hspace{1cm} (1.6)

In [20], Wu et al. studied the following multiple-set split equality problem for finite families of multi-valued quasi-nonexpansive mappings:

Find $x \in C = \bigcap_{j=1}^{N} F(R^j_1)$ and $y \in Q = \bigcap_{j=1}^{N} F(R^j_2)$ such that $Ax = By,$  \hspace{1cm} (1.7)

where $N$ is a positive integer, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, $R^j_i : H_i \to CB(H_i), \ i = 1, 2, \ j = 1, 2, \ldots, N$ is a family of multi-valued quasi-nonexpansive mappings. They obtained strong convergence results to a solution of (1.7).

Definition 1.3 (see e.g., [5]) A mapping $T : H \to H$ is said to be $(\{\nu_n\}, \{\mu_n\}, \varphi)$-total asymptotically strict pseudocontractive if there exist a constant $k \in [0, 1)$ and nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \to 0$ and $\mu_n \to 0$ and a strictly increasing continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that for all $n \geq 1,$

$$\|T^n x - T^n y\| \leq \|x - y\|^2 + k\|x - y - (T^n x - T^n y)\|^2 + \nu_n \varphi(\|x - y\|) + \mu_n, \ \forall \ x, y \in H$$  \hspace{1cm} (1.8)

Chang et al. [5], (see also Jinfang [17]) introduced and studied the following multiple-set split feasibility problem for a countable family of multi-valued quasi-nonexpansive mappings, $S_i$, and a total asymptotically strict pseudo-contractive mapping $T$:

Find $x^* \in C = \bigcap_{i=1}^{\infty} F(S_i)$ such that $Ax^* \in Q = F(T),$  \hspace{1cm} (1.9)

where $A : H_1 \to H_2$ is a bounded linear map, $S_i : H_1 \to CB(H_1), \ i = 1, 2, \ldots$ is a family of multi-valued quasi-nonexpansive mappings, $T : H_2 \to H_2$ is a total asymptotically strict pseudo-contractive mapping, and $F(S_i)$ and $F(T)$ denote the fixed point sets of $S_i$ and $T$, respectively. Clearly, the class of multi-valued demi-contractive mappings properly contains that of multi-valued quasi-nonexpansive mappings.

Motivated by the works of Chang et al. [5], Wu et al. [20] and Chidume et al. [9], it is our purpose in this paper to introduce the multiple-sets split equality fixed point problem for countable families of multi-valued demi-contractive mappings defined as follows:

Find $x \in C = \bigcap_{i=1}^{\infty} F(S_i)$ and $y \in Q = \bigcap_{j=1}^{\infty} F(T_j)$ such that $Ax = By,$  \hspace{1cm} (1.10)
where \( A : H_1 \to H_3 \) and \( B : H_2 \to H_3 \) are two bounded linear maps, \( S_i : H_1 \to CB(H_1), \ i = 1, 2, \cdots \) and \( T_j : H_2 \to CB(H_2), \ j = 1, 2, \cdots \) are two families of multi-valued demi-contractive mappings, and \( F(S_i) \) and \( F(T_j) \) denote the fixed point sets of \( S_i \) and \( T_j \), respectively. Our theorems extend and complement the results of Chang et al. [5] and those of a host of other authors.

2 Preliminaries

We recall some definitions and lemmas which will be needed in the proof of our theorems.

In the sequel, we denote strong and weak convergence by “\( \to \)” and “\( \rightharpoonup \)”, respectively. Moreover, the fixed point set of a mapping \( T \) is denoted by \( F(T) \) and the solution set of problem (1.10) by \( \Omega \), where,

\[
\Omega := \left\{ (x^*, y^*) \in \bigcap_{i=1}^{\infty} F(S_i) \times \bigcap_{j=1}^{\infty} F(T_j) : Ax^* = By^* \right\}.
\]

**Definition 2.1** Let \( H \) be a real Hilbert space.

1. Let \( T : H \to CB(H) \) be a multi-valued mapping. Then, \( T \) is said to be demi-closed at zero if for any sequence \( \{x_n\} \subset H \) with \( x_n \rightharpoonup x^* \), and \( d(x_n, T(x_n)) \to 0 \), we have \( x^* \in Tx^* \).

