Strong Convergence Theorems for a Common Fixed Point of a Finite Family of Multivalued Mappings in Certain Banach Spaces

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Abstract
In this paper, we introduce an new iterative algorithm for finding an element of the set of common fixed points for a finite family of multivalued $k$-strictly pseudo contractive mappings in $q$-uniformly smooth real Banach spaces. Further, we prove strong convergence theorems of the iterative sequence generated by the proposed algorithm under suitable conditions. Our results extend and improve some important recent results.

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1 Introduction

Let $(X, d)$ be a metric space, $K$ be a nonempty subset of $X$ and $T : K \rightarrow 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of $T$ if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of $T$ is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

For several years, the study of fixed point theory for multi-valued nonlinear mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [2], Kakutani [3], Downing and Kirk [4], Nash [7, 8], Geanakoplos [16], Nadla [21]).

Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential Inclusions, Optimization theory. We describe briefly the connection of fixed point theory for multi-valued mappings with these applications.

Optimization problems with constraints. Let $H$ be a real Hilbert space $H$ and $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\varphi : H \rightarrow 2^H$ be a multivalued mapping. Consider the following optimization problem:

$$(P) \quad \left\{ \begin{array}{l} \min f(x) \\ 0 \in \varphi(x). \end{array} \right.$$ 

It is known that the multivalued map, $\partial f$, the subdifferential of $f$, is maximal monotone (see, e.g., [6]), where for $x, w \in H$,

$$w \in \partial f(x) \iff f(y) - f(x) \geq \langle y - x, w \rangle \quad \forall \ y \in H$$

$$\iff x \in \arg\min(f - \langle \cdot, w \rangle).$$

It is easily seen that, for $x \in H$, with $0 \in \varphi(x)$, $x$ is a solution of $(P)$ if and only if $0 \in \partial f(x) \cap \varphi(x)$, or equivalently

$$x \in T_1x \cap T_2x,$$

with $T_1 := I - \partial f$ and $T_2 := I - \varphi$, where $I$ is the identity map of $H$. Therefore, $x$ is a solution of $(P)$ if and only if $x$ is a common fixed point of the
multivalued mappings $T_1$ and $T_2$.

**Game Theory and Market Economy.** In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [7, 8] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [2] or Kakutani [3] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued map* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, e.g., [7]).

From the point of view of social recognition, game theory is perhaps the most successful area of application of *fixed point theory of multi-valued mappings*. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. *This is part of the problem that is being addressed by iterative methods for fixed point of multi-valued mappings.*

**Differential Inclusions, [9].** For $\Omega = (0, \pi)$, consider the following differential inclusion:

$$
\begin{cases}
-\frac{d^2u}{dt^2} \in u - \frac{1}{4} - \frac{1}{4} \text{sgn}(u - 1), & t \in \Omega; \\
u(0) = 0; \\
u(\pi) = 0,
\end{cases}
$$

where

$$
\text{sgn}(x) := \begin{cases}
-1 & \text{if } x < 0; \\
[-1, 1] & \text{if } x = 0; \\
1 & \text{if } x > 0.
\end{cases}
$$

Let $H := H^1_0(\Omega)$ and $(\cdot, \cdot)_H$ the inner product on $H$ defined by:

$$(u, v)_H = \int_{\Omega} u'v' \, dt \quad \forall \ u, v \in H.$$

From Riesz Theorem, there exists an operator $A : H \to H$ satisfying

$$(Au, v)_H = \int_{\Omega} u'v' \, dt, \quad \forall \ v \in H.$$ 

For $u \in H$, let $E(u) := u - \frac{1}{4} - \frac{1}{4} \text{sgn}(u - 1)$. For $w \in E(u)$, let $L^w_u : H \to \mathbb{R}$ the map defined by

$$L^w_u(v) := \int_{\Omega} wv \, dt, \quad \forall \ v \in H.$$
Then $L_w^u$ is linear and continuous on $H$. Therefore, using again Riesz Theorem, there exists a unique vector $b^w_u \in H$ such that:

