An Extension of Hadamard Fractional Integral

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Abstract

In this paper, we present an extension of the classical Hadamard fractional integral. The extension is based on \( k \)-gamma function. We discuss the properties, involving the semigroup property, commutative law and boundedness of the extended operator.

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1 Introduction

We start with some basic definitions, symbols, notations and results that will be used throughout the discussion.

The set of real numbers is denoted by \( \mathbb{R} \). We write \( \log a \) for \( \log_e a \) or \( \ln a \). We simply write \( \alpha, \beta > a \) for \( \alpha > a, \beta > a \), etc. Likewise if \( x \in (a,b) \) and \( y \in (a,b) \), we write it simply by \( x, y \in (a,b) \), etc.

Large dedicated literature is available to study fractional integrals. We refer [5], [4] and [9]. We, also, refer [1], [11] and [2] to study special functions and [8] for mathematical analysis.
Hadamard [6] has defined fractional integral, so far called Hadamard fractional integral, of order \(\alpha > 0\), over the interval \([a, t]\), as

\[
I^\alpha (f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) d\tau,
\]

which differs from Riemann-Liouville and Caputo’s definition in the sense that the kernel of the integral (1.1) contains logarithmic function of arbitrary exponent. We can denote \(I^\alpha (f(x))\), defined in (1.1), by \(I^\alpha_a f(t), I^\alpha_{a,b} f\) or \(D^\alpha_{a,b}(f(x))\) too, instead of the formal notation \(D^\alpha_{a,b} f(t)\).

Diaz and Pariguan [7] have introduced \(k\) – gamma and \(k\) – beta functions as

\[
\Gamma_k(x) = \lim_{n \to \infty} \frac{n!k^n (nk)^{x-1}}{(k)_{n}}, \quad x \in C - kZ^-, \quad k > 0,
\]

where \((k)_{n}\) is the Pochhammer \(k\) – symbol for factorial function, which is defined as

\[
(k)_{n} = \prod_{j=0}^{n-1} (x + jk),
\]

\[
B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt.
\]

It is easy to see that

\[
\Gamma_k(x) = \int_0^\infty e^{-t} t^{x-1} dt = k^{\frac{x}{k}} \Gamma_k\left(\frac{x}{k}\right), \quad \text{Re}(x) > 0,
\]

\[
B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \frac{1}{k} B\left(\frac{x}{k}, \frac{y}{k}\right), \quad \text{Re}(x) > 0, \text{Re}(y) > 0,
\]

\[
\Gamma_k(k) = 1,
\]

\[
\Gamma_k(x + k) = x\Gamma_k(x),
\]

\[
(k)_{n} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)}.
\]

Obviously, for \(k = 1\), we have the classical gamma and beta functions

\[
\Gamma_1(x) = \Gamma(x),
\]

\[
B_1(x, y) = B(x, y).
\]
Mubeen and Habibullah [10] have used $k$-gamma function to introduce an extension of fractional integral operator. While Farid, Habibullah and Shahzeen [3] have presented some inequalities, on the basis of which, fractional integral inequalities can be established.

The classical gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt = \int_0^{\infty} e^{-t} t^{\alpha-1} dt + \int_{\infty}^{\infty} e^{-t} t^{\alpha-1} dt = \gamma(\alpha, x) + \Gamma(\alpha, x),$$

(1.10)

where

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt,$$

(1.11)

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$$

(1.12)

are called upper and lower incomplete gamma functions, respectively.

Substituting $y = \frac{t - x}{t - x}$, it is easy to see that

$$\int_x^t (t - \tau)^{\alpha-1} (\tau - x)^{\beta-1} d\tau = (t - x)^{\alpha+\beta-1} B(p, q).$$

(1.13)

Likewise, let

$$I_k = \int_0^1 \left( \log \frac{t}{\tau} \right)^{\alpha-1} \left( \log \frac{\tau}{u} \right)^{\beta-1} \frac{d\tau}{\tau},$$

(1.14)

then if we put $y = \frac{\log \tau}{\log u}$, $1 - y = \frac{\log t}{\log u}$, $d\tau = \log \frac{t}{u} dy$ and $y : 0 \to 1$. Thus,