2. A multi-valued mapping \( T : H \to CB(H) \) is said to be hemi-compact if for any bounded sequence \( \{x_n\} \subset H \) with \( d(x_n, T(x_n)) \to 0 \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to some \( x^* \in H \).

**Definition 2.2** Let \( H \) be a real Hilbert space.

1. A multi-valued mapping \( T : D(T) \subseteq H \to CB(H) \) is said to be nonexpansive if

\[
D(Tx, Ty) \leq \|x - y\| \quad \forall \ x, y \in D(T).
\]  

2. A multi-valued mapping \( T : D(T) \subseteq H \to CB(H) \) is said to be quasi-nonexpansive if \( F(T) \neq \emptyset \) and

\[
D(Tx, Tx^*) \leq \|x - x^*\| \quad \forall x^* \in F(T), \ x \in D(T).
\]

3. A multi-valued mapping \( T : D(T) \subseteq H \to CB(H) \) is said to be uniformly continuous if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y \in D(T) \)

\[
\|x - y\| < \delta \implies D(Tx, Ty) < \epsilon.
\]
Chidume et al. [7] introduced a class of *multi-valued k-strictly pseudo-contractive* mappings defined on a real Hilbert space $H$ as follows.

**Definition 2.3** A multi-valued mapping $T : D(T) \subseteq H \rightarrow CB(H)$ is said to be $k$-strictly pseudo-contractive if there exists a constant $k \in (0, 1)$ such that for all $x, y \in D(T)$,

$$\left( D(Tx, Ty) \right)^2 \leq \|x - y\|^2 + k\|x - y - (u - v)\|^2 \quad \forall u \in Tx, v \in Ty. \quad (D.1)$$

**Definition 2.4** A multi-valued mapping $T : D(T) \subseteq H \rightarrow CB(H)$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that for all $x \in D(T), x^* \in F(T)$,

$$\left( D(Tx, Tx^*) \right)^2 \leq \|x - x^*\|^2 + k\left( d(x, Tx) \right)^2. \quad (D.2)$$

The class of *demi-contractive* mappings is important because several common types of operators arising in optimization problems belong to this class, see for example, Chidume and Maruster [10], Maruster and Popirlan [11] and references therein.

**Remark 2.5** The following inclusion is obvious.

$$\text{Quasi-nonexpansive} \subset \text{Demi-contractive}. \quad (R.1)$$

We give an example showing that the above inclusion is proper.

**Example 2.6** ([6])

Let $H = \mathbb{R}$ with the usual metric. Let $T : H \rightarrow 2^H$ be defined by

$$Tx = \begin{cases} 
[-3x, -\frac{5x}{2}], & x \in [0, \infty) \\
\left[-\frac{5x}{2}, -3x\right], & x \in (-\infty, 0] 
\end{cases} \quad (2.4)$$

We have that $F(T) = \{0\}$ and $T$ is a multi-valued demi-contractive mapping which is not quasi-nonexpansive.

In fact, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$\left( D(Tx, T0) \right)^2 = | -3x - 0|^2 = 9|x - 0|^2,$$

which implies that $T$ is not quasi-nonexpansive.

Also, we have that

$$\left( d(x, Tx) \right)^2 = \left| x - \left( -\frac{5}{2}x \right) \right|^2 = \frac{49}{4} |x|^2.$$


Thus,
\[
(D(Tx, T0))^2 = |x - 0|^2 + 8|x - 0|^2 = |x - 0|^2 + \frac{32}{49}(d(x, Tx))^2.
\]
Therefore, \(T\) is a demi-contractive mapping with constant \(k = \frac{32}{49} \in (0, 1)\).
For simplicity, we shall write \(D^2(A, B)\) for \((D(A, B))^2\) for all \(A, B \in CB(H)\).

**Lemma 2.7** (Opial’s Lemma [15]) Let \(H\) be a real Hilbert space and \(\{\mu_n\}\) be a sequence in \(H\) such that there exists a nonempty set \(W \subset H\) satisfying the following conditions:

(i) For every \(\mu \in W\), \(\lim_{n \to \infty} \|\mu_n - \mu\|\) exists;

(ii) Any weak-cluster point of the sequence \(\{\mu_n\}\) belongs to \(W\).