$$(b^w_u, v)_H = \int_\Omega wv \, dt, \quad \forall \ v \in H.$$ 

Let $B : H \to 2^H$ be the multivalued map defined by:

$$Bu = \{b^w_u : w \in E(u)\}.$$ 

Then, $u$ is a solution of (1) if and only if $Au \in Bu$. Further, the operator $A : H \to H$ is strongly monotone. If we introduce the multivalued map $T : H \to 2^H$ defined by:

$$Tu = u - Au - Bu \quad \forall \ u \in H,$$

Then $u \in H$ is a solution of (1) if and only if $u \in Tu$, that is $u$ is a fixed point of $T$.

Let $K$ be a nonempty subset of a normed space $E$. The set $K$ is called proximinal (see, e.g., [29, 28, 31]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x, u) = \inf \{\|x - y\| : y \in K\} = d(x, K),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets and nonempty, proximinal and bounded subsets of $K$, respectively. The Hausdorff metric on $CB(K)$ is defined by:

$$D(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for all $A, B \in CB(K)$. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called $L$-Lipschitzian if there exists $L > 0$ such that

$$D(Tx, Ty) \leq L\|x - y\| \quad \forall x, y \in D(T).$$

When $L \in (0, 1)$ in (2), we say that $T$ is a contraction, and $T$ is called nonexpansive if $L = 1$.

Different iterative processes have been developed to approximate fixed points of multi-valued nonexpansive mappings (see, e.g., [33, 32, 29, 28, 31], and the references therein) and their generalizations (see, e.g., [34]).

The important class of single-valued $k$-strictly pseudo-contractive maps on Hilbert spaces was introduced by Browder and Petryshyn [1] as a generalization of the class of nonexpansive mappings.
Definition 1.1 Let $K$ be a nonempty subset of a real Hilbert space $H$. A map $T : K \to H$ is called $k$-strictly pseudo-contractive if there exists $k \in (0, 1)$ such that
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \quad \forall x, y \in K.
\] (3)

Motivated by approximating fixed points of multivalued mappings, Chidume et al. [26] introduced the following important class of multi-valued strictly pseudo-contractive mappings in real Hilbert spaces which is more general than the class of multi-valued nonexpansive mappings.

Definition 1.2 A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be $k$-strictly pseudo-contractive if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$ we have:
\[
\left(D(Tx, Ty)\right)^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \quad \forall u \in Tx, v \in Ty.
\] (4)

If $k = 1$ in (4), the map $T$ is said to be pseudo-contractive.

Remark 1.3 It is easily seen that any multi-valued nonexpansive mapping is $k$-strictly pseudocontractive for any $k \in (0, 1)$. Moreover the inverse is not true (see, e.g., Djitte and Sene [17]).

Then, they proved strong convergence theorems for this class of mappings. The recursion formula used in [26] is of Krasnosel'kii-type [36] which is known to be superior (see, e.g., Remark 4 in [26]) to the recursion formula of Mann [24] or Ishikawa [18]. In fact, they proved the following theorem.

Theorem CA1 (Chidume et al. [26]). Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Suppose that $T : K \to CB(K)$ is a multi-valued $k$-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$, $x_{n+1} = (1 - \lambda)x_n + \lambda y_n$, $n \geq 0$, where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then, $\lim_{n \to \infty} d(x_n, Tx_n) = 0$.