(1.14) becomes

$$I_k = \left( \log \frac{t}{u} \right)^{\alpha+\beta-1} \int_0^1 (1 - y)^{\alpha-1} y^{\beta-1} dy = \left( \log \frac{t}{u} \right)^{\alpha+\beta-1} B(\alpha, \beta).$$

By use of (1.4), it becomes

$$\int_0^1 \left( \log \frac{t}{\tau} \right)^{\alpha-1} \left( \log \frac{\tau}{u} \right)^{\beta-1} d\tau = k \left( \log \frac{t}{u} \right)^{\alpha+\beta-1} B_k(\alpha, \beta).$$

(1.15)

Definition 1.1:

Definition 1.2: For any tuple $(k, \alpha, a)$ of positive real numbers, $t > a$, we define Hadamard $k$–fractional integral, of order $\alpha$, of a function $f(x)$ as

$$\frac{1}{k} \Gamma_k(\alpha) \int_a^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) d\tau.$$ 

(1.16)
We can denote \( k \frac{D}{H} \overline{I}_a(t, \alpha) \), which is \( k \frac{D}{H} \overline{I}_a(t, \alpha) (f(x)) \) in fact, by \( k \frac{D}{H} \overline{I}_a(t, \alpha) (f)(t) \), \( k \frac{D}{H} \overline{I}_a(t, \alpha) (f(x))(t) \) or \( k \frac{D}{H} \overline{D}_a(t, \alpha) f(t) \) too. It is easy to see that

\[
\frac{D}{H} \overline{I}_a(t, \alpha) (l) = \frac{1}{\alpha \Gamma_k(\alpha)} (\log \frac{t}{\alpha})^\alpha.
\]  \hspace{1cm} (1.17)

Furthermore, we define \( k \frac{D}{H} \overline{I}_a(t, \alpha) \) as in (1.15) for the function \( f \) which is continuous on the open interval \((a, t)\) for some \( t \geq a \) subjecting to growth condition at \( a \), we prove the properties, involving commutative law and semigroup property, of the extended integral operator. Moreover, we define \( k \frac{D}{H} \overline{I}_a(t, \alpha) \), for \( \alpha < 0 \), as the inverse operation to \( k \frac{D}{H} \overline{I}_a(t, \alpha) \), that is, we define \( g(t) = k \frac{D}{H} \overline{I}_a(t, \alpha) f(t) \) to be the solution, if it exists, of the integral equation \( f(t) = k \frac{D}{H} \overline{I}_a(t, \alpha) g(t) \). For imaginary \( \alpha \), Kober [5] has investigated an extension of \( I^\alpha \).

**Theorem 2.1:** For each fixed tuple \((k, \alpha, \beta, a)\) of positive real numbers, \( t \geq a \), we have

\[
k \frac{D}{H} \overline{I}_a(t, \alpha) k \frac{D}{H} \overline{I}_a(t, \alpha - \beta) f(t) = k \frac{D}{H} \overline{I}_a(t, \alpha) f(t).
\]  \hspace{1cm} (2.1)

**Proof:** Using (1.16) and applying Fubini’s theorem, we find that

\[
k \frac{D}{H} \overline{I}_a(t, \alpha) k \frac{D}{H} \overline{I}_a(t, \alpha - \beta) f(t) = \frac{1}{k^2 \Gamma_k(\alpha) \Gamma_k(\beta)} \int_a^t \left( \int_a^u \frac{1}{\tau} \left( \log \frac{\tau}{\alpha} \right)^{\alpha - 1} \left( \log \frac{u}{\beta} \right)^{\beta - 1} d\tau \right) f(u) du.
\]  \hspace{1cm} (2.2)

Using (1.15), it becomes
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$$\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, f(t) = \frac{B_k(\alpha, \beta)}{k\Gamma_k(\alpha)\Gamma_k(\beta)} \int \frac{1}{u} (\log \frac{t}{u})^{\alpha+\beta-1} f(u) du.$$ 

Eventually, using (1.4), we obtain

$$\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, f(t) = \frac{1}{k\Gamma_k(\alpha + \beta)} \int \frac{1}{u} (\log \frac{t}{u})^{\alpha+\beta-1} f(u) du = \frac{k}{\beta} I_{(a,t)}^{\alpha + \beta}, f(t).$$