Then, there exists \(w^* \in W\) such that \(\{\mu_n\}\) converges weakly to \(w^*\).

The proof of the following lemma can be found in Chidume and Ezeora [8].

**Lemma 2.8** ([8]) Let \(H\) be a real Hilbert space. Let \(\{x_i, i = 1, 2, \cdots, m\} \subset H\). For \(\alpha_i \in (0, 1), i = 1, 2, \cdots, m\) such that \(\sum_{i=1}^m \alpha_i = 1\). Then, the following identity holds:
\[
\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j \|x_i - x_j\|^2.
\]

**Remark 2.9** It follows easily from lemma 2.8 that the following identity holds.
\[
\left\| \sum_{i=1}^\infty \alpha_i x_i \right\|^2 = \sum_{i=1}^\infty \alpha_i \|x_i\|^2 - \sum_{i,j=1, i \neq j}^\infty \alpha_i \alpha_j \|x_i - x_j\|^2,
\]

for \(\sum_{i=1}^\infty \alpha_i = 1\), provided that \(\{x_i\}\) is bounded.

The following lemma shows a Lipschitz-type property of multi-valued demi-contractive mappings.

**Lemma 2.10** Let \(K\) be a nonempty subset of a real Hilbert space \(H\) and let \(T : K \to CB(K)\) be a multi-valued \(k\)-demicontractive mapping. Assume that for every \(p \in F(T), \ Tp = \{p\}\). Then, there exists \(L > 0\) such that
\[
D(Tx,Tp) \leq L\|x - p\|, \forall x \in K, p \in F(T).
\]
Proof. We have that
\[
D^2(Tx, Tp) \leq \|x - p\|^2 + kd^2(x, Tx)
\]
\[
\leq \|x - p\|^2 + kD^2(\{x\}, Tx)
\]
\[
\leq \left(\|x - p\| + \sqrt{k}D(\{x\}, Tx)\right)^2,
\]
so that,
\[
D(Tx, Tp) \leq \|x - p\| + \sqrt{k}\left(\|x - p\| + D(Tx, Tp)\right)
\]
\[
\leq \|x - p\| + \sqrt{k}\|x - p\| + \sqrt{k}D(Tx, Tp)
\]
This implies that
\[
D(Tx, Tp) \leq \frac{1 + \sqrt{k}}{1 - \sqrt{k}}\|x - p\|.
\]
Hence, \( L := \frac{1 + \sqrt{k}}{1 - \sqrt{k}} \).

3 Main Results

To approximate a solution of the multiple-sets split equality fixed point problem (1.10), we make the following assumptions:

(A_1) \( H_1, H_2 \) and \( H_3 \) are real Hilbert spaces, \( A : H_1 \to H_3 \) and \( B : H_2 \to H_3 \) are bounded linear maps.

(A_2) \( S_i : H_1 \to CB(H_1), \ i = 1, 2, 3, \ldots, \ T_j : H_2 \to CB(H_2), \ j = 1, 2, 3, \ldots \) are multi-valued demi-contractive mappings with constants \( k_i \) and \( k_j \), respectively, such that \( k_1 = \sup_{i \geq 1} \{k_i\}, k_2 = \sup_{j \geq 1} \{k_j\} \in (0, 1) \).

(A_3) \( S_i : H_1 \to CB(H_1), \ i = 1, 2, 3, \ldots, \ T_j : H_2 \to CB(H_2), \ j = 1, 2, 3, \ldots \) are demi-closed at zero and are uniformly continuous.

(A_4) \( k \in (0, 1) \), where \( k = \max\{k_1, k_2\} \)

We define an iterative algorithm as follows:
For,
\[
x_1 \in H_1, \ y_1 \in H_2, \ z^i_n \in S_i\left(x_n + \gamma A^*(Ax_n - By_n)\right), \text{ and } w^j_n \in T_j\left(y_n + \gamma B^*(Ax_n - By_n)\right),
\]
define a sequence \( \{(x_n, y_n)\} \) by:

\[
\begin{align*}
x_{n+1} &= \alpha_0 \left( x_n - \gamma A^*(Ax_n - By_n) \right) + \sum_{i=1}^{\infty} \alpha_i z_n^i, \\
y_{n+1} &= \alpha_0 \left( y_n + \gamma B^*(Ax_n - By_n) \right) + \sum_{i=1}^{\infty} \alpha_j w_n^j, \quad \forall n \geq 1,
\end{align*}
\]