They also proved in [26] that under some conditions or on the map $T$ or on the set $K$, the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

Let $E$ be a real normed linear space of dimension $\geq 2$. The modulus of smoothness of $E$, $\rho_E$, is defined by:
\[
\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\} ; \quad \tau > 0.
\]
A normed linear space $E$ is called \textit{uniformly smooth} if

$$\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0.$$ 

It is well known (see, e.g., [25] p. 16, [30]) that $\rho_E$ is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c \tau^q$, then $E$ is said to be \textit{$q$-uniformly smooth}. Typical examples of such spaces are the $L_p$, $\ell_p$ and $W^m_p$ spaces for $1 < p < \infty$ where,

$$L_p \text{ (or } \ell_p \text{) or } W^m_p \text{ is } \begin{cases} 2 \text{- uniformly smooth if } 2 \leq p < \infty; \\ p \text{- uniformly smooth if } 1 < p < 2. \end{cases}$$

Let $J_q$ denote the \textit{generalized duality mapping} from $E$ to $2^{E^*}$ defined by

$$J_q(x) := \{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. $J_2$ is called the \textit{normalized duality mapping} and is denoted by $J$. It is well known that if $E$ is smooth, $J_q$ is single-valued.

Recently, in [27], Chidume et. al. extended the definition of multivalued-strictly pseudo-contractive maps from Hilbert spaces to Banach spaces, class much more larger than that of Hilbert spaces. They proposed the following definition.

**Definition 1.4** Let $E$ be a normed linear space. A multi-valued map $T : D(T) \subset E \to CB(E)$ is called \textit{$k$-strictly pseudo-contractive} if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$,

$$k \left( D(Ax, Ay) \right)^2 \leq \langle u - v, j(x - y) \rangle \quad \forall u \in Ax, v \in Ay. \quad (5)$$

where $A := I - T$ and $I$ is the identity map on $E$.

With this definition at hand, Chidume \textit{et al.}[27] proved some strong convergence theorems for approximating fixed points multi-valued $k$-strictly pseudo-contracative mappings defined on \textit{$q$-uniformly smooth space}.

On the other hand, Abbas \textit{et al.} [33] introduced a new one-step iterative process for approximating a common fixed point of two multivalued \textit{nonexpansive mappings} in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic
boundary conditions. Let \( S, T : K \rightarrow CB(K) \) be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

\[
\begin{cases}
  x_1 \in K \\
  x_{n+1} = a_n x_n + b_n y_n + c_n z_n
\end{cases}
\]  

(6)

where \( y_n \in Tx_n \) and \( z_n \in Sx_n \) are such that:

\[
\begin{cases}
  \| y_n - p \| \leq d(p, Sx_n); \\
  \| z_n - p \| \leq d(p, Tx_n),
\end{cases}
\]  

(7)

and \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are real sequences in \((0, 1)\) satisfying \( a_n + b_n + c_n = 1 \).

Following Abbas et. al. [33], Rashwan and Altwqi [11] introduced a new scheme for approximation a common fixed point of three multivalued nonexpansive mappings in uniformly convex Banach spaces. Let \( S, T, R : K \rightarrow CB(K) \) be three multivalued nonexpansive mappings. They employed the following iterative process:

\[
\begin{cases}
  x_1 \in K \\
  x_{n+1} = a_n y_n + b_n z_n + c_n w_n, \ n \geq 1
\end{cases}
\]  

(8)

where \( y_n \in Tx_n \), \( z_n \in Sx_n \) and \( w_n \in Rx_n \) are such that:

\[
\begin{align*}
  \| y_n - y_{n+1} \| & \leq D(Tx_n, Tx_{n+1}) + \eta_n; \\
  \| z_n - z_{n+1} \| & \leq D(Sx_n, Sx_{n+1}) + \eta_n; \\
  \| w_n - w_{n+1} \| & \leq D(Rx_n, Rx_{n+1}) + \eta_n
\end{align*}
\]

and \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are real sequences in \((0, 1)\) satisfying \( a_n + b_n + c_n = 1 \).

Then, they prove the following result.