**Theorem 2.2:** For each fixed pair \((k,a)\) of positive real numbers, \(t > a\), and fixed pair \((\alpha, \beta)\) of non-negative real numbers, we have

$$\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, f(t) = \frac{k}{\beta} I_{(a,t)}^{\alpha + \beta}, f(t). \quad (2.3)$$

**Proof:** The result (2.3) follows immediately from (1.18) and (2.1).

**Theorem 2.3:** (Commutative Law)

For each fixed pair \((k,a)\) of positive real numbers, \(t > a\), and fixed pair \((\alpha, \beta)\) of real numbers, one has

$$\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, f(t) = \frac{k}{\beta} I_{(a,t)}^{\alpha + \beta}, f(t). \quad (2.4)$$

**Proof:** Case-1: If \(\alpha, \beta \geq 0\), the relation (2.4) follows directly from (2.3) as \(+\) is commutative in \(R\).

Case-2: If \(\alpha, \beta < 0\), assuming that \(\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, (f(x)) = g(x)\), using result of case-1, we obtain

$$f(x) = \frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, g(x) = \frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, g(x),$$

implying that \(\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, f(x) = g(x)\) and consequently, (2.4) holds for \(\alpha, \beta < 0\) too.

Case-3: If one of \(\alpha, \beta\) is positive and the other is negative, without loss of generality we assume that \(\alpha < 0\), \(\beta > 0\). Now if we let \(\frac{k}{\beta} I_{(a,t)}^\alpha, f(x) = g(x)\),

$$f(x) = \frac{k}{\beta} I_{(a,t)}^\alpha, g(x),$$

implying, by use of result of case-1, that

$$\frac{k}{\beta} I_{(a,t)}^\alpha, f(x) = \frac{k}{\beta} I_{(a,t)}^\alpha, \frac{k}{\alpha} I_{(a,t)}^\beta, g(x) = \frac{k}{\beta} I_{(a,t)}^\alpha, \frac{k}{\alpha} I_{(a,t)}^\beta, g(x),$$

and eventually, we obtain

$$\frac{k}{\beta} I_{(a,t)}^\alpha, \frac{k}{\alpha} I_{(a,t)}^\beta, f(x) = \frac{k}{\beta} I_{(a,t)}^\alpha, \frac{k}{\alpha} I_{(a,t)}^\beta, g(x) = \frac{k}{\beta} I_{(a,t)}^\alpha, \frac{k}{\alpha} I_{(a,t)}^\beta, f(x).$$

Hence, (2.4) holds for all \(\alpha, \beta \in R\).

**Theorem 2.4:** (Semi group Property)

For each fixed pair \((k,a)\) of positive real numbers, \(t > a\), and fixed pair \((\alpha, \beta)\) of real numbers, we have

$$\frac{k}{\beta} I_{(a,t)}^\alpha + \frac{k}{\alpha} I_{(a,t)}^\beta, (f(x)) = \frac{k}{\beta} I_{(a,t)}^{\alpha + \beta}, (f(x)). \quad (2.5)$$
Proof: Case-1: If $\alpha, \beta \geq 0$, the relation (2.5) holds as we have already shown (see (2.3)).

Case-2: If $\alpha, \beta < 0$, assuming that $\int_0^\infty I_{\alpha}^\frac{\alpha + \beta}{\alpha} (f(x)) = g(x)$, using (2.1) and (2.4), we obtain

$$f(x) = \int_0^\infty I_{\alpha}^\frac{\alpha}{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (g(x)) = \int_0^\infty I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (g(x)) = \int_0^\infty I_{\alpha} I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (g(x)),$$

implying that $\int_0^\infty I_{\alpha}^\frac{\alpha + \beta}{\alpha} f(x) = g(x)$, and thus (2.5) holds for $\alpha, \beta < 0$ too.

Case-3: If one of $\alpha, \beta$ is positive and the other is negative, without loss of generality we assume that $\alpha < 0, \beta > 0$. Then

(i) if $\alpha + \beta > 0$, assuming that $\int_0^\infty I_{\alpha}^\frac{\alpha + \beta}{\alpha} (f(x)) = g(x)$,

using (2.1), we have $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (f(x)) = \int_0^\infty I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (g(x))$, implying that $g(x) = \int_0^\infty I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (f(x))$, and consequently, (2.5) remain valid for this case,

(ii) and if $\alpha + \beta < 0$, assuming that $\int_0^\infty I_{\alpha}^\frac{\alpha + \beta}{\alpha} (f(x)) = g(x)$, we have

$$f(x) = \int_0^\infty I_{\alpha} I_{\alpha} I_{\alpha} I_{\alpha}^\frac{-\beta}{\alpha} (g(x))$$

and eventually, (2.5) remain valid for this case too.