(3.1)

where \( \alpha_0 \in (k, 1) \), \( \alpha_i \in (0, 1), \ i = 0, 1, 2, \cdots \) such that \( \sum_{i=0}^{\infty} \alpha_i = 1 \) and

\( \gamma \in \left(0, \frac{2}{(\lambda_{A^*A}+\lambda_{B^*B})}\right) \), where \( \lambda_{A^*A} \) and \( \lambda_{B^*B} \) denote the spectral radii of \( A^*A \) and \( B^*B \), respectively.

We now prove the following theorem.

**Theorem 3.1** Suppose assumptions \( (A_1)-(A_4) \) hold such that \( \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset \) and \( \bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset \).

Assume that for \( x^* \in \bigcap_{i=1}^{\infty} F(S_i) \), \( S_ix^* = \{x^*\} \) and for \( y^* \in \bigcap_{j=1}^{\infty} F(T_j) \), \( T_jy^* = \{y^*\} \). If \( \Omega := \left\{(x^*, y^*) \in \bigcap_{i=1}^{\infty} F(S_i) \times \bigcap_{j=1}^{\infty} F(T_j) : Ax^* = By^* \right\} \neq \emptyset \), then the sequence \( \{(x_n, y_n)\} \) generated by (3.1) converges weakly to a solution of problem (1.10).

**Proof.** Let \((x^*, y^*) \in \Omega\).

We first show that for each \( i = 1, 2, 3, \cdots \), \( \{z_n^i\} \) is bounded. By applying lemma 2.10, we have that

\[
\left\| z_n^i - x^* \right\| \leq D \left( S_ix^*, S_i(x_n - \gamma A^*(Ax_n - By_n)) \right) \leq \frac{1 + \sqrt{k_1}}{1 - \sqrt{k_1}} \left\| x_n - \gamma A^*(Ax_n - By_n) - x^* \right\| = M_n.
\]

Thus \( \{z_n^i\}_{i \geq 1} \) is bounded.
Now, using lemma 2.8 and assumption $A_2$, we have

\[
\begin{align*}
\|x_{n+1} - x^*\|^2 &= \|\alpha_0\left(x_n - \gamma A^*(Ax_n - By_n)\right) + \sum_{i=1}^{\infty} \alpha_i z^i_n - x^*\|^2 \\
&= \|\alpha_0\left(x_n - \gamma A^*(Ax_n - By_n) - x^*\right) + \sum_{i=1}^{\infty} \alpha_i (z^i_n - x^*)\|^2 \\
&= \alpha_0 \|x_n - \gamma A^*(Ax_n - By_n) - x^*\|^2 + \sum_{i=1}^{\infty} \alpha_i \|z^i_n - x^*\|^2 \\
&\quad - \sum_{i=1}^{\infty} \alpha_0 \alpha_i \|x_n - \gamma A^*(Ax_n - By_n) - z^i_n\|^2 - \sum_{i,j=1, i\neq j}^{\infty} \alpha_i \alpha_k \|z^i_n - z^k_n\|^2 \\
&\leq \alpha_0 \|x_n - \gamma A^*(Ax_n - By_n) - x^*\|^2 + \sum_{i=1}^{\infty} \alpha_i \|z^i_n - x^*\|^2 \\
&\quad - \sum_{i=1}^{\infty} \alpha_0 \alpha_i d^2 \left(\langle x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \rangle \right) \\
&\leq \alpha_0 \|x_n - \gamma A^*(Ax_n - By_n) - x^*\|^2 + \sum_{i=1}^{\infty} \alpha_i d^2 \left(\langle x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \rangle \right) \\
&\quad - \sum_{i=1}^{\infty} \alpha_0 \alpha_i d^2 \left(\langle x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \rangle \right) \\
&= \|x_n - \gamma A^*(Ax_n - By_n) - x^*\|^2 \\
&\quad - \sum_{i=1}^{\infty} \alpha_i (\alpha_0 - k_i) d^2 \left(\langle x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \rangle \right) \\
&\leq \|x_n - x^*\|^2 - 2\gamma \langle Ax_n - By_n, Ax_n - Ax^* \rangle + \gamma^2 \lambda_{A^*A} \|Ax_n - By_n\|^2 \\
&\quad - (\alpha_0 - k_1) \sum_{i=1}^{\infty} \alpha_i d^2 \left(\langle x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \rangle \right).
\end{align*}
\]
That is
\[
\left\| x_{n+1} - x^* \right\|^2 \leq \left\| x_n - x^* \right\|^2 - 2\gamma \langle Ax_n - By_n, Ax_n - Ax^* \rangle + \gamma^2 \lambda_{A^*A} \left\| Ax_n - By_n \right\|^2
\]
\[
-\left( \alpha_0 - k_1 \right) \sum_{i=1}^{\infty} \alpha_i d^2 \left( x_n - \gamma A^* (Ax_n - By_n), S_i(x_n - \gamma A^* (Ax_n - By_n)) \right).
\]