**Theorem RA** (Rashwan and Altwqi [11]). Let \( E \) be a uniformly convex real Banach space and \( K \) be a nonempty closed and convex subset of \( E \). Let \( T, S, R : KCB(K) \) be multivalued nonexpansive mappings satisfying condition \((C)\) and \( \{x_n\} \) be the sequence as defined in (8). If \( F \neq \emptyset \) and \( Tp = Sp = Rp = \{p\} \) for any \( p \in F \), then

\[
\lim_{n \to \infty} d(x_n, Tx_n) = \lim_{n \to \infty} d(x_n, Sx_n) = \lim_{n \to \infty} d(x_n, Rx_n) = 0.
\]

It is our purpose in this paper to construct a new iteration process, simpler that those of Abbas et. al. [33] and Rashwan and Altwqi [11] and prove that the corresponding sequence \( \{x_n\} \) converges strongly to a common fixed point of a finite family of *multivalued strictly pseudo contractive mappings* defined in \( q \)-uniformly smooth real Banach spaces. This class of mappings is much more general than that of mutivalued nonexpansive mappings. Our theorems generalize and extend those of Abbas et. al. [33], Rashwan and Altwqi [11], and many other important results.
2 Preliminaries

In the sequel, we shall need the following definitions and results.

Definition 2.1 Let $E$ be a real Banach space and $T$ be a multi-valued mapping. The multi-valued map $(I - T)$ is said to be strongly demiclosed at 0 (see e.g., [34]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges strongly to $x^*$ and $d(x_n, Tx_n)$ converges to 0, then $d(x^*, Tx^*) = 0$.

Lemma 2.2 (Chidume et. al., [26]) Let $E$ be a reflexive real Banach space and let $A, B \in CB(X)$. Assume that $B$ is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that
\[ \|a - b\| \leq D(A, B). \] (9)

Proposition 2.3 Let $K$ be a nonempty subset of a real Banach space $E$ and let $T : K \to CB(K)$ be a multi-valued $k$-strictly pseudo-contractive mapping. Assume that for every $x \in K$, $Tx$ is weakly closed. Then, $T$ is Lipschitzian.

Lemma 2.4 (Chidume et. al., [27]) Let $q > 1$, $E$ be a $q$-uniformly smooth real Banach space, $k \in (0, 1)$. Suppose $T : D(T) \subseteq E \to CB(E)$ is a multi-valued map with $F(T) \neq \emptyset$, and such that for all $x \in D(T)$, $x^* \in F(T)$,
\[ k (D(Ax, Ax^*))^2 \leq \langle u - v^*, j(x - x^*) \rangle \quad \forall u \in Ax, v^* \in Ax^*, \] (10)
where $A := I - T$, $I$ is the identity map on $E$. If $Tx^* = \{x^*\}$ for all $x^* \in F(T)$. Then, the inequality holds
\[ \langle x - y, j_q(x - x^*) \rangle \geq k^{q-1}\|x - y\|^q, \quad \forall x \in D(T), \forall y \in Tx. \]

Lemma 2.5 (Chidume et. al., [27]) Let $K$ be a nonempty closed subset of a real Banach space $E$ and let $T : K \to P(K)$ be a $k$-strictly pseudo-contractive mapping. Assume that for every $x \in K$, $Tx$ is weakly closed. Then, $(I - T)$ is strongly demiclosed at zero.

Lemma 2.6 (H.K. Xu, [38]) Let $q > 1$ and $E$ be a smooth real Banach space. Then the following are equivalent.

(i) $E$ is $q$-uniformly smooth.

(ii) There exists a constant $d_q > 0$ such that for all $x, y \in E$,
\[ \|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + d_q\|y\|^q. \] (11)

From here, $d_q$ is the constant appearing in Lemma 2.6.
3 Main results

In what follows, $E$ is a $q$-uniformly smooth real Banach space, $q > 1$, $K$ is a nonempty closed convex subset of a $E$, $m \geq 1$, and $T_1, \ldots, T_m : K \to CB(K)$ be multivalued $k_i$-strictly pseudocontractive mappings. Let $\{x_n\}$ be a sequence defined iteratively as follows:

$$\begin{aligned}
\begin{cases}
x_1 \in K \\
x_{n+1} = \lambda_0 x_n + \lambda_1 u_1^n + \cdots + \lambda_m u_m^n,
\end{cases}
\end{aligned} \quad (12)$$

where $u_i^n \in T_i x_n$, $i = 1, \ldots, m$, $\lambda_i \in (0, k)$ with $k = \min_{1 \leq i \leq m} \left\{ 1, \left( \frac{qk^i - 1}{2^{(m-1)q}d^i} \right)^{\frac{1}{q-1}} \right\}$ and such that $\lambda_0 + \lambda_1 + \cdots \lambda_m = 1$.