Hence, (2.5) holds for all $\alpha, \beta \in \mathbb{R}$.

Definition 2.1:

Definition 2.2: We define the class $C_\alpha$ as

$$C_\alpha = \{ f : f \text{ is continuous on } (0,t) \text{ and } f \text{ is integrable at } 0 \}$$

and the class $C_\alpha$ as

$$C_\alpha = \{ f \in C_\alpha : \exists g \text{ in } C_\alpha \text{ such that } \int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f = g \}.$$

Theorem 2.5: If $f \in C_\alpha$, $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f$ exists and belongs to $C_\alpha$ for each fixed pair $(k,a)$ of positive real numbers, $t > a$ and fixed non-negative real number $\alpha$.

Proof: The result follows due to continuity of the function $f$ (see Theorem-6.2.7 of [8]).

Theorem 2.6: If $f \in C_\alpha$, $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f \in \beta_f$ for each fixed pair $(k,a)$ of positive real numbers, $t > a$ and fixed pair $(\alpha, \beta)$ of non-negative real numbers, $\beta \leq \alpha$.

Proof: Since $\alpha, \alpha - \beta \geq 0$ and $f \in C_\alpha$, it follows from Theorem 2.5 that $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f$ and $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{-\alpha}{\alpha} f$ exist and belong to $C_\alpha$. Then letting $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f = g$, it follows, by use of semigroup property, that $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f = \int_0^\infty I_{\alpha} I_{\alpha} I_{\alpha}^\frac{-\alpha}{\alpha} g$ implying that $\int_0^\infty I_{\alpha} I_{\alpha}^\frac{\alpha}{\alpha} f \in \beta_f$. 

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Theorem 2.7: If \( n \) is a positive integer and \( \alpha = -n \), for each fixed tuple \((k, \alpha, a)\) of positive real numbers, \( t > a \), \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \) if and only if \( f \in C_n \).

Proof: If we assume that \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \), letting \( \frac{k}{\alpha} I^{n}_{(a,t)} f = g \), we have \( f = \frac{k}{\alpha} I^{n}_{(a,t)} g \), which implies that \( f \in C_n \).

Conversely, if \( f \in C_n \), by definition there exists a function \( g \in C_0 \) such that \( \frac{k}{\alpha} I^{n}_{(a,t)} g = f \), which implies that \( g = \frac{k}{\alpha} I^{n}_{(a,t)} f \) and so \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \).

Theorem 2.8: Let \( n \) be a positive integer, \( 0 < p < 1 \) and \( \alpha = -n + p \). Then for each fixed pair \((k, \alpha)\) of positive real numbers, \( t > a \),

(i) if \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \), \( f \in C_{n-1} \),

(ii) if \( f \in C_n \), \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \),

(iii) the necessary and sufficient condition for existence of \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) in \( C_0 \) is that \( \frac{k}{\alpha} I^{p}_{(a,t)} f \in C_n \).

Proof:

(i) If we assume that \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \), letting \( \frac{k}{\alpha} I^{n}_{(a,t)} f = g \), we find, by semigroup property, that \( f = \frac{k}{\alpha} I^{n-1}_{(a,t)} h \), where \( h = \frac{k}{\alpha} I^{p}_{(a,t)} g \). But it follows by Theorem 2.5 that \( h \in C_0 \) as \( 1 - p > 0 \). Thus, we conclude that \( f \in C_{n-1} \).

(ii) Now if \( f \in C_n \), by definition there exists a function \( g \in C_0 \) such that \( \frac{k}{\alpha} I^{n}_{(a,t)} g = f \), which, by semigroup property, leads to \( h = \frac{k}{\alpha} I^{p}_{(a,t)} f \), where \( h = \frac{k}{\alpha} I^{p}_{(a,t)} g \). But it follows by Theorem 2.5 that \( h \in C_0 \) as \( p > 0 \). Hence, we accomplish that \( \frac{k}{\alpha} I^{p}_{(a,t)} f \) exists and belongs to \( C_0 \).