Similarly, we have that
\[
\left\| y_{n+1} - y^* \right\|^2 \leq \left\| y_n - y^* \right\|^2 + 2\gamma \langle Ax_n - By_n, By_n - By^* \rangle + \gamma^2 \lambda_{B^*B} \left\| Ax_n - By_n \right\|^2
\]
\[
-\left( \alpha_0 - k_2 \right) \sum_{j=1}^{\infty} \alpha_j d^2 \left( y_n + \gamma B^* (Ax_n - By_n), T_j(y_n + \gamma B^* (Ax_n - By_n)) \right)^2.
\]

Adding the above two inequalities and using \( k = \max \{ k_1, k_2 \} \) and the fact that \( Ax^* = By^* \), we have that
\[
\left\| x_{n+1} - x^* \right\|^2 + \left\| y_{n+1} - y^* \right\|^2
\]
\[
\leq \left\| x_n - x^* \right\|^2 + \left\| y_n - y^* \right\|^2 + 2\gamma \left( \lambda_{A^*A} + \lambda_{B^*B} \right) \left\| Ax_n - By_n \right\|^2
\]
\[
-2\gamma \left\| Ax_n - By_n \right\|^2
\]
\[
-\left( \alpha_0 - k \right) \sum_{i=1}^{\infty} \alpha_i d^2 \left( x_n - \gamma A^* (Ax_n - By_n), S_i(x_n - \gamma A^* (Ax_n - By_n)) \right)
\]
\[
-\left( \alpha_0 - k \right) \sum_{j=1}^{\infty} \alpha_j d^2 \left( y_n + \gamma B^* (Ax_n - By_n), T_j(y_n + \gamma B^* (Ax_n - By_n)) \right).
\]

That is,
\[
\left\| x_{n+1} - x^* \right\|^2 + \left\| y_{n+1} - y^* \right\|^2
\]
\[
\leq \left\| x_n - x^* \right\|^2 + \left\| y_n - y^* \right\|^2 - \gamma \left( 2 - \gamma \left( \lambda_{A^*A} + \lambda_{B^*B} \right) \right) \left\| Ax_n - By_n \right\|^2
\]
\[
-\left( \alpha_0 - k \right) \sum_{i=1}^{\infty} \alpha_i d^2 \left( x_n - \gamma A^* (Ax_n - By_n), S_i(x_n - \gamma A^* (Ax_n - By_n)) \right)
\]
\[
-\left( \alpha_0 - k \right) \sum_{j=1}^{\infty} \alpha_j d^2 \left( y_n + \gamma B^* (Ax_n - By_n), T_j(y_n + \gamma B^* (Ax_n - By_n)) \right).
\]
Now, set \( \Omega_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \). Then, it follows that

\[
\Omega_{n+1}(x^*, y^*) \leq \Omega_n(x^*, y^*) - \gamma \left( 2 - \gamma(\lambda_{A^*A} + \lambda_{B^*B}) \right) \|Ax_n - By_n\|^2
\]

\[
- (\alpha_0 - k) \sum_{i=1}^{\infty} \alpha_i d^2\left( x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \right)
\]

\[
- (\alpha_0 - k) \sum_{j=1}^{\infty} \alpha_j d^2\left( y_n + \gamma B^*(Ax_n - By_n), T_j(y_n + \gamma B^*(Ax_n - By_n)) \right).
\]