We now prove the following theorem.

**Theorem 3.1** Let $E$ be a $q$-uniformly smooth real Banach space, $q > 1$ and $K$ a nonempty, closed and convex subset of $E$. Let $T_i : K \to CB(K)$, $i = 1, \ldots, m$ be multi-valued $k_i$-strictly pseudo-contractive mappings. Assume that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^m F(T_i)$. Let $\{x_n\}$ be a sequence defined by (12). Then,

$$\lim_{n \to \infty} d(x_n, T_i x_n) = 0 \quad \forall \ i = 1, \ldots, m.$$  

**Proof.** Let $x^* \in \bigcap_{i=1}^m F(T_i)$. Using Lemma 2.4 and inequality (11) of Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \left\| \lambda_0 (x_n - x^*) + \sum_{i=1}^m \lambda_i (u_i^n - x^*) \right\|^q \\
&= \left\| \lambda_0 (x_n - x^*) + \sum_{i=1}^m \lambda_i (u_i^n - x_n) + \sum_{i=1}^m \lambda_i (x_n - x^*) \right\|^q \\
&= \left\| x_n - x^* + \sum_{i=1}^m \lambda_i (u_i^n - x_n) \right\|^q \\
&\leq \|x_n - x^*\|^q - q \sum_{i=1}^m \lambda_i (x_n - u_i^n, j(x_n - x^*)) + d_q \left\| \sum_{i=1}^m \lambda_i (u_i^n - x_n) \right\|^q \\
&\leq \|x_n - x^*\|^q - q \sum_{i=1}^m \lambda_i k_i^{q-1} \|x_n - u_i^n\|^q + d_q \left\| \sum_{i=1}^m \lambda_i (u_i^n - x_n) \right\|^q. \quad (13)
\end{aligned}$$

We have
\[ \left\| \sum_{i=1}^{m} \lambda_i (u_i^n - x_n) \right\|^q \leq 2^{(m-1)q} \sum_{i=1}^{m} \lambda_i^q \left\| u_i^n - x_n \right\|^q. \]  
(14)

Combining inequalities (13) and (14), it then follows that
\[ \left\| x_{n+1} - x^* \right\|^q \leq \left\| x_n - x^* \right\|^q - q \sum_{i=1}^{m} \lambda_i k_i^{q-1} \left\| x_n - u_i^n \right\|^q + d_q 2^{(m-1)q} \sum_{i=1}^{m} \lambda_i^q \left\| u_i^n - x_n \right\|^q. \]
(15)

So,
\[ \sum_{i=1}^{m} \lambda_i \left[ k_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} \right] \left\| x_n - u_i^n \right\|^q \leq \left\| x_n - x^* \right\|^q - \left\| x_{n+1} - x^* \right\|^q. \]

Therefore,
\[ \sum_{n=1}^{\infty} \left( \sum_{i=1}^{m} \lambda_i \left[ k_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} \right] \left\| x_n - u_i^n \right\|^q \right) < \infty. \]  
(16)

Using the definition of \( k \) and the fact that \( \lambda_i \in (0, k) \) for all \( i \), it follows that \( k_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} > 0 \) for all \( i = 1, \cdots, m \). Therefore, from (16), we have
\[ \lim_{n \to \infty} \sum_{i=1}^{m} \lambda_i \left[ k_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} \right] \left\| x_n - u_i^n \right\|^q = 0, \]
which gives
\[ \lim_{n \to \infty} \left\| x_n - u_i^n \right\| = 0 \ \forall \ i = 1, \cdots, m. \]

Since \( u_i^n \in T_i x_n \), it follows that
\[ \lim_{n \to \infty} d(x_n, T_i x_n) = 0 \ \forall \ i = 1, \cdots, m. \]

This completes the proof. We have the following corollaries.