(iii) If we assume that \( \frac{k}{\alpha} I^{p}_{(a,t)} f \) exists and belongs to \( C_0 \), taking \( \frac{k}{\alpha} I^{n}_{(a,t)} f = g \), we obtain, by semigroup property, that \( \frac{k}{\alpha} I^{p}_{(a,t)} f = \frac{k}{\alpha} I^{n}_{(a,t)} g \), implying that \( \frac{k}{\alpha} I^{p}_{(a,t)} f \in C_n \).

And if \( \frac{k}{\alpha} I^{p}_{(a,t)} f \in C_n \), by definition, there exists a function \( g \in C_0 \) such that \( \frac{k}{\alpha} I^{p}_{(a,t)} g = \frac{k}{\alpha} I^{n}_{(a,t)} f \), which, by semigroup property, leads to \( g = \frac{k}{\alpha} I^{n}_{(a,t)} f \). Thus, we find that \( \frac{k}{\alpha} I^{n}_{(a,t)} f \) exists and belongs to \( C_0 \).

Theorem 2.9: If \( f \in C_n \), for each fixed pair \((k, \alpha)\) of positive real numbers, \( t > a \), we have

(i) \( \frac{k}{\alpha} I^{p}_{(a,t)} f \in C_{n+p} \) for each \( \beta \in R \),

(ii) \( \frac{k}{\alpha} I^{p}_{(a,t)} f \in C_{n+p-\gamma} \) for each \( \beta, \gamma \in R \).
Proof: Since \( f \in C_{\alpha} \), by definition, there exists a function \( g \in C_{0} \) such that
\[
f = \frac{1}{k} I_{(a,d)}^{\alpha} g,
\]
which, by semigroup property, gives
(i) \( \frac{1}{k} I_{(a,d)}^{\alpha+\beta} g = \frac{1}{k} I_{(a,d)}^{\beta} f \), implying that \( \frac{1}{k} I_{(a,d)}^{\beta} f \in C_{\alpha+\beta} \),
(ii) \( \frac{1}{k} I_{(a,d)}^{\alpha+\beta-\gamma} g = \frac{1}{k} I_{(a,d)}^{\beta-\gamma} f \), implying that \( \frac{1}{k} I_{(a,d)}^{\beta-\gamma} f \in C_{\alpha+\beta-\gamma} \).

**Theorem 2.10:** For any \( f, g \in C_{0} \), if we define \( \frac{1}{k} I_{(a,d)}^{\alpha,(g,\beta)} \) as
\[
\frac{1}{k} I_{(a,d)}^{\alpha,(g,\beta)} f(t) = \left( \frac{g(t)}{k} \right) \frac{1}{k^{\alpha+\beta}} \int_{a}^{t} \frac{1}{\tau} \left( \log \frac{t}{\tau} \right)^{\alpha-1} \left( \frac{g(\tau)}{\tau} \right)^{\beta} f(\tau) d\tau
\]
(2.6)
then
\[
\frac{1}{k} I_{(a,d)}^{\gamma,(g,\alpha+\beta)} \frac{1}{k} I_{(a,d)}^{\alpha,(g,\beta)} f(t) = \frac{1}{k} I_{(a,d)}^{\gamma+(\alpha,\beta)} f(t).
\]
(2.7)

Proof: Using (2.6) and applying Fubini’s theorem, we find that
\[
\frac{1}{k} I_{(a,d)}^{\gamma,(g,\alpha+\beta)} \frac{1}{k} I_{(a,d)}^{\alpha,(g,\beta)} f(t) = \left( \frac{g(t)}{k} \right) \frac{1}{k^{\alpha+\beta}} \int_{a}^{t} \frac{1}{u} \left( \frac{g(u)}{u} \right)^{\beta} f(u) \left( \frac{1}{u} \right) \left( \frac{1}{\tau} \right)^{\alpha-1} \left( \log \frac{t}{\tau} \right)^{\beta} d\tau du
\]
Substitution \( y = \frac{\log \tau}{u} \), and using (1.4) and (1.5), it becomes
\[
\frac{1}{k} I_{(a,d)}^{\gamma,(g,\alpha+\beta)} \frac{1}{k} I_{(a,d)}^{\alpha,(g,\beta)} f(t) = \left( \frac{g(t)}{k} \right) \frac{1}{k^{\alpha+\beta}} \int_{a}^{t} \frac{1}{u} \left( \frac{g(u)}{u} \right)^{\beta} f(u) \left( \frac{1}{u} \right) \left( \frac{1}{\tau} \right)^{\alpha-1} \left( \log \frac{t}{\tau} \right)^{\beta} d\tau du
\]
\[
= \frac{1}{k} I_{(a,d)}^{\gamma+(\alpha,\beta)} f(t).
\]