(3.2)

Since \( \alpha_0 \in (k, 1) \) and \( \gamma \in \left( 0, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} \right) \), we have \( 2 - \gamma(\lambda_{A^*A} + \lambda_{B^*B}) > 0 \) and \( (\alpha_0 - k) > 0 \). It follows that

\[
\Omega_{n+1}(x^*, y^*) \leq \Omega_n(x^*, y^*).
\]

So, the sequence \( \{\Omega_n(x^*, y^*)\} \) is non-increasing and bounded below, therefore, it converges. On the other hand, it follows from inequality (3.2) and the convergence of the sequence \( \{\Omega_n(x^*, y^*)\} \) that

\[
\lim_{n \to \infty} \|Ax_n - By_n\| = 0,
\]

(3.3)

\[
\lim_{n \to \infty} d\left( x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \right) = 0,
\]

(3.4)

and

\[
\lim_{n \to \infty} d\left( y_n + \gamma B^*(Ax_n - By_n), T_j(y_n + \gamma B^*(Ax_n - By_n)) \right) = 0.
\]

(3.5)

Furthermore, since \( \{\Omega_n(x^*, y^*)\} \) converges, we have that \( \{x_n\} \) and \( \{y_n\} \) are bounded. Let \( x^{**} \) and \( y^{**} \) be the weak-cluster points of the sequences \( \{x_n\} \) and \( \{y_n\} \), respectively. Then, there exists a subsequence of \( \{\langle x_n, y_n \rangle \} \) (without loss of generality, still denoted by \( \{\langle x_n, y_n \rangle \} \)) such that \( x_n \to x^{**} \) and \( y_n \to y^{**} \). Next, we show that \( x^{**} \in S_i x^{**}, i = 1, 2, \cdots \) and \( y^{**} \in T_j y^{**}, j = 1, 2, \cdots \). Since for each \( i = 1, 2, \cdots, S_i \) is uniformly continuous, it follows from (3.3) that

\[
\lim_{n \to \infty} D\left( S_i(x_n - \gamma A^*(Ax_n - By_n)), S_i x_n \right) = 0.
\]

(3.6)

Similarly, since, for each \( j = 1, 2, \cdots, T_j \) is uniformly continuous, we have that

\[
\lim_{n \to \infty} D\left( T_j(y_n + \gamma B^*(Ax_n - By_n)), T_j y_n \right) = 0.
\]

(3.7)
We now show that for each \( i = 1, 2, \ldots \), \( \lim_{n \to \infty} d(x_n, S_ix_n) = 0 \).

Take \( z^i_n \in S_i(x_n - \gamma A^*(Ax_n - By_n)) \) such that

\[
\left\| x_n - \gamma A^*(Ax_n - By_n) - z^i_n \right\| \leq d(x_n, S_i(x_n - \gamma A^*(Ax_n - By_n))) + \frac{1}{n}.
\]

Then, using (3.3), (3.4) and (3.6), we have

\[
d(x_n, S_ix_n) \leq \left\| x_n - \gamma A^*(Ax_n - By_n) - x_n \right\| + \left\| x_n - \gamma A^*(Ax_n - By_n) - z^i_n \right\|
+ \left( z^i_n, S_i(x_n - \gamma A^*(Ax_n - By_n)) \right) \\
\leq \gamma \left\| Ax^* \right\| \left\| Ax_n - By_n \right\|
+ \left( x_n - \gamma A^*(Ax_n - By_n), S_i(x_n - \gamma A^*(Ax_n - By_n)) \right) + \frac{1}{n}
+ D \left( S_i(x_n - \gamma A^*(Ax_n - By_n)), S_ix_n \right) \longrightarrow 0 \text{ as } n \to \infty.
\]

This implies that

\[
\lim_{n \to \infty} d(x_n, S_ix_n) = 0, \text{ for each } i = 1, 2, 3, \ldots. (3.8)
\]

Similarly, we have that

\[
\lim_{n \to \infty} d(y_n, T_jy_n) = 0, \text{ for each } j = 1, 2, 3, \ldots. (3.9)
\]

Now, since \( x_n \to x^* \), \( S_i \) is demi-closed at zero, for each \( i = 1, 2, 3, \ldots \), and \( \lim_{n \to \infty} d(x_n, S_ix_n) = 0 \), for each \( i = 1, 2, 3, \ldots \), we have that \( x^* \in S_ix^* \) for each \( i = 1, 2, 3, \ldots \), which shows that \( x^* \in \bigcap_{i=1}^{\infty} F(S_i) \). Similarly, we have that \( y^* \in \bigcap_{j=1}^{\infty} F(T_j) \).