**Corollary 3.2** Let \( E = L_p, 1 < p < \infty \) and \( q := \min \{2, p\} \). Let \( K \) be a nonempty, closed and convex subset of \( E \). Suppose that \( T_i : K \to CB(K), i = 1, \cdots, m \) are multi-valued \( k_i \)-strictly pseudo-contractive mappings such that \( \bigcap_{i=1}^{m} F(T_i) \neq \emptyset \) and \( T_i p = \{p\} \) for all \( p \in \bigcap_{i=1}^{m} F(T_i) \). Let \( \{x_n\} \) be a sequence defined (12). Then,
\[ \lim_{n \to \infty} d(x_n, T_i x_n) = 0 \ \forall \ i = 1, \cdots, m. \]
Proof. Since $L_p$ spaces, $1 < p < \infty$ are $q$-uniformly smooth real Banach spaces with $q := \min\{2, p\}$, the proof follows from Theorem 3.1.

Corollary 3.3 Let $K$ be a nonempty, closed and convex subset of a 2-uniformly smooth real Banach space. Suppose that $T : K \to CB(K)$ is a multi-valued $k_1$-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $T_p = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by (12). Then, $\lim_{n \to \infty} d(x_n, Tx_n) = 0$.

Proof. Just take $m = 1$ in Theorem 3.1. We now approximate common fixed points of the mappings $T_i$ through strong convergence of the sequence $\{x_n\}$ defined by (12). We start with the following definition.

Definition 3.4 A mapping $T : K \to CB(K)$ is called hemicompact if, for any sequence $\{x_n\}$ in $K$ such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. We note that if $K$ is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Theorem 3.5 Let $E$ be a $q$-uniformly smooth real Banach space and $K$ be a nonempty, closed and convex subset of $E$. Let $T_i : K \to CB(K), i = 1 \cdots, m$ be multi-valued continuous, $k_i$-strictly pseudo-contractive mappings with $\bigcap_{i=1}^m F(T_i) \neq \emptyset$ and such that $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^m F(T_i)$. Assume that $T_{i_0}$ is hemicompact for some $i_0$. Then the sequence $\{x_n\}$ defined by (12) converges strongly to a common fixed point of the $T_i$'s.

Proof. From Theorem 3.1, we have $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$. Since $T_{i_0}$ is hemicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ for some $p \in K$. Using the fact that, for all $i$, $T_i$ is continuous, we have $d(x_{n_j}, T_i x_{n_j}) \to d(p, T_i p)$. Therefore, $d(p, T_i p) = 0$ and so $p \in F(T_i)$. Setting $x^* = p$ in the proof of Theorem 3.1, it follows from inequality (13) that the sequence $\{\|x_n - p\|\}$ is decreasing and bounded from below. Therefore, $\lim_{n \to \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to $p$. This completes the proof.

Corollary 3.6 Let $E$ be a $q$-uniformly smooth real Banach space and $K$ be a nonempty, compact and convex subset of $E$. Let $T_i : K \to CB(K), i = 1 \cdots, m$ be multi-valued continuous, $k_i$-strictly pseudo-contractive mappings with $\bigcap_{i=1}^m F(T_i) \neq \emptyset$ and such that $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^m F(T_i)$. Then the sequence $\{x_n\}$ defined by (12) converges strongly to a common fixed point of the $T_i$’s.