**Theorem 2.11:** If \( \frac{1}{k} I_{(a,d)}^{\alpha} f = \frac{1}{k} I_{(a,d)}^{\alpha} g \), \( f = g \).

Proof: Since \( \frac{1}{k} I_{(a,d)}^{\alpha} f = \frac{1}{k} I_{(a,d)}^{\alpha} g \), it follows by (1.6), due to linearity of integral, that
\[
\int_{a}^{t} \frac{1}{u} \left( \frac{1}{\tau} \right)^{\alpha-1} \left( f(\tau) - g(\tau) \right) d\tau = 0
\]
implying that \( f = g \) (see p. 105 of [4]).
Example 2.1: For each fixed tuple \((k, \alpha, \beta)\) of positive real numbers, \(t > 1\), we have

\[
\hat{I}_{H}^{\alpha}(\log x)^{\beta-1}(t) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha + \beta)} (\log t)^{\alpha-1}. \tag{2.8}
\]

Solution: The result follows, by use of (1.16) and (1.15) for \(u = 1\), as

\[
\hat{I}_{H}^{\alpha}(\log x)^{\beta-1}(t) = \frac{1}{\Gamma_k(\alpha)} (\log t)^{\alpha-1} B_k(\alpha, \beta) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha + \beta)} (\log t)^{\alpha-1}. \tag{2.9}
\]

and generally, for each \(\gamma > -1\), we have

\[
\hat{I}_{H}^{\alpha}(\log x)^{\gamma}(t) = \frac{\Gamma_k((\gamma+1)k)}{\Gamma_k(\alpha + (\gamma+1)k)} (\log t)^{\frac{\alpha}{k}}. \tag{2.10}
\]

Solution: The results (2.9) and (2.10) follow directly from (2.8) by using \(\beta = 2k\) and \(\beta = (\gamma+1)k\), respectively.

Remark 2.1: One may verify (1.17) from (2.10), for \(\gamma = 0\), with collaboration of (1.5) and (1.6).

Example 2.2: For each fixed pair \((k, \alpha)\) of positive real numbers, \(t > 1\), one has

\[
\hat{I}_{H}^{\alpha}(\log x)^{\gamma}(t) = \frac{\Gamma_k((\gamma+1)k)}{\Gamma_k(\alpha + (\gamma+1)k)} (\log t)^{\frac{\alpha}{k}}. \tag{2.11}
\]

Solution: Using \(f(x) = x\) in (1.16), we find that

\[
\hat{I}_{H}^{\alpha}(\log x)^{\gamma}(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{a}^{t} (\log \frac{t}{\tau})^{\frac{a-1}{k}} \frac{d\tau}{\tau} = \frac{1}{k\Gamma_k(\alpha)} \int_{a}^{t} (\log \frac{t}{\tau})^{\frac{a-1}{k}} d\tau. \tag{2.12}
\]
If we put \( y = \log\left(\frac{t}{\tau}\right) \), \( y : \log(\frac{t}{a}) \to 0 \) as \( \tau : a \to t \) and \( e^y = e^{\log\left(\frac{t}{\tau}\right)} = \frac{t}{\tau} \), implying that \( \tau = te^{-y} \) and thus \( d\tau = -te^{-y} dy \). Thus, (2.12) gives

\[
_{\alpha}I^a_k(x) = \frac{t}{k\Gamma_k(\alpha)} \int_0^y e^{-\gamma} y^{k-1} dy,
\]

which implies (2.11) by use of (1.12).