Since \( A : H_1 \to H_3 \) and \( B : H_2 \to H_3 \) are bounded linear maps, and \( \{x_n\} \) and \( \{y_n\} \) converge weakly to \( x^* \) and \( y^* \), respectively, we have that for arbitrary \( f \in H_3^* \),

\[
f(Ax_n) = (f \circ A)(x_n) \longrightarrow (f \circ A)(x^*) = f(Ax^*).
\]

Similarly,

\[
f(By_n) = (f \circ B)(y_n) \longrightarrow (f \circ B)(y^*) = f(By^*).
\]

These convergences imply that

\[
Ax_n - By_n \to Ax^* - By^*,
\]

which, in turn, implies that

\[
\left\| Ax^* - By^* \right\| \leq \liminf_{n \to \infty} \left\| Ax_n - By_n \right\| = 0,
\]

so that \( Ax^* = By^* \). Hence, we have \( (x^*, y^*) \in \Omega \).

Summing up, we have proved that:
(1) for each \((x^*, y^*) \in \Omega, \lim_{n \to \infty} \left( \left\| x_n - x^* \right\|^2 + \left\| y_n - y^* \right\|^2 \right) \) exists;

(2) each weak cluster point of the sequence \(\{(x_n, y_n)\}\) belongs to \(\Omega\).

Taking \(H = H_1 \times H_2\) with the norm \(\left\| (x, y) \right\| = \left( \left\| x \right\|^2 + \left\| y \right\|^2 \right)^{\frac{1}{2}}\), \(W = \Omega, \mu_n = (x_n, y_n)\), and \(\mu = (x^*, y^*)\) in lemma 2.7, we have that there exists \((\bar{x}, \bar{y}) \in \Omega\) such that \(x_n \rightharpoonup \bar{x}\) and \(y_n \rightharpoonup \bar{y}\). Hence, the sequence \(\{(x_n, y_n)\}\) generated by the iterative scheme (3.1) converges weakly to a solution of problem (1.10) in \(\Omega\). This completes the proof. \(\square\)

We now prove the following strong convergence theorem.

**Theorem 3.2** Suppose assumptions (A₁)–(A₄) hold such that \(\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset\) and \(\bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset\). Assume that for \(x^* \in \bigcap_{i=1}^{\infty} F(S_i)\), \(S_ix^* = \{x^*\}\) and for \(y^* \in \bigcap_{j=1}^{\infty} F(T_j)\), \(T_jy^* = \{y^*\}\). Let \(\{x_n\}\) and \(\{y_n\}\) be as in theorem 3.1. If \(\Omega \neq \emptyset\), and the mappings \(S_i, i = 1, 2, 3, \ldots\) and \(T_j, j = 1, 2, 3, \ldots\) are hemi-compact, then, the sequence \(\{(x_n, y_n)\}\) generated by (3.1) converges strongly to a solution of problem (1.10) in \(\Omega\).

**Proof.** Since \(S_i, i = 1, 2, 3, \ldots\) and \(T_j, j = 1, 2, 3, \ldots\) are hemi-compact, \(\{x_n\}\) and \(\{y_n\}\) are bounded (by theorem 3.1), and \(\lim_{n \to \infty} d\left( x_n, S_ix_n \right) = 0\) for each \(i = 1, 2, 3, \ldots\), and \(\lim_{n \to \infty} d\left( y_n, T_jy_n \right) = 0\) for each \(j = 1, 2, 3, \ldots\), there exist (without loss of generality) subsequences \(\{x_{n_k}\}\) of \(\{x_n\}\) and \(\{y_{n_k}\}\) of \(\{y_n\}\) such that \(\{x_{n_k}\}\) and \(\{y_{n_k}\}\) converge strongly to some points \(x^*\) and \(y^*\), respectively. It follows from the demi-closedness of \(S_i, i = 1, 2, 3, \ldots\) and \(T_j, j = 1, 2, 3, \ldots\) that \(x^* \in \bigcap_{i=1}^{\infty} F(S_i)\) and \(y^* \in \bigcap_{j=1}^{\infty} F(T_j)\).