Proof. Observing that if $K$ is compact, every map $T : K \to CB(K)$ is hemicompact, the proof follows from Theorem 3.5.
**Definition 3.7** The mappings $T_1, \ldots, T_m : K \to CB(K)$ satisfy Condition $(I^*)$, if there exists a strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and $i_0$, $1 \leq i_0 \leq m$ such that

$$d(x, T_{i_0}x) \geq f(d(x, F)) \forall x \in K, \text{ with } F := \cap_{i=1}^{m} F(T_i).$$

**Theorem 3.8** Let $E$ be a $q$-uniformly smooth real Banach space and $K$ be a nonempty, closed and convex subset of $E$. Let $T_i : K \to CB(K), i = 1 \ldots, m$ be multi-valued $k_i$-strictly pseudo-contractive mappings with $\cap_{i=1}^{m} F(T_i) \neq \emptyset$ and such that $\forall \ x \in K, \ Tx$ is weakly closed and $T_i p = \{p\}$ for all $p \in \cap_{i=1}^{m} F(T_i)$. Assume $T_1, \ldots, T_m$ satisfy the condition $(I^*)$. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the $T_i$’s.

**Proof.** From Theorem 3.1, we have $\lim_{n \to \infty} d(x_n, T_ix_n) = 0$. Using the fact that $T_i$ satisfies condition $(I)$, it follows that $\lim_{n \to \infty} f(d(x_n, F(T_i))) = 0$. Thus there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T_i)$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j} \quad \forall \ j \in \mathbb{N}.$$

By setting $x^* = p_j$ and following the same arguments as in the proof of Theorem 3.1, we obtain from inequality (13) that

$$\|x_{n_j+1} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in $K$. Notice that

$$\|p_{j+1} - p_j\| \leq \|p_{j+1} - x_{n_j+1}\| + \|x_{n_j+1} - p_j\| < \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}.$$

This shows that $\{p_j\}$ is a Cauchy sequence in $K$ and thus converges strongly to some $p \in K$. Using the fact that $T_i$ is $L$-Lipschitzian and $p_j \to p$, we have

$$d(p_j, T_ip) \leq D(T_ip_j, T_ip) \leq L\|p_j - p\|,$$

so that $d(p, T_ip) = 0$ and thus $p \in T_ip$. Therefore, $p \in F(T_i)$ and $\{x_{n_j}\}$ converges strongly to $p$. Setting $x^* = p$ in the proof of Theorem 3.1, it follows from inequality (13) that $\lim_{n \to \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to $p$. This completes the proof.
Remark 3.9 The recursion formula (12) used in our theorems is of the Krasnoselkii type (see e.g., [36]) and it is superior to the recursion formula of Abbas et. al. (6) and the one of Rashwan and Altawqin [11] the following sense:

- In the recursion formula (6), $y_n \in Tx_n$ and $z_n \in Sx_n$ are required to satisfy (7) and in the the recursion formula (8), $y_n \in Tx_n$, $z_n \in Sx_n$ and $w_n \in Rx_n$ are required to satisfy (8). In the recursion formula (12), used in our theorems, the $u^n_i, (i = 0, 1, 2)$ are chose freely in $T_ix_n$ without any additional conditions.

- The recursion formula (12) requires less computation time than the formula (6) because the parameters $\lambda_i$ in formula (12) are fixed in $(k, 1)$ whereas in the algorithms (6) and (8), the $\lambda_i$ are replaced by sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ in $(0, 1)$ satisfying the condition: $a_n + b_n + c_n = 1$. The parameters $a_n, b_n$ and $c_n$ must be chose at each step of the iteration process.

- The Krasnoselskii-type algorithm usually yields rate of convergence as fast as that of a geometric progression whereas the algorithm (6), usually has order of convergence of the form o(1/n).

Remark 3.10 In the Hilbert spaces setting, our theorems in this papers are important generalizations of several important recent results in the following sense: Our theorems extend results proved for multi-valued nonexpansive mappings in real Hilbert spaces (see e.g., [28, 29, 31, 32, 33]) to the much more larger class of multi-valued strictly pseudo-contractive mappings.

References


Iterative algorithm


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