**Example 2.4:** For each fixed tuple \((k, \alpha, a)\) of positive real numbers, \( t > a \), and fixed \( \lambda \in R - \{0\} \), we have

\[
_{\alpha}I^a_k(x^\lambda) = \frac{t^\lambda}{k\Gamma_k(\alpha)} y^{\left(\frac{\alpha}{k}\right)} \log\left(\frac{t}{a}\right). \tag{2.13}
\]

**Solution:** Using \( f(x) = x^\lambda \) in (1.16), we find that

\[
_{\alpha}I^a_k(x^\lambda) = \frac{1}{k\Gamma_k(\alpha)} \int_a^y (\log\left(\frac{t}{\tau}\right))^{\frac{\alpha}{k}} \tau^{k-1} d\tau. \tag{2.14}
\]

If we put \( y = \log\left(\frac{t}{\tau}\right) \), \( y : \log(\frac{t}{a}) \to 0 \) as \( \tau : a \to t \) and \( e^y = e^{\log\left(\frac{t}{\tau}\right)} = \frac{t}{\tau} \), implying that \( \tau = te^{-y} \) and thus \( d\tau = -te^{-y} dy \). Thus, (2.14) gives

\[
_{\alpha}I^a_k(x^\lambda) = \frac{t^\lambda}{k\Gamma_k(\alpha)} \int_0^y e^{-\lambda y} y^{k-1} dy. \tag{2.15}
\]

If we substitute \( u = \lambda y, dy = \frac{1}{\lambda} du \) and \( u : 0 \to \lambda \log\left(\frac{t}{a}\right) \) as \( y : 0 \to \log\left(\frac{t}{a}\right) \).

Thus, (2.15) becomes

\[
_{\alpha}I^a_k(x^\lambda) = \frac{t^\lambda}{k\Gamma_k(\alpha)} \int_0^{\lambda \log\left(\frac{t}{a}\right)} e^{-u} u^{\frac{\alpha}{k}-1} du,
\]

which implies (2.13), by use of (1.12).

**Remark 2.2:** One may notice that for \( \lambda = 1 \), we find (2.11) from (2.13).
Theorem 2.12: (Boundedness of $\frac{k}{H}^\alpha I^\alpha_{(a,t)}$)

Let $p \geq 1$. Then, for each fixed tuple $(k, \alpha, a)$ of positive real numbers, $t > a$, the operator $\frac{k}{H}^\alpha I^\alpha_{(a,t)}$, defined in (1.16), is bounded in the sense $\frac{k}{H}^\alpha I^\alpha_{(a,t)}: L^p \longrightarrow L^p$ such that

$$\|\frac{k}{H}^\alpha I^\alpha_{(a,t)}f\|_p \leq K \|f\|_p, \quad K = \frac{1}{k \Gamma_k(\alpha)} \int \frac{\log \frac{\theta}{t}}{\log_a} \left(-\theta\right)^{\alpha-1} d\theta.$$

Proof: In (1.16), for any fixed $t > a$ if

$$\psi(t, \tau) = \begin{cases} \frac{1}{k \Gamma_k(\alpha)} \left(\log \frac{\tau}{t}\right)^{\alpha-1}, & \tau \in (a, t), \\ 0, & \tau \notin (a, t) \end{cases}$$

then $|\psi(ct, ct)| = |c|^{-1} |\psi(t, \tau)|$, $c \neq 0$. Therefore, $\mu = 0$ and by Theorem 4.2.6 of [4], we conclude that $\frac{k}{H}^\alpha I^\alpha_{(a,t)}: L^p \longrightarrow L^p$ such that

$$\|\frac{k}{H}^\alpha I^\alpha_{(a,t)}f\|_p \leq K \|f\|_p, \quad K = K(p, \alpha, k, a, t),$$

$$K = \int_a^t \left|\frac{\tau^{\mu-1}}{\tau^{\mu'}} \frac{1}{\psi(1, \tau)} \right|^{\frac{1}{\mu'}} d\tau = \frac{1}{k \Gamma_k(\alpha)} \int_a^t \left|\frac{1}{\psi(1, \tau)} \right| d\tau = \frac{1}{k \Gamma_k(\alpha)} \int_a^t e^{-\frac{\theta}{t}} \left(-\theta\right)^{\alpha-1} d\theta.$$

References


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