Thus,

\[
\left\| Ax^* - By^* \right\| = \lim_{k \to \infty} \left\| Ax_{n_k} - By_{n_k} \right\| = 0.
\]

This implies that \(Ax^* = By^*\). Hence, \((x^*, y^*) \in \Omega\). On the other hand, since \(\Omega_n(x, y) = \left\| x_n - x \right\|^2 + \left\| y_n - y \right\|^2\) for any \((x, y) \in \Omega\), we know that \(\lim_{k \to \infty} \Omega_n(x^*, y^*) = 0\). From theorem 3.1, we have \(\lim_{n \to \infty} \Omega_n(x^*, y^*)\) exists, therefore \(\lim_{n \to \infty} \Omega_n(x^*, y^*) = 0\). So, the iterative scheme converges strongly to a solution of problem (1.10) in \(\Omega\). The proof is complete. \(\square\)

**Corollary 3.3** Suppose assumptions (A₁)–(A₄) hold such that \(\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset\) and \(\bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset\). Assume that for \(x^* \in \bigcap_{i=1}^{\infty} F(S_i)\), \(S_ix^* = \{x^*\}\) and for \(y^* \in \bigcap_{j=1}^{\infty} F(T_j)\), \(T_jy^* = \{y^*\}\) and let \(\{x_n\}\) and \(\{y_n\}\) be as in theorem 3.1. If \(\Omega \neq \emptyset\), and the mappings \(S_i, i = 1, 2, 3, \ldots\) and \(T_j, j = 1, 2, 3, \ldots\) have convex and compact domain, then, the sequence \(\{(x_n, y_n)\}\) generated by (3.1) converges strongly to a solution of problem (1.10) in \(\Omega\).
Proof. Since every multi-valued mapping $T : \mathcal{D}(T) \subseteq H \to CB(H)$, with $\mathcal{D}(T)$ compact, is hemi-compact, the proof follows from theorem 3.2. □

Corollary 3.4 Suppose assumptions (A₁), (A₃) and (A₄) hold such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $\bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Assume that for $x^* \in \bigcap_{i=1}^{\infty} F(S_i)$, $S_ix^* = \{x^*\}$ and for $y^* \in \bigcap_{j=1}^{\infty} F(T_j)$, $T_jy^* = \{y^*\}$ and let $\{x_n\}$ and $\{y_n\}$ be as in theorem 3.1. If $\Omega \neq \emptyset$, and the mappings $S_i$, $i = 1, 2, 3, \cdots$ and $T_j$, $j = 1, 2, 3, \cdots$ are quasi-nonexpansive and hemi-compact, then, the sequence $\{(x_n, y_n)\}$ generated by (3.1) converges strongly to a solution of problem (1.10) in $\Omega$.

Remark 3.5

Our theorems 3.1 and 3.2 extend and complement the results of Chang et al. [5], Wu et al. [20], Chidume et al. [9] and Tang Jinfang [17] in the following sense:

- The class of operators considered in this paper is larger than the class considered in [5], [17], and [20].

- The Multiple-sets Split Equality Fixed Point Problem considered in this paper is a more general problem than the Split Feasibility Problem considered in [5] and [17].

- In [20], the authors considered finite families of multi-valued maps, whereas in this paper, we consider countable families of multi-valued maps.

- In [9], the authors considered single-valued maps, whereas in this paper, we considered countable families of multi-valued maps.

Remark 3.6

The recursion formula considered in this paper is, in some sense, of Krasnoselskii-type (since $\sum_{i=0}^{\infty} \alpha_i = 1$) which, in general, converges as fast as a geometric progression (see, for example [18]).

Remark 3.7

Prototypes for our iteration process are $\alpha_i = \frac{1}{2^i+1}$, $i = 0, 1, 2, 3, \cdots$, and $\alpha_j = \frac{1}{2^j+1}$, $j = 0, 1, 2, 3, \cdots$. 
References


Received: January 10, 2015; Published: February 12, 